

Non-Existence of Entities (Sci.American 1980s)

There are objects/entities which one can describe but which can't exist (maybe because their description is “faulty”), one example:

Describe really large numbers, using n symbols, e.g. $n = 3$. Maybe this could be 999, better 9^{9^9} , or (hexadecimal) F^{F^F} , ...

LARGEST $n \in \mathbb{N}$ DESCRIBED BY AT MOST 43 SYMBOLS

$$7 + 3 + 9 + 2 + 2 + 4 + 2 + 7 = 36 + 7 \text{ spaces} \Rightarrow 43 \text{ symbols}$$

Thus, we can't have the largest number described with 45 symbols:

LARGEST $n \in \mathbb{N}$ DESCRIBED BY AT MOST 45 SYMBOL $S+1$

Halting Problem for Register Machines

Definition. A register machine H **decides the Halting Problem** if for all $e, a_1, \dots, a_n \in \mathbb{N}$, starting H with

$$R_0 = 0 \quad R_1 = e \quad R_2 = \ulcorner [a_1, \dots, a_n] \urcorner$$

and all other registers zeroed, the computation of H always halts with R_0 containing 0 or 1; moreover when the computation halts, $R_0 = 1$ if and only if

the register machine program with index e eventually halts when started with $R_0 = 0, R_1 = a_1, \dots, R_n = a_n$ and all other registers zeroed.

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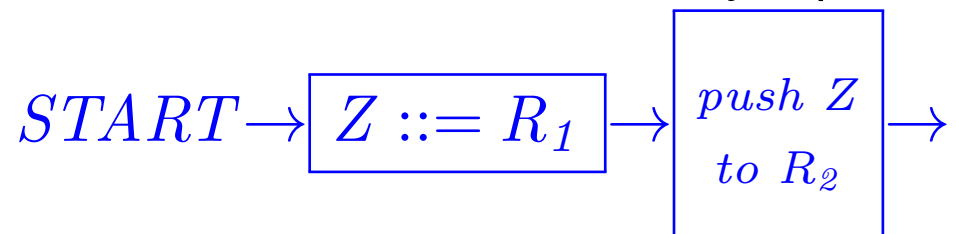
the register machine program with index e eventually halts when started with $R_0 = 0, R_1 = a_1, \dots, R_n = a_n$ and all other registers zeroed.

Theorem No such register machine H can exist.

Proof of the theorem

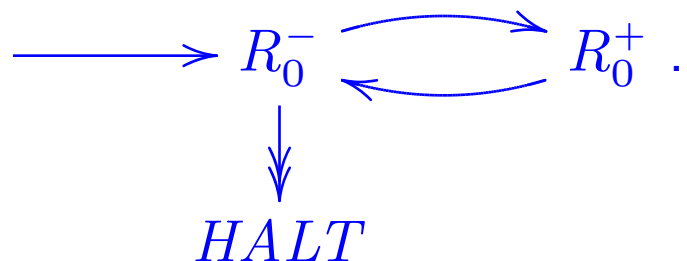
Assume we have a RM H that decides the Halting Problem and derive a contradiction, as follows:

- Let H' be obtained from H by replacing $START \rightarrow$ by



(where Z is a register not mentioned in H 's program).

- Let C be obtained from H' by replacing each $HALT$ (& each erroneous halt) by



- Let $c \in \mathbb{N}$ be the index of C 's program.

Proof of the theorem

Assume we have a RM H that decides the Halting Problem and derive a contradiction, as follows: (assuming $R_0 = 0$ and $R_2 = 0$)

C started with $R_1 = c$ eventually halts

if and only if

H' started with $R_1 = c$ halts with $R_0 = 0$

Proof of the theorem

Assume we have a RM H that decides the Halting Problem and derive a contradiction, as follows:

C started with $R_1 = c$ eventually halts

if and only if

H' started with $R_1 = c$ halts with $R_0 = 0$

if and only if

H started with $R_1 = c, R_2 = \lceil c \rceil$ halts with $R_0 = 0$

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H started with $R_1 = c, R_2 = \ulcorner c \urcorner$ halts with $R_0 = 0$

if and only if

$prog(c)$ started with $R_1 = c$ does not halt

$prog(c)$ means the program given by the number c .

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$prog(c)$ started with $R_1 = c$ does not halt

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C started with $R_1 = c$ does not halt

Contradiction!

Enumerating computable functions

For each $e \in \mathbb{N}$, let $\varphi_e \in \mathbb{N} \rightarrow \mathbb{N}$ be the unary partial function computed by the RM with program $prog(e)$. So for all $x, y \in \mathbb{N}$:

$\varphi_e(x) = y$ holds iff the computation of $prog(e)$ started with $R_0 = 0, R_1 = x$ and all other registers zeroed eventually halts with $R_0 = y$.

Thus

$$e \mapsto \varphi_e$$

defines an **onto** function from \mathbb{N} to the collection of all computable partial functions from \mathbb{N} to \mathbb{N} .

An uncomputable function

Let $f \in \mathbb{N} \rightarrow \mathbb{N}$ be the partial function $\{(x, 0) \mid \varphi_x(x) \uparrow\}$.

$$\text{Thus } f(x) = \begin{cases} 0 & \text{if } \varphi_x(x) \uparrow \\ \text{undefined} & \text{if } \varphi_x(x) \downarrow \end{cases}$$

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f is not computable, because if it were, then $f = \varphi_e$ for some $e \in \mathbb{N}$ and hence

- if $\varphi_e(e) \uparrow$, then $f(e) = 0$ (by def. of f); so $\varphi_e(e) = 0$ (by def. of e), i.e. $\varphi_e(e) \downarrow$
- if $\varphi_e(e) \downarrow$, then $f(e) \uparrow$ (by def. of e); so $\varphi_e(e) \uparrow$ (by def. of f)

Contradiction! So f cannot be computable.

(Un)decidable sets of numbers

Given a subset $S \subseteq \mathbb{N}$, its **characteristic function** $\chi_S \in \mathbb{N} \rightarrow \mathbb{N}$ is

$$\text{given by: } \chi_S(x) \triangleq \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

(Un)decidable sets of numbers

Definition. $S \subseteq \mathbb{N}$ is called (register machine) **decidable** if its characteristic function $\chi_S \in \mathbb{N} \rightarrow \mathbb{N}$ is a register machine computable function. Otherwise it is called **undecidable**.

So S is decidable iff there is a RM M with the property: for all $x \in \mathbb{N}$, M started with $R_0 = 0$, $R_1 = x$ and all other registers zeroed eventually halts with R_0 containing 1 or 0; and $R_0 = 1$ on halting iff $x \in S$.

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Basic strategy: to prove $S \subseteq \mathbb{N}$ undecidable, try to show that decidability of S would imply decidability of the Halting Problem.

For example...

Claim: $S_0 \triangleq \{e \mid \varphi_e(0) \downarrow\}$ is undecidable.

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Proof (sketch): Suppose M_0 is a RM computing χ_{S_0} . From M_0 's program (using the same techniques as for constructing a universal RM) we can construct a RM H to carry out:

let $e = R_1$ and $\ulcorner [a_1, \dots, a_n] \urcorner = R_2$ in

$R_1 ::= \ulcorner (R_1 ::= a_1) ; \dots ; (R_n ::= a_n) ; \text{prog}(e) \urcorner ;$

$R_2 ::= 0 ;$

run M_0

Then by assumption on M_0 , H decides the Halting Problem. **Contradiction.**
So no such M_0 exists, i.e. χ_{S_0} is uncomputable, i.e. S_0 is undecidable.

Claim: $S_1 \triangleq \{e \mid \varphi_e \text{ total function}\}$ is undecidable.

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Proof (sketch): Suppose M_1 is a RM computing χ_{S_1} . From M_1 's program we can construct a RM M_0 to carry out:

$let\ e = R_1\ in\ R_1 ::= \lceil R_1 ::= 0 ; prog(e) \rceil ;$
 $run\ M_1$

Then by assumption on M_1 , M_0 decides membership of S_0 from previous example (i.e. computes χ_{S_0}). **Contradiction.** So no such M_1 exists, i.e. χ_{S_1} is uncomputable, i.e. S_1 is undecidable.