Program Analysis (70020)

Probabilistic Programs

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Probabilistic Problem I: Guards and Conditionals

```
[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
    [n := n − 1]^4
end while
[stop]^5
```

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1$; $\quad \triangleright\ P(m = 1), P(m = 2), \ldots$ — $P(n = 1), \ldots$
while $[n > 1]^2$ **do**
  $[m := m \times n]^3$;
  $[n := n - 1]^4$
end while
$[\textbf{stop}]^5$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

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[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
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end while
[stop]^5
```

$\triangleright\ (p_1, p_2, p_3, \ldots) — (q_1, q_2, \ldots)$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1$;
while $[n > 1]^2$ **do**
    $[m := m \times n]^3$;
    $[n := n - 1]^4$
end while
$[\textbf{stop}]^5$

$\triangleright\ (p_1, p_2, p_3, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

```
[m := 1]^1;                      ▷ (p_1, p_2, p_3, ...) — (1/2, 1/2, ...)
while [n > 1]^2 do               ▷ (1, 0, 0, ...) — (1/2, 1/2, ...)
    [m := m × n]^3;
    [n := n − 1]^4
end while
[stop]^5
```

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

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[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
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end while
[stop]^5
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$\triangleright\ (p_1, p_2, p_3, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$

$\triangleright\ (1, 0, 0, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1$;	$\triangleright\ (p_1, p_2, p_3, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$
while $[n > 1]^2$ **do**	$\triangleright\ (1, 0, 0, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$
$\quad [m := m \times n]^3$;	$\triangleright\ (1, 0, 0, \ldots)$ — $(0, \frac{1}{2}, \ldots)$
$\quad [n := n - 1]^4$	
end while	
$[$**stop**$]^5$	$\triangleright\ (1, 0, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1;$ $\triangleright\ (p_1, p_2, p_3, \ldots) — (\frac{1}{2}, \frac{1}{2}, \ldots)$

while $[n > 1]^2$ **do** $\triangleright\ (1, 0, 0, \ldots) — (\frac{1}{2}, \frac{1}{2}, \ldots)$

    $[m := m \times n]^3;$ $\triangleright\ (1, 0, 0, \ldots) — (0, \frac{1}{2}, \ldots)$

    $[n := n - 1]^4$ $\triangleright\ (0, 1, 0, \ldots) — (0, \frac{1}{2}, \ldots)$

end while

$[$**stop**$]^5$ $\triangleright\ (1, 0, 0, \ldots) — (\frac{1}{2}, 0, \ldots)$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1;$

while $[n > 1]^2$ **do**

  $[m := m \times n]^3;$

  $[n := n - 1]^4$

end while

$[\textbf{stop}]^5$

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$\triangleright\ (1, 0, 0, \ldots) — (\frac{1}{2}, \frac{1}{2}, \ldots)$

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Concrete Probabilities

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[m := 1]^1;
while [n > 1]^2 do
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end while
[stop]^5
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$\triangleright\ (0, \frac{1}{2}, 0, \ldots) — (0, \frac{1}{2}, \ldots)$

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Concrete Probabilities

Perhaps better this way?

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1$;
while $[n > 1]^2$ **do**
    $[m := m \times n]^3$;
    $[n := n - 1]^4$
end while
$[\textbf{stop}]^5$

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Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1$; $\triangleright\ (p_1, p_2, p_3, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$

while $[n > 1]^2$ **do** $\triangleright\ (0, 1, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$

    $[m := m \times n]^3$;

    $[n := n - 1]^4$

end while

$[\textbf{stop}]^5$ $\triangleright\ (1, 0, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1$; ▷ $(p_1, p_2, p_3, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$

while $[n > 1]^2$ **do** ▷ $(0, 1, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$

$\quad [m := m \times n]^3$;

$\quad [n := n - 1]^4$

end while

$[\textbf{stop}]^5$ ▷ $(1, 0, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$
$(0, 1, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

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$\triangleright\ (p_1, p_2, p_3, \ldots) — (\frac{1}{2}, \frac{1}{2}, \ldots)$

$\triangleright\ (0, 1, 0, \ldots) — (\frac{1}{2}, 0, \ldots)$

$\triangleright\ (1, 1, 0, \ldots) — (1, 0, \ldots)$

Concrete Probabilities

Correct? How to justify this?

Probabilistic Problem I: Guards and Conditionals

```
[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
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end while
[stop]^5
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$[m := 1]^1;$ $\triangleright$ $(p_1, p_2, p_3, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$

while $[n > 1]^2$ **do** $\triangleright$ $(0, 1, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$

$\quad [m := m \times n]^3;$

$\quad [n := n - 1]^4$

end while

$[\textbf{stop}]^5$ $\triangleright$ $(1, 0, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$
$(0, 1, 0, \ldots)$ — $(\frac{1}{2}, 0, \ldots)$

Concrete Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := 1]^1$;
while $[n > 1]^2$ **do**
    $[m := m \times n]^3$;
    $[n := n - 1]^4$
end while
$[\textbf{stop}]^5$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

```
[m := 1]^1;                 ▷ P(m = 2k), P(m ≠ 2k) — P(n = 1), ...
while [n > 1]^2 do
    [m := m × n]^3;
    [n := n − 1]^4
end while
[stop]^5
```

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := 1]^1;$
while $[n > 1]^2$ **do**
    $[m := m \times n]^3;$
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end while
$[\textbf{stop}]^5$

$\triangleright\ (p_e, p_o)$ — $(q_1, q_2, \ldots)$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := 1]^1$;
while $[n > 1]^2$ **do**
    $[m := m \times n]^3$;
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end while
$[\textbf{stop}]^5$

$\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

```
[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
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[stop]^5
```

▷ (p_e, p_o) — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
▷ $(0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := 1]^1;$
while $[n > 1]^2$ **do**
  $[m := m \times n]^3;$
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end while
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$\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

Abstract Probabilities

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$[m := 1]^1;$	$\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
while $[n > 1]^2$ **do**	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\quad [m := m \times n]^3;$	$\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
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end while	
$[\textbf{stop}]^5$	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Abstract Probabilities

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$[m := 1]^1;$ $\quad\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

while $[n > 1]^2$ **do** $\quad\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

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$\quad [n := n - 1]^4$ $\quad\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

end while

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Abstract Probabilities

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while $[n > 1]^2$ **do**	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
$[m := m \times n]^3$;	$\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
$[n := n - 1]^4$	$\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
end while	$\triangleright\ (1, 0)$ — $(\frac{1}{3}, \frac{1}{3}, 0, \ldots)$
$[$**stop**$]^5$	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

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end while

$[\textbf{stop}]^5$ $\quad \triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$
$(1, 0)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Abstract Probabilities

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$\triangleright\ (1, 0)$ — $(\frac{1}{3}, \frac{1}{3}, 0, \ldots)$
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$(1, 0)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

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[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
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$\triangleright\ (1, 0) — (\frac{1}{3}, \frac{1}{3}, 0, \ldots)$

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$(1, 0) — (\frac{1}{3}, 0, 0, \ldots)$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

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while $[n > 1]^2$ **do** $\quad \triangleright\ (1, 0)$ — $(\frac{1}{3}, 0, 0, \ldots)$

$\quad [m := m \times n]^3;$

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Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := 1]^1;$ $\quad \triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

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Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

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[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
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end while
[stop]^5
```

$\triangleright$ (p_e, p_o) — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

$\triangleright$ $(1, 0)$ — $(\frac{1}{3}, \frac{1}{3}, 0, \ldots)$

$\triangleright$ $(0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

$\triangleright$ $(1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

$\triangleright$

Abstract Probabilities

Correct?

Probabilistic Problem II: Abstract Evaluation

```
[m := 1]^1;
while [n > 1]^2 do
    [m := m × n]^3;
    [n := n − 1]^4
end while
[stop]^5
```

- $\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
  - $\triangleright\ (1, 0)$ — $(\frac{1}{3}, \frac{1}{3}, 0, \ldots)$
  - $\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
  - $\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
- $\triangleright$

Abstract Probabilities

How to justify this?

Probabilistic Problem III: Relational Dependency

Given an (input) distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$ for n one would expect an (output) distribution $(\frac{2}{3}, \frac{1}{3})$ for $even(m)$ and $odd(m)$.

Probabilistic Problem III: Relational Dependency

Given an (input) distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$ for n one would expect an (output) distribution $(\frac{2}{3}, \frac{1}{3})$ for $even(m)$ and $odd(m)$.

For every pair (m, n) we can write the probabilities to observe it as $P(m = i \wedge n = j) = P(m = i)P(n = j)$ – assume perhaps that n does not change.

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For every pair (m, n) we can write the probabilities to observe it as $P(m = i \wedge n = j) = P(m = i)P(n = j)$ – assume perhaps that n does not change.

The available data thus suggest this probability distribution:

	$n = 1$	$n = 2$	$n = 3$
$even(m)$	$\frac{1}{3} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{2}{3}$
$odd(m)$	$\frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{1}{3}$

Probabilistic Problem III: Relational Dependency

Given an (input) distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$ for n one would expect an (output) distribution $(\frac{2}{3}, \frac{1}{3})$ for $even(m)$ and $odd(m)$.

For every pair (m, n) we can write the probabilities to observe it as $P(m = i \wedge n = j) = P(m = i)P(n = j)$ – assume perhaps that n does not change.

The available data thus suggest this probability distribution:

	$n = 1$	$n = 2$	$n = 3$
$even(m)$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
$odd(m)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

Probabilistic Problem III: Relational Dependency

Given an (input) distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$ for n one would expect an (output) distribution $(\frac{2}{3}, \frac{1}{3})$ for $even(m)$ and $odd(m)$.

For every pair (m, n) we can write the probabilities to observe it as $P(m = i \wedge n = j) = P(m = i)P(n = j)$ – assume perhaps that n does not change.

In fact, we have the following joint probability distribution:

	$n = 1$	$n = 2$	$n = 3$
$even(m)$	0	$\frac{1}{3}$	$\frac{1}{3}$
$odd(m)$	$\frac{1}{3}$	0	0

Problems in Probabilistic Program Analysis

$[m := 1]^1$; $\quad\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

while $[n > 1]^2$ **do** $\quad\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

  $[m := m \times n]^3$; $\quad\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

  $[n := n - 1]^4$ $\quad\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

end while

[**stop**]^5 $\quad\triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Problems in Probabilistic Program Analysis

$[m := 1]^1;$	$\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
while $[n > 1]^2$ **do**	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\quad [m := m \times n]^3;$	$\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\quad [n := n - 1]^4$	$\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
end while	
$[$**stop**$]^5$	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Problems in Probabilistic Program Analysis

$[m := 1]^1;$ $\quad \triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

while $[n > 1]^2$ **do** $\quad \triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

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$\quad [n := n - 1]^4$ $\quad \triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

end while

$[$**stop**$]^5$ $\quad \triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Splitting: How to distribute information along branches?

Problems in Probabilistic Program Analysis

$[m := 1]^1;$ $\qquad \triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

while $[n > 1]^2$ **do** $\qquad \triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

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end while

$[$**stop**$]^5$ $\qquad \triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Splitting: How to distribute information along branches?

Transforming: How computing changes the information?

Problems in Probabilistic Program Analysis

$[m := 1]^1;$	$\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
while $[n > 1]^2$ **do**	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\quad [m := m \times n]^3;$	$\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\quad [n := n - 1]^4$	$\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
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$[\textbf{stop}]^5$	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$ $(1, 0)$ — $(\frac{1}{3}, 0, 0, \ldots)$ $(1, 0)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Splitting: How to distribute information along branches?

Transforming: How computing changes the information?

Joining: How to combine information along branches?

Probability and Computation

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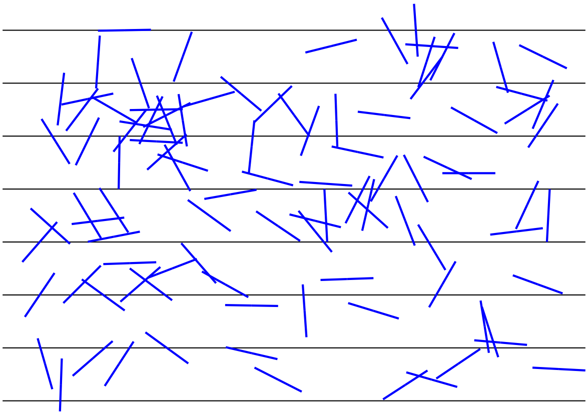
Las Vegas Algorithms
: are randomised algorithms that always give correct results (with non-deterministic running time), e.g. QuickSort (with random pivoting).

Monte Carlo Algorithms
: produce (with deterministic running time) an output which may be incorrect with a certain probability, e.g. Buffon's Needle.

(Georges-Louis Leclerc, Comte de) Buffon's Needle



$$\Pr(\text{cross}) = \frac{2}{\pi} \text{ or } \pi = \frac{2}{\Pr(\text{cross})}$$

The Monty Hall Problem

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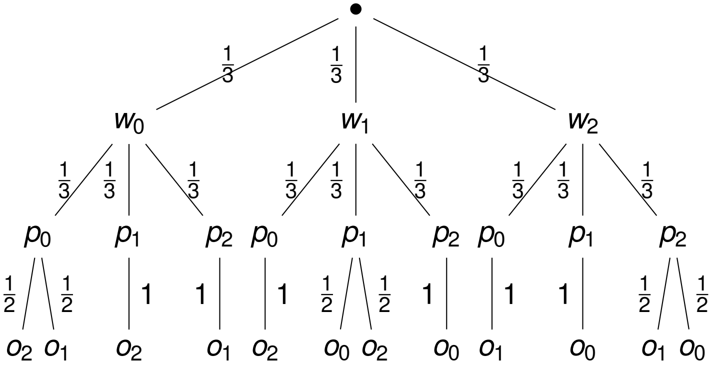
- Instead, the host – legendary Monty Hall – opens one of the other doors which is empty.
- After that the contestant is given a last chance to stick with his/her door or to switch to the other closed one.
- Note that the host (knowing where the prize is) has always at least one door he can open.

Optimal Strategy: To Switch or not to Switch



$w_i =$ win behind i $\quad p_i =$ pick door i $\quad o_i =$ Monty opens door i

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Model: Definite Value
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Probability — Probabilistic Non-Determinism
Model: Distribution (Measure)
e.g. $(0, 0, \frac{1}{5}, 0, \frac{1}{5}, 0, \ldots) \in \mathcal{V}(\mathbb{N})$

Structures: Power Sets

Given a finite set (universe) Ω (of states) we can construct the power set $\mathcal{P}(\Omega)$ of Ω easily as:

$$\mathcal{P}(\Omega) = \{X \mid X \subseteq \Omega\}$$

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$$\mathcal{V}(\Omega) = \{\langle \omega, x_\omega \rangle \mid \omega \in \Omega, x_\omega \in \mathbb{K}\} = \{(x_\omega)_{\omega \in \Omega} \mid x_\omega \in \mathbb{K}\}$$

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As function spaces $\mathcal{V}(\Omega)$ and $\mathcal{P}(\Omega)$ are not so different:

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However, there are major topological problems when Ω is (un)countable infinite.

Tuple Spaces

Theorem

All finite dimensional vector spaces are isomorphic to the (finite) Cartesian product of the underlying field $\mathbb{K}^n$ *(e.g.* $\mathbb{R}^n$ *or* $\mathbb{C}^m$*).*

Finite dimensional vectors can always be represented via their coordinates with respect to a given base, e.g.

$$
\begin{aligned}
x &= (x_1, x_2, x_3, \ldots, x_n) \\
y &= (y_1, y_2, y_3, \ldots, y_n)
\end{aligned}
$$

Algebraic Structure

$$
\begin{aligned}
\alpha x &= (\alpha x_1, \alpha x_2, \alpha x_3, \ldots, \alpha x_n) \\
x + y &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n)
\end{aligned}
$$

Introducing Probability in Programs

Various ways for introducing probabilities into programs:

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Random Assignment The value a variable is assigned to is chosen randomly (according to some, e.g. uniform, probability distribution) from a set:

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Random Assignment The value a variable is assigned to is chosen randomly (according to some, e.g. uniform, probability distribution) from a set:

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Probabilistic Choice There is a probabilistic choice between different instructions:

$$\textbf{choose } 0.5 : (x := 0) \textbf{ or } 0.5 : (x := 1) \textbf{ ro}$$

Syntactic Sugar

One can show that a single “coin flipping” is enough.

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Alternatively we also have

$$\textbf{choose } 0.5 : S_1 \textbf{ or } 0.5 : S_2 \textbf{ ro}$$

is equivalent to (also with other probability distributions):

$$x \mathrel{?}= \{0, 1\}; \ \textbf{if } (x > 0) \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi}$$

Probabilities as Ratios

Consider integer "weights" to express relative probabilities, e.g.

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In general, for constant "weights" p and q (`int`), we translate

$$\textbf{choose } p : S_1 \textbf{ or } q : S_2 \textbf{ ro}$$

(by exploiting an implicit normalisation) into

$$\textbf{choose } \frac{p}{p+q} : S_1 \textbf{ or } \frac{q}{p+q} : S_2 \textbf{ ro}$$

pWHILE – Concrete Syntax

The syntax of statements S is as follows:

$$
\begin{array}{rcl}
S & ::= & \textbf{stop} \\
 & \mid & \textbf{skip} \\
 & \mid & x \;:=\; e \\
 & \mid & x \;?\!\!= r \\
 & \mid & S_1 \,;\; S_2 \\
 & \mid & \textbf{choose } p_1 : S_1 \textbf{ or } p_2 : S_2 \textbf{ ro} \\
 & \mid & \textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi} \\
 & \mid & \textbf{while } b \textbf{ do } S \textbf{ od}
\end{array}
$$

We also allow for boolean expressions, i.e. e is an arithmetic expression a or a boolean expression b. The **choose** statement can be generalised to more than two alternatives.

pWHILE – Labelled Syntax

$$
\begin{array}{rcl}
S & ::= & [\textbf{stop}]^{\ell} \\
 & | & [\textbf{skip}]^{\ell} \\
 & | & [x \ := e]^{\ell} \\
 & | & [x \ ?\!\!= r]^{\ell} \\
 & | & S_1 \, ; \ \ S_2 \\
 & | & \textbf{choose}^{\ell} \ p_1 : S_1 \ \textbf{or} \ p_2 : S_2 \ \textbf{ro} \\
 & | & \textbf{if} \ [b]^{\ell} \ \textbf{then} \ S_1 \ \textbf{else} \ S_2 \ \textbf{fi} \\
 & | & \textbf{while} \ [b]^{\ell} \ \textbf{do} \ S \ \textbf{od}
\end{array}
$$

Where the p_i are constants, representing choice probabilities. By r we denote a range/set, e.g. $\{-1, 0, 1\}$, from which the value of x is chosen (based on a uniform distribution).

Evaluation of Expressions [Not for Exam]

$$\sigma \ni \textbf{State} = (\textbf{Var} \to \textbf{Z} \uplus \textbf{B})$$

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Evaluation $\mathcal{E}$ of expressions *e* in state σ:

$$\begin{array}{rcl} \mathcal{E}(n)\sigma & = & n \\ \mathcal{E}(x)\sigma & = & \sigma(x) \\ \mathcal{E}(a_1 \odot a_2)\sigma & = & \mathcal{E}(a_1)\sigma \odot \mathcal{E}(a_2)\sigma \end{array}$$

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$$\begin{array}{rcl}
\mathcal{E}(\textbf{true})\sigma & = & \textbf{tt} \\
\mathcal{E}(\textbf{false})\sigma & = & \textbf{ff} \\
\mathcal{E}(\textbf{not}\ b)\sigma & = & \neg\mathcal{E}(b)\sigma \\
\ldots & = & \ldots
\end{array}$$

pWhile – SOS Semantics I [Provided in Exam]

R0 $\langle \textbf{skip}, \sigma \rangle \Rightarrow_1 \langle \textbf{stop}, \sigma \rangle$

R1 $\langle \textbf{stop}, \sigma \rangle \Rightarrow_1 \langle \textbf{stop}, \sigma \rangle$

R2 $\langle \mathtt{x} := e, \sigma \rangle \Rightarrow_1 \langle \textbf{stop}, \sigma[x \mapsto \mathcal{E}(e)\sigma] \rangle$

R3' $\langle \mathtt{x} ?= r, \sigma \rangle \Rightarrow_{\frac{1}{|r|}} \langle \textbf{stop}, \sigma[x \mapsto r_i \in r] \rangle$

R3$_1$ $\dfrac{\langle S_1, \sigma \rangle \Rightarrow_p \langle S_1', \sigma' \rangle}{\langle S_1; S_2, \sigma \rangle \Rightarrow_p \langle S_1'; S_2, \sigma' \rangle}$

R3$_2$ $\dfrac{\langle S_1, \sigma \rangle \Rightarrow_p \langle \textbf{stop}, \sigma' \rangle}{\langle S_1; S_2, \sigma \rangle \Rightarrow_p \langle S_2, \sigma' \rangle}$

pWhile – SOS Semantics II [Provided in Exam]

R4$_1$	$\langle \textbf{choose } p_1 : S_1 \textbf{ or } p_2 : S_2, \sigma\rangle \Rightarrow_{p_1} \langle S_1, \sigma\rangle$	
R4$_2$	$\langle \textbf{choose } p_1 : S_1 \textbf{ or } p_2 : S_2, \sigma\rangle \Rightarrow_{p_2} \langle S_2, \sigma\rangle$	
R5$_1$	$\langle \textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2, \sigma\rangle \Rightarrow_1 \langle S_1, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{tt}$
R5$_2$	$\langle \textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2, \sigma\rangle \Rightarrow_1 \langle S_2, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{ff}$
R6$_1$	$\langle \textbf{while } b \textbf{ do } S, \sigma\rangle \Rightarrow_1 \langle S;\ \textbf{while } b \textbf{ do } S, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{tt}$
R6$_2$	$\langle \textbf{while } b \textbf{ do } S, \sigma\rangle \Rightarrow_1 \langle \textbf{stop}, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{ff}$

DTMC Semantics

Given a pWHILE program, consider any enumeration of all its configurations (= pairs of statements and state) $C_1, C_2, C_3, \ldots \in \textbf{Conf}$. Then

$$(\mathbf{T})_{ij} = \begin{cases} p & \text{if } C_i \Rightarrow_p C_j \\ 0 & \text{otherwise} \end{cases}$$

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Example Program

Let us investigate the possible transitions of the following labelled program (with $\mathbf{x} \in \{0, 1\}$):

$$
\begin{array}{l}
\textbf{if } [\mathbf{x} == 0]^1 \textbf{ then} \\
\quad [\mathbf{x} := 0]^2; \\
\textbf{else} \\
\quad [\mathbf{x} := 1]^3; \\
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$$

Record transitions using labelling to simplify notation, i.e.

$$\langle S, \sigma \rangle \ \Rightarrow_p \langle S', \sigma' \rangle \ \text{ becomes } \ \langle \sigma, \mathit{init}(S) \rangle \ \Rightarrow_p \langle \sigma', \mathit{init}(S') \rangle$$

Stating also the initial statement together with $\ell = \mathit{init}(s)$.

Example DTMC

$$
\begin{array}{rc}
\langle x \mapsto 0, [\mathbf{x} == 0]^1 \rangle & \ldots \\
\langle x \mapsto 0, [\mathbf{x} := 0]^2 \rangle & \ldots \\
\langle x \mapsto 0, [\mathbf{x} := 1]^3 \rangle & \ldots \\
\langle x \mapsto 0, [\mathbf{stop}]^4 \rangle & \ldots \\
\langle x \mapsto 1, [\mathbf{x} == 0]^1 \rangle & \ldots \\
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\end{array}
\quad
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
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$$

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\end{pmatrix}
$$

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$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We get: $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

This represents the (deterministic) transition step:

$$\langle x \mapsto 0, [\mathbf{x}{:=}1]^3 \rangle \Rightarrow_1 \langle x \mapsto 1, [\mathbf{stop}]^4 \rangle$$

Linear Operator Semantics (LOS)

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The matrix representation of the SOS semantics of a pWHILE program is not 'compositional'.

In order to be able to analyse programs by analysing its parts, a more useful semantics is one resulting from the composition of different linear operators each expressing a particular operation contributing to the overall behaviour of the program.

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$$\mathbf{State} = (\mathbf{Z} + \mathbf{B})^v = \mathbf{Value}_1 \times \mathbf{Value}_2 \ldots \times \mathbf{Value}_v$$

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Thus, we can represent the space of configurations as

$$\begin{aligned} &\textbf{Dist}(\textbf{Value}_1 \times \ldots \times \textbf{Value}_v \times \textbf{Lab}) \subseteq \\ \subseteq \quad &\mathcal{V}(\textbf{Value}_1 \times \ldots \times \textbf{Value}_v \times \textbf{Lab}) \\ = \quad &\mathcal{V}(\textbf{Value}_1) \otimes \ldots \otimes \mathcal{V}(\textbf{Value}_v) \otimes \mathcal{V}(\textbf{Lab}). \end{aligned}$$

Tensor Product or Kronecker Product

Given a $n \times m$ matrix $\mathbf{A}$ and a $k \times l$ matrix $\mathbf{B}$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1l} \\ \vdots & \ddots & \vdots \\ b_{k1} & \dots & b_{kl} \end{pmatrix}$$

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Special cases are square matrices ($n = m$ and $k = l$) and vectors (row $n = k = 1$, column $m = l = 1$).

Tensor Product Spaces

The tensor product $\mathcal{V} \otimes \mathcal{W}$ of two vector spaces is generated by all linear combinations of the form $v \otimes w$ with $v \in \mathcal{V}$ and $w \in \mathcal{W}$.

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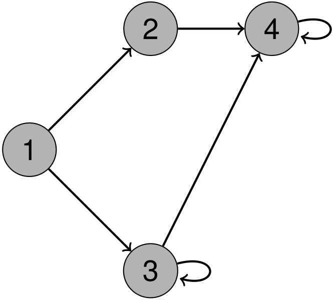
It is possible to construct a base of $\mathcal{V} \otimes \mathcal{W}$ using just base vectors of $\mathcal{V}$ and $\mathcal{W}$ and $\dim(\mathcal{V} \otimes \mathcal{W}) = \dim(\mathcal{V}) \dim(\mathcal{W})$.

Represent joint distributions on $X \times Y$ in $\mathcal{V}(x) \otimes \mathcal{V}(Y)$; e.g.

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} \otimes (1 \;\; 0 \;\; 0) + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} \otimes (0 \;\; \frac{1}{2} \;\; \frac{1}{2})$$

but no two (marginal) distribution exist such that a single tensor product gives this (joint) distribution (non-independence).

Transitions and Generator of DTMC (1) - Deterministic



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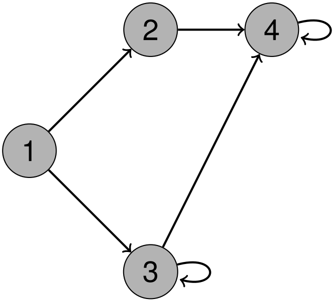
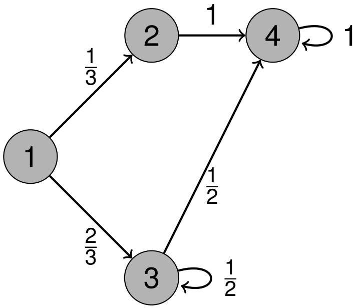


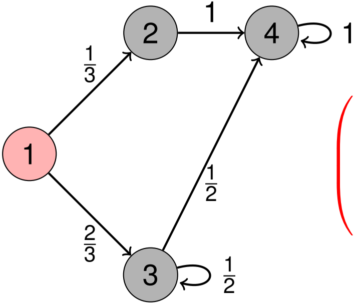
$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{T}$$

Transitions and Generator of DTMC (2) - Probabilistic



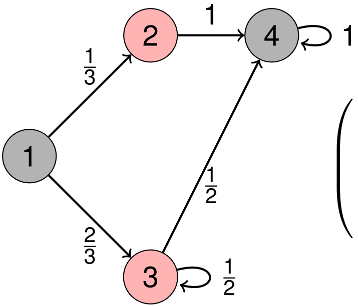
$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{T}$$

Transitions and Generator of DTMC (3)



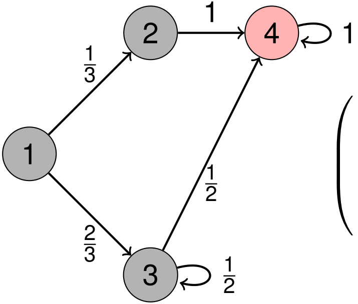
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transitions and Generator of DTMC (4)



$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix}^t$$

Transitions and Generator of DTMC (5)



$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}^\infty = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^t$$

Combination of Steps

We can combine single steps to construct a transition graph.

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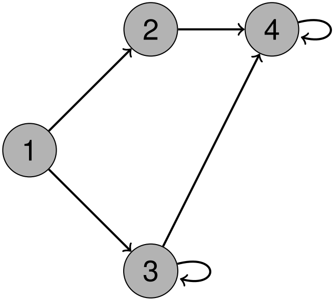

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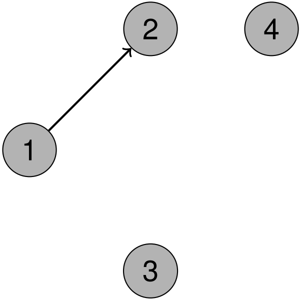

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{array}{l} \mathbf{E}(1,2) \\ \\ \\ \\ \end{array} \right.$$

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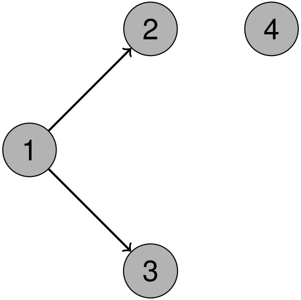

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{array}{ll} & \mathbf{E}(1,2) \\ + & \mathbf{E}(1,3) \end{array} \right.$$

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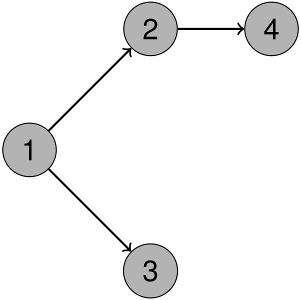

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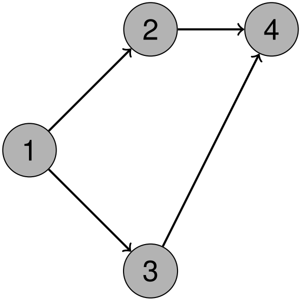

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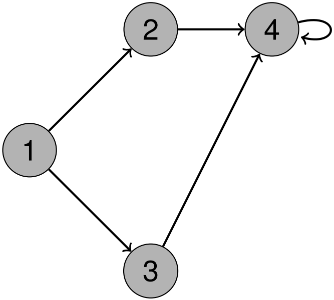

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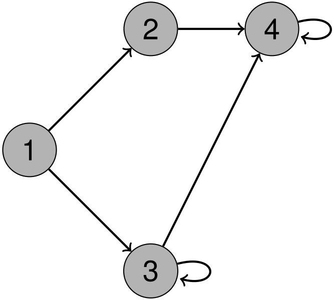

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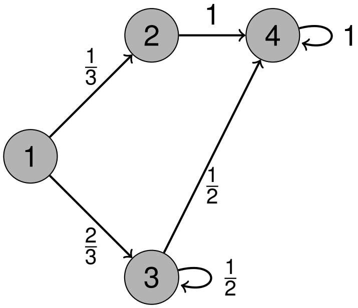
Probabilistic Transitions

Constructing the matrix for probabilistic transitions:



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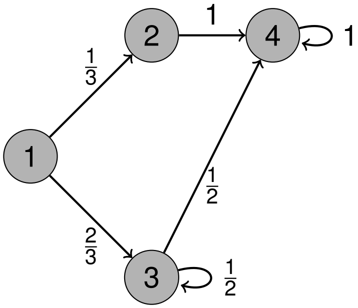

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T

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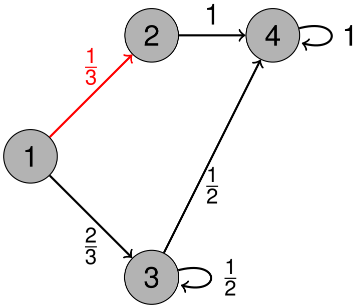

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$$\mathbf{T} = \frac{1}{3}\mathbf{E}(1,2)$$

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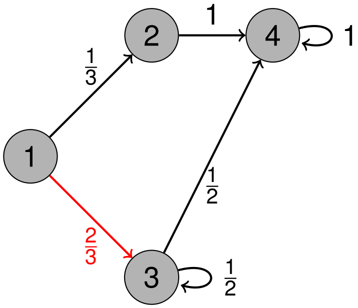

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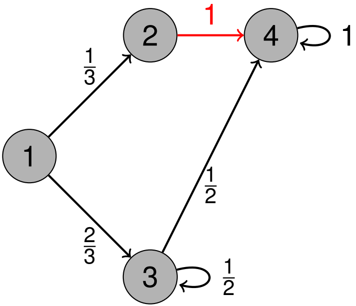

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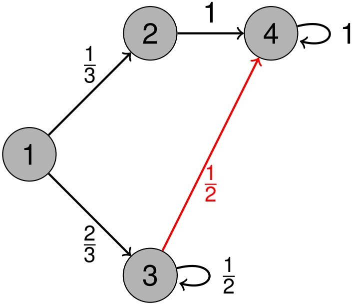

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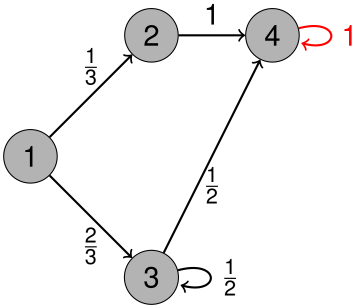

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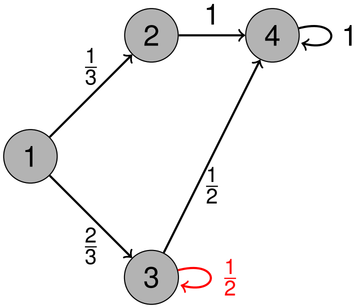

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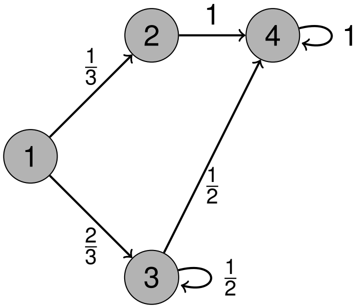

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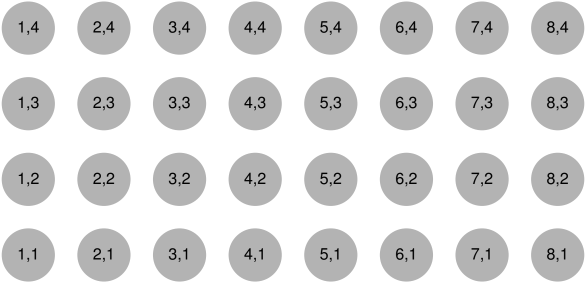
"Turtle" Graphics

Consider a "(probabilistic) turtle graphics" with up/down and left/right moves done simultaneously and probabilistically.



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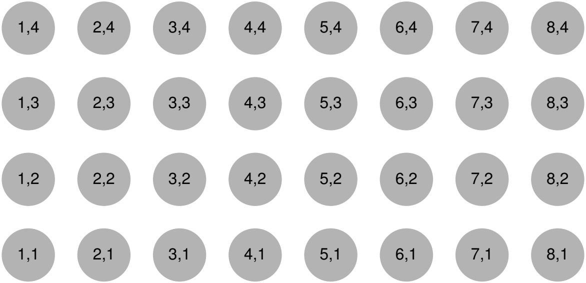

The (classical) configuration space is $\{1, \ldots, 8\} \times \{1, \ldots, 4\}$.
To describe any probabilistic situation, i.e. joint distribution, we need $8 \times 4 = 32$ probabilities, not just $8 + 4 = 12$.

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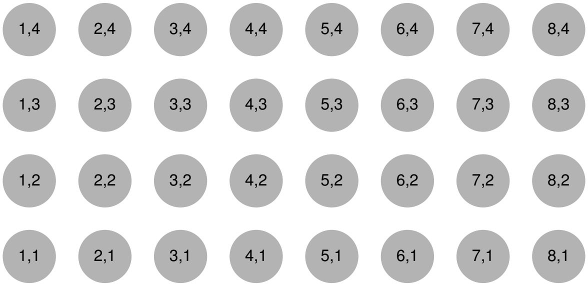

The (classical) configuration space is $\{1, \ldots, 8\} \times \{1, \ldots, 4\}$.
To describe any probabilistic situation, i.e. joint distribution, we need $8 \times 4 = 32$ probabilities, not just $8 + 4 = 12$.
We consider $\mathbb{R}^8 \otimes \mathbb{R}^4 = \mathbb{R}^{32}$ as probabilistic configuration space rather than $\mathbb{R}^8 \oplus \mathbb{R}^4 = \mathbb{R}^{12}$, i.e. just the marginal distributions.

Moves in "Turtle" Graphics

Consider only horizontal moves over eight possible positions.

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Move from 1 to 2: $\mathbf{E}(1, 2)$

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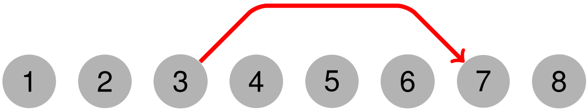
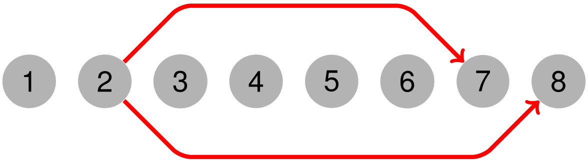


Move from 3 to 7: $\mathbf{E}(3, 7)$

Moves in "Turtle" Graphics

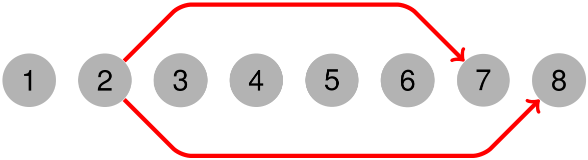
Consider only horizontal moves over eight possible positions.



Move from 2 to 7 or 8: $\mathbf{E}(2,7) + \mathbf{E}(2,8)$

Moves in "Turtle" Graphics

Consider only horizontal moves over eight possible positions.



Move from 2 to 7 or 8: $\mathbf{E}(2,7) + \mathbf{E}(2,8)$ or $\frac{1}{2}\mathbf{E}(2,7) + \frac{1}{2}\mathbf{E}(2,8)$

Moves in "Turtle" Graphics

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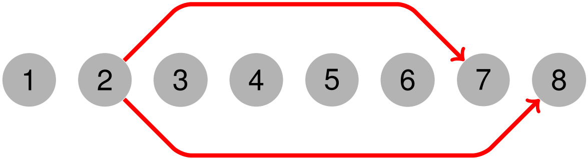
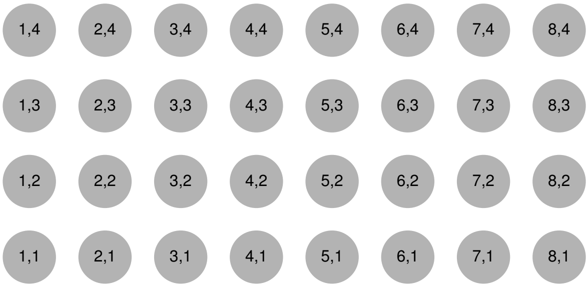


Move from 2 to 7 or 8: $\mathbf{E}(2,7) + \mathbf{E}(2,8)$ or $\frac{1}{2}\mathbf{E}(2,7) + \frac{1}{2}\mathbf{E}(2,8)$

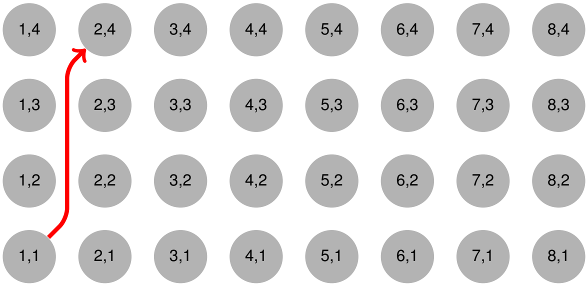
Similar representation also for vertical moves.

"Parallel" Execution: $x \in \{1, \ldots, 8\}$ and $y \in \{1, \ldots, 4\}$



Describe the effect **M** on x and the change of y described by **N**, then the combined effect on $\langle x, y \rangle$ is given by $\mathbf{M} \otimes \mathbf{N}$.

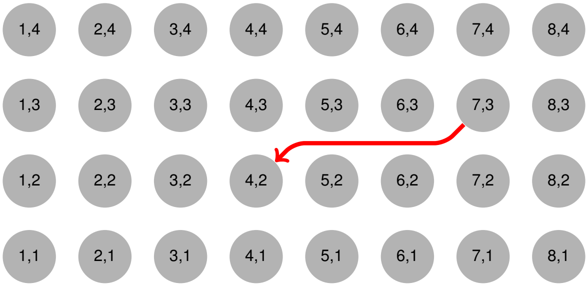
"Parallel" Execution: $x \in \{1, \ldots, 8\}$ and $y \in \{1, \ldots, 4\}$



Describe the effect **M** on x and the change of y described by **N**, then the combined effect on $\langle x, y\rangle$ is given by $\mathbf{M} \otimes \mathbf{N}$.

From $(1, 1)$ move 1 left and 3 up: $\mathbf{E}(1, 2) \otimes \mathbf{E}(1, 4)$

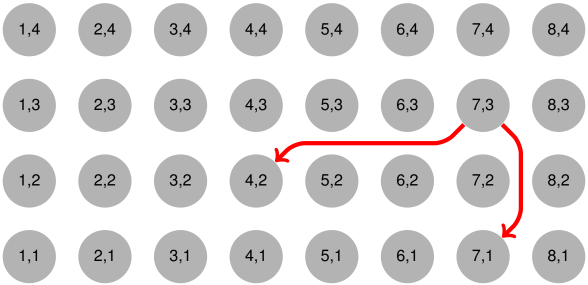
"Parallel" Execution: $x \in \{1, \ldots, 8\}$ and $y \in \{1, \ldots, 4\}$



Describe the effect **M** on x and the change of y described by **N**, then the combined effect on $\langle x, y \rangle$ is given by $\mathbf{M} \otimes \mathbf{N}$.

From $(7, 3)$ move $(4, 2)$: $\mathbf{E}(7, 4) \otimes \mathbf{E}(3, 2)$

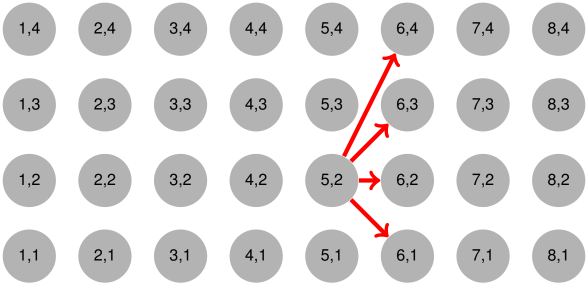
"Parallel" Execution: $x \in \{1, \ldots, 8\}$ and $y \in \{1, \ldots, 4\}$



Describe the effect **M** on x and the change of y described by **N**, then the combined effect on $\langle x, y \rangle$ is given by $\mathbf{M} \otimes \mathbf{N}$.

From $(7, 3)$ to $(4, 2)/(7, 2)$: $\mathbf{E}(7, 4) \otimes \mathbf{E}(3, 2) + \mathbf{E}(7, 7) \otimes \mathbf{E}(3, 1)$

"Parallel" Execution: $x \in \{1, \ldots, 8\}$ and $y \in \{1, \ldots, 4\}$



Describe the effect **M** on x and the change of y described by **N**, then the combined effect on $\langle x, y \rangle$ is given by $\mathbf{M} \otimes \mathbf{N}$.

From $(5, 2)$ move to all one right: $\mathbf{E}(5, 6) \otimes (\sum_{i=1}^{4} \mathbf{E}(2, i))$

Transfer Functions (Edge Effects): Assignment

Assume $x \in 1, .., 8$; How do statements change its value?

Transfer Functions (Edge Effects): Assignment

Assume $x \in 1, .., 8$; How do statements change its value?

$x := 4$

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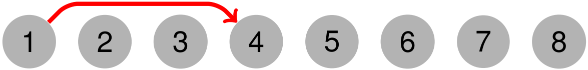

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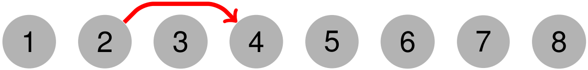

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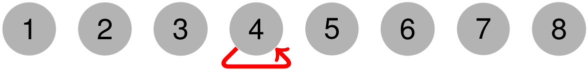

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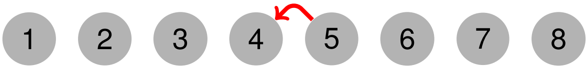

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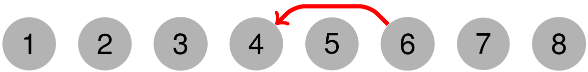

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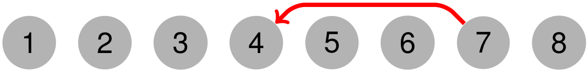

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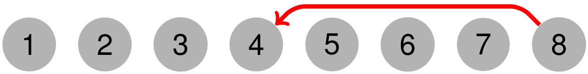

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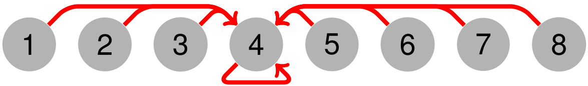

$$x \;:=\; 4 \quad \text{gives} \quad \mathbf{U}(x \leftarrow 4) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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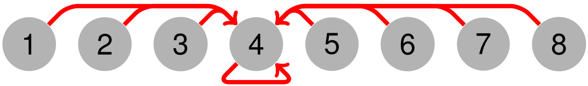

Thus, the LOS of the statement is $[\![x\ :=\ 4]\!] = \mathbf{U}(x \leftarrow 4)$.

Transfer Functions (Edge Effects): Shift

Assume $x \in 1, .., 8$; How do statements change its value?

Transfer Functions (Edge Effects): Shift

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$x := x + 1$

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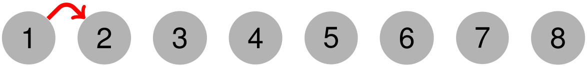

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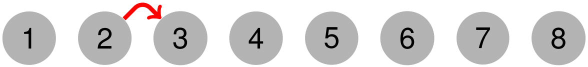

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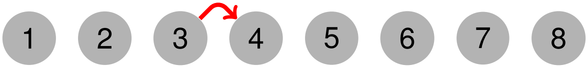

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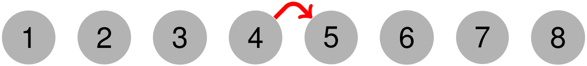

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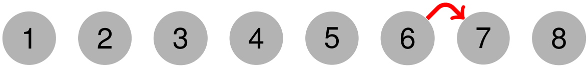

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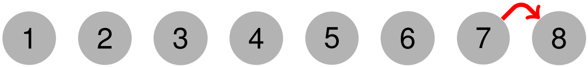

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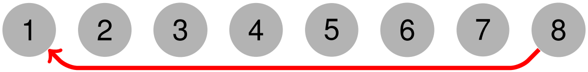

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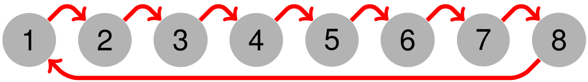

$$x := x + 1 \quad \text{gives} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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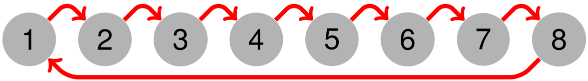

The LOS of the statement is $[\![x \ := x + 1]\!] = \mathbf{U}(x \leftarrow x + 1)$.
To avoid "overflow": actually $[\![x \ := ((x - 1) + 1 \bmod 8) + 1]\!]$.

Transfer Functions (Edge Effects): Random Assign

Assume $x \in 1, .., 8$; How do statements change its value?

1 2 3 4 5 6 7 8

Transfer Functions (Edge Effects): Random Assign

Assume $x \in 1, .., 8$; How do statements change its value?

$x \: ? = \{4, 5\}$

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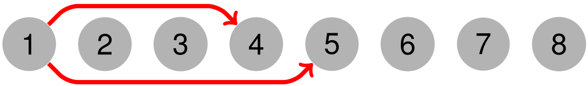

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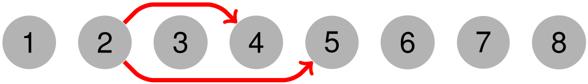

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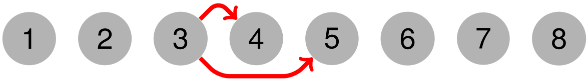

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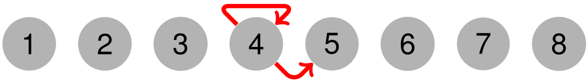

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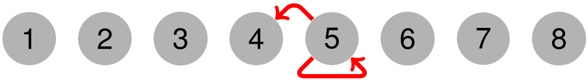

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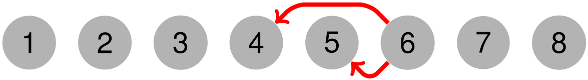

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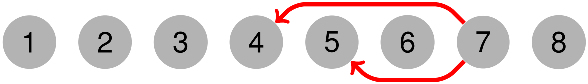

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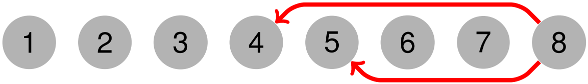

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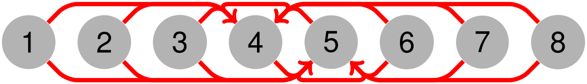
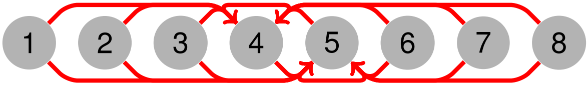


$x \; ? = \; \{4, 5\}$ gives
$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

Transfer Functions (Edge Effects): Random Assign

Assume $x \in 1, .., 8$; How do statements change its value?



So the LOS is $[\![x \;?\!= \{4, 5\}]\!] = \frac{1}{2}\mathbf{U}(x \leftarrow 4) + \frac{1}{2}\mathbf{U}(x \leftarrow 5)$.

Using the Linear Operators

We have now as states probability distributions over possible values $\sigma \in \mathcal{D}(\textbf{Value})$ rather than classical states $s \in \textbf{Value}$

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We can compute what happens to classical states, e.g.

$$
\begin{aligned}
(0,1,0,0,0,0,0,0) \cdot [\![x := 4]\!] &= (0,0,0,1,0,0,0,0) \\
(0,1,0,0,0,0,0,0) \cdot [\![x? = \{4,5\}]\!] &= (0,0,0,\frac{1}{2},\frac{1}{2},0,0,0)
\end{aligned}
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$$

but also what happens with distributions, e.g.

$$
(0,\frac{2}{3},0,0,\frac{1}{3},0,0,0) \cdot [\![x := x+1]\!] = (0,0,\frac{2}{3},0,0,\frac{1}{3},0,0)
$$

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$$\begin{aligned}(0,1,0,0,0,0,0,0) \cdot [\![x := 4]\!] &= (0,0,0,1,0,0,0,0)\\ (0,1,0,0,0,0,0,0) \cdot [\![x? = \{4,5\}]\!] &= (0,0,0,\frac{1}{2},\frac{1}{2},0,0,0)\end{aligned}$$

but also what happens with distributions, e.g.

$$(0,\frac{2}{3},0,0,\frac{1}{3},0,0,0) \cdot [\![x := x+1]\!] = (0,0,\frac{2}{3},0,0,\frac{1}{3},0,0)$$

and we can combine effects (to the same variable), e.g.

$$[\![x? = \{4,5\}]\!] = \frac{1}{2}[\![x := 4]\!] + \frac{1}{2}[\![x := 5]\!]$$

Putting Things Together

We can use the tensor product construction to combine the effects on different variables. For $x \in \{1..8\}$ and $y \in \{1, ..4\}$

$$
\begin{aligned}
[\![x? = \{2,4,6,8\}]\!] &= \frac{1}{4}\sum_{k=1}^{4} \mathbf{U}(x \leftarrow 2k) \otimes \mathbf{I} \\
[\![y := 3]\!] &= \mathbf{I} \otimes \mathbf{U}(y \leftarrow 3)
\end{aligned}
$$

The execution of "$x? = \{2,4,6,8\};\ y := 3$" is implemented by

$$
\begin{aligned}
[\![x? = \{2,4,6,8\};\ y := 3]\!] &= (\frac{1}{4}\sum_{k=1}^{4} \mathbf{U}(x \leftarrow 2k) \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{U}(y \leftarrow 3)) \\
&= \frac{1}{4}\sum_{k=1}^{4} \mathbf{U}(x \leftarrow 2k) \otimes \mathbf{U}(y \leftarrow 3)
\end{aligned}
$$

"Turtle" Execution

$$
\begin{aligned}
[\![x? = \{2,4,6,8\};\; y := 3]\!] =\\
&= \frac{1}{4}\sum_{k=1}^{4} \mathbf{U}(x \leftarrow 2k) \otimes \mathbf{U}(y \leftarrow 3)\\
&= \frac{1}{4}\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
0 & 0 & 1 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 1 & 0
\end{pmatrix}
\end{aligned}
$$

Conditionals

Consider conditional jumps or statements, e.g.

$$\textbf{if } \mathit{even}(x) \textbf{ then } x := x/2 \textbf{ else } y := y + 1 \textbf{ fi}$$

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Consider conditional jumps or statements, e.g.

$$\textbf{if } \mathit{even}(x) \textbf{ then } x := x/2 \textbf{ else } y := y + 1 \textbf{ fi}$$

The branches have the following LOS:

$$[\![x := x/2]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathbf{I}$$

Conditionals

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$$[\![y := y + 1]\!] = \mathbf{I} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Conditionals

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$$\textbf{if } \mathit{even}(x) \textbf{ then } x := x/2 \textbf{ else } y := y + 1 \textbf{ fi}$$

Note: To avoid errors $a/b = \lceil a/b \rceil$ and $a + b = a + b \bmod n$.

Tests and Distribution Splitting

We represent the filter for testing if x is even by a projection:

$$\mathbf{P}(\mathit{even}(x)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbf{I}$$

Its negation is represented by:

$$\mathbf{P}(\neg \mathit{even}(x)) = \mathbf{P}(\mathit{even}(x))^{\perp} = \mathbf{I} - \mathbf{P}(\mathit{even}(x)).$$

Using Tests

The semantics of a conditional is given by applying the semantics of the branches to the filtered (probabilistic) states and to combine the results. In our example:

$$
\begin{aligned}
&[\![\textbf{if } \mathit{even}(x) \textbf{ then } x := x/2 \textbf{ else } y+1 \textbf{ fi}]\!] = \\
&= \quad \mathbf{P}(\mathit{even}(x)) \cdot [\![x := x/2]\!] + \mathbf{P}(\mathit{even}(x))^{\perp} \cdot [\![y := y+1]\!]
\end{aligned}
$$

Given state where x has with probability $\frac{1}{2}$ values 3 and 6, and y value 2, i.e. $\sigma_0 = (0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0) \otimes (0, 1, 0, 0)$ then

$$
\begin{aligned}
\sigma_0 \cdot \mathbf{P}(\mathit{even}(x)) &= (0, 0, 0, 0, 0, \frac{1}{2}, 0, 0) \otimes (0, 1, 0, 0) \\
&= \frac{1}{2} \cdot (0, 0, 0, 0, 0, 1, 0, 0) \otimes (0, 1, 0, 0) \\
\sigma_0 \cdot \mathbf{P}(\mathit{even}(x))^{\perp} &= (0, 0, \frac{1}{2}, 0, 0, 0, 0, 0) \otimes (0, 1, 0, 0) \\
&= \frac{1}{2} \cdot (0, 0, 1, 0, 0, 0, 0, 0) \otimes (0, 1, 0, 0)
\end{aligned}
$$

Semantics of Conditionals

Applying the semantics of both branches gives us:

$$\sigma_0 \cdot \mathbf{P}(even(x)) \cdot [\![x := x/2]\!] =$$
$$= \quad (0, 0, \frac{1}{2}, 0, 0, 0, 0) \otimes (0, 1, 0, 0)$$
$$\sigma_0 \cdot \mathbf{P}(even(x))^{\perp} \cdot [\![y := y + 1]\!] =$$
$$= \quad (0, 0, \frac{1}{2}, 0, 0, 0, 0, 0) \otimes (0, 0, 1, 0)$$

The sum of both branches is now, maybe somewhat surprising:

$$\sigma = (0, 0, 1, 0, 0, 0, 0, 0) \otimes (0, \frac{1}{2}, \frac{1}{2}, 0)$$

Though we have started with a definitive value for y and a distribution for x, the opposite is now the case.

Probabilistic Control Flow

Consider the following labelled program:

while [**z** $<$ 100]^1 **do**
    choose^2 $\frac{1}{3}$: [**x**:=3]^3 **or** $\frac{2}{3}$: [**x**:=1]^4 **ro**
end while
[**stop**]^5

Probabilistic Control Flow

Consider the following labelled program:

while $[z < 100]^1$ **do**
  choose^2 $\frac{1}{3} : [x := 3]^3$ **or** $\frac{2}{3} : [x := 1]^4$ **ro**
end while
$[\textbf{stop}]^5$

Its probabilistic control flow is given by:

$$\mathit{flow}(P) = \{\langle 1, 1, 2\rangle, \langle 1, 1, 5\rangle, \langle 2, \frac{1}{3}, 3\rangle, \langle 2, \frac{2}{3}, 4\rangle, \langle 3, 1, 1\rangle, \langle 4, 1, 1\rangle\}.$$

Init Label

$$
\begin{array}{rcl}
\mathit{init}([\textbf{skip}]^\ell) & = & \ell \\
\mathit{init}([\textbf{stop}]^\ell) & = & \ell \\
\mathit{init}([\mathtt{x}{:=}e]^\ell) & = & \ell \\
\mathit{init}([\mathtt{x}?{=}e]^\ell) & = & \ell \\
\mathit{init}(S_1; S_2) & = & \mathit{init}(S_1) \\
\mathit{init}(\textbf{choose}^\ell\ p_1 : S_1\ \textbf{or}\ p_2 : S_2) & = & \ell \\
\mathit{init}(\textbf{if}\ [b]^\ell\ \textbf{then}\ S_1\ \textbf{else}\ S_2) & = & \ell \\
\mathit{init}(\textbf{while}\ [b]^\ell\ \textbf{do}\ S) & = & \ell
\end{array}
$$

Final Labels

$$
\begin{array}{rcl}
\mathit{final}([\textbf{skip}]^\ell) & = & \{\ell\} \\
\mathit{final}([\textbf{stop}]^\ell) & = & \{\ell\} \\
\mathit{final}([\mathtt{x}{:=}e]^\ell) & = & \{\ell\} \\
\mathit{final}([\mathtt{x}?{=}e]^\ell) & = & \{\ell\} \\
\mathit{final}(S_1; S_2) & = & \mathit{final}(S_2) \\
\mathit{final}(\textbf{choose}^\ell\ p_1 : S_1\ \textbf{or}\ p_2 : S_2) & = & \mathit{final}(S_1) \cup \mathit{final}(S_2) \\
\mathit{final}(\textbf{if}\ [b]^\ell\ \textbf{then}\ S_1\ \textbf{else}\ S_2) & = & \mathit{final}(S_1) \cup \mathit{final}(S_2) \\
\mathit{final}(\textbf{while}\ [b]^\ell\ \textbf{do}\ S) & = & \{\ell\}
\end{array}
$$

Flow I — Control Transfer

The probabilistic control flow is defined by the function:

$$\mathit{flow} : \mathbf{Stmt} \to \mathcal{P}(\mathbf{Lab} \times [0,1] \times \mathbf{Lab})$$

Flow I — Control Transfer

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$$\begin{array}{rcl}
\textit{flow}([\textbf{skip}]^{\ell}) & = & \emptyset \\
\textit{flow}([\textbf{stop}]^{\ell}) & = & \{\langle \ell, 1, \ell \rangle\} \\
\textit{flow}([\mathrm{x} := e]^{\ell}) & = & \emptyset \\
\textit{flow}([\mathrm{x} ?= e]^{\ell}) & = & \emptyset \\
\textit{flow}(S_1 ; S_2) & = & \textit{flow}(S_1) \cup \textit{flow}(S_2) \cup \\
 & \cup & \{(\ell, 1, \textit{init}(S_2)) \mid \ell \in \textit{final}(S_1)\}
\end{array}$$

Flow II — Control Transfer

$$
\begin{array}{rcl}
\mathit{flow}(\textbf{choose}^{\ell}\ p_1 : S_1\ \textbf{or}\ p_2 : S_2) & = & \mathit{flow}(S_1) \cup \mathit{flow}(S_2) \cup \\
& \cup & \{(\ell, p_1, \mathit{init}(S_1)), (\ell, p_2, \mathit{init}(S_2))\} \\
\mathit{flow}(\textbf{if}\ [b]^{\ell}\ \textbf{then}\ S_1\ \textbf{else}\ S_2) & = & \mathit{flow}(S_1) \cup \mathit{flow}(S_2) \cup \\
& \cup & \{(\ell, 1, \mathit{init}(S_1)), (\ell, 1, \mathit{init}(S_2))\} \\
\mathit{flow}(\textbf{while}\ [b]^{\ell}\ \textbf{do}\ S) & = & \mathit{flow}(S) \cup \\
& \cup & \{(\ell, 1, \mathit{init}(S))\} \\
& \cup & \{(\ell', 1, \ell) \mid \ell' \in \mathit{final}(S)\}
\end{array}
$$

A Linear Operator Semantics (LOS) based on *flow*

Using the *flow*(S) we construct a linear operator/matrix/DTMC generator in a compositional way, essentially as:

$$\mathbf{T}(S) = \sum_{\langle i, p_{ij}, j\rangle \in \mathit{flow}(S)} p_{ij} \cdot \mathbf{T}(\langle \ell_i, p_{ij}, \ell_j \rangle),$$

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where

$$\mathbf{T}(\langle \ell_i, p_{ij}, \ell_j \rangle) = \mathbf{N}_{\ell_i} \otimes \mathbf{E}(\ell_i, \ell_j),$$

With $\mathbf{N}_{\ell_1}$ the operator representing a state update (change of variable values) at the block with label ℓ_i and the second factor implementing the transfer of control from label ℓ_i to label ℓ_j.

Transfer Operators [Provided in Exam]

For all the blocks in S we have transfer operators which change the state and (then/simultanously) perform a control transfer to another bloc/ or program points:

$$
\begin{array}{rcll}
\mathbf{T}(\langle \ell_1, p, \ell_2 \rangle) & = & \mathbf{I} \otimes \mathbf{E}(\ell_1, \ell_2) & \text{for } [\mathbf{skip}]^{\ell_1} \\
\mathbf{T}(\langle \ell_1, p, \ell_2 \rangle) & = & \mathbf{U}(\mathtt{x} \leftarrow a) \otimes \mathbf{E}(\ell_1, \ell_2) & \text{for } [\mathtt{x} \leftarrow a]^{\ell_1} \\
\mathbf{T}(\langle \ell_1, p, \ell_2 \rangle) & = & \sum_{i \in r} \frac{1}{|r|} \mathbf{U}(\mathtt{x} \leftarrow i) \otimes \mathbf{E}(\ell_1, \ell_2) & \text{for } [\mathtt{x} \mathrel{?=} r]^{\ell_1} \\
\mathbf{T}(\langle \ell, p, \ell_t \rangle) & = & \mathbf{P}(b = \mathbf{true}) \otimes \mathbf{E}(\ell, \ell_t) & \text{for } [b]^{\ell} \\
\mathbf{T}(\langle \ell, p, \ell_f \rangle) & = & \mathbf{P}(b = \mathbf{false}) \otimes \mathbf{E}(\ell, \ell_f) & \text{for } [b]^{\ell} \\
\mathbf{T}(\langle \ell, p_k, \ell_k \rangle) & = & \mathbf{I} \otimes \mathbf{E}(\ell, \ell_k) & \text{for } [\mathbf{choose}]^{\ell} \\
\mathbf{T}(\langle \ell, p, \ell \rangle & = & \mathbf{I} \otimes \mathbf{E}(\ell, \ell) & \text{for } [\mathbf{stop}]^{\ell}
\end{array}
$$

For $[b]^{\ell}$ the label ℓ_t denotes the label to the '**true**' situation (e.g. **then** branch) and ℓ_f the situation where b is '**false**'.

In the case of a **choose** statement the different alternatives are labeled with (initial) label ℓ_k.

Tests and Filters

Select a value $c \in \mathbf{Value}_k$ for variable x_k (with $k = 1, \ldots, v$):

$$(\mathbf{P}(c))_{ij} = \begin{cases} 1 & \text{if } i = c = j \\ 0 & \text{otherwise.} \end{cases}$$

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Select a certain classical state $\sigma \in \mathbf{State} = \mathbf{Value}^v$:

$$\mathbf{P}(\sigma) = \bigotimes_{i=1}^{v} \mathbf{P}(\sigma(\mathrm{x}_i))$$

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Select states where expression $e = a \mid b$ evaluates to c:

$$\mathbf{P}(e = c) = \sum_{\mathcal{E}(e)\sigma = c} \mathbf{P}(\sigma)$$

Updates

Modify the value of variable x_k to a constant $c \in \textbf{Value}_k$:

$$(\mathbf{U}(c))_{ij} = \begin{cases} 1 & \text{if } j = c \\ 0 & \text{otherwise.} \end{cases}$$

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Set value of variable $\mathtt{x}_k \in \textbf{Var}$ to constant $c \in \textbf{Value}$:

$$\mathbf{U}(\mathtt{x}_k \leftarrow c) = \left(\bigotimes_{i=1}^{k-1} \mathbf{I} \right) \otimes \mathbf{U}(c) \otimes \left(\bigotimes_{i=k+1}^{v} \mathbf{I} \right)$$

Updates

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Set value of variable $\mathtt{x}_k \in \mathbf{Var}$ to value given by $e = a \mid b$:

$$\mathbf{U}(\mathtt{x}_k \leftarrow e) = \sum_{c} \mathbf{P}(e = c)\mathbf{U}(\mathtt{x}_k \leftarrow c)$$

An Example

if $[\mathrm{x} == 0]^1$ **then**
    $[x \leftarrow 0]^2$;
else
    $[x \leftarrow 1]^3$;
end if;
$[\textbf{stop}]^4$

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    $[x \leftarrow 0]^2$;
else
    $[x \leftarrow 1]^3$;
end if;
$[\textbf{stop}]^4$

$$
\begin{aligned}
\mathbf{T}(S) &= \mathbf{P}(\mathrm{x} = 0) \otimes \mathbf{E}(1,2) + \\
&+ \mathbf{P}(\mathrm{x} \neq 0) \otimes \mathbf{E}(1,3) + \\
&+ \mathbf{U}(x \leftarrow 0) \otimes \mathbf{E}(2,4) + \\
&+ \mathbf{U}(x \leftarrow 1) \otimes \mathbf{E}(3,4) + \\
&+ \mathbf{I} \otimes \mathbf{E}(4,4)
\end{aligned}
$$

An Example

$$
\begin{aligned}
\mathbf{T}(S) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{E}(1,2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{E}(1,3) + \\
&+ \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{E}(2,3) \right) + \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{E}(3,4) \right) + \\
&+ (\mathbf{I} \otimes \mathbf{E}(4,4))
\end{aligned}
$$

An Example

$$
\begin{aligned}
\mathbf{T}(S) \quad = \quad & \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\
+ \quad & \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\
+ \quad & \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \\
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+ \quad & \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)
\end{aligned}
$$

LOS and DTMC

We can compare this $\mathbf{T}(S)$ with the directly extracted operator, and indeed the two coincide.

$$
\begin{array}{cc}
\langle x \mapsto 0, [\mathbf{x} == 0]^1 \rangle & \ldots \\
\langle x \mapsto 0, [\mathbf{x} := 0]^2 \rangle & \ldots \\
\langle x \mapsto 0, [\mathbf{x} := 1]^3 \rangle & \ldots \\
\langle x \mapsto 0, [\mathbf{stop}]^4 \rangle & \ldots \\
\langle x \mapsto 1, [\mathbf{x} == 0]^1 \rangle & \ldots \\
\langle x \mapsto 1, [\mathbf{x} := 0]^2 \rangle & \ldots \\
\langle x \mapsto 1, [\mathbf{x} := 1]^3 \rangle & \ldots \\
\langle x \mapsto 1, [\mathbf{stop}]^4 \rangle & \ldots
\end{array}
\quad
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Factorial

Consider the program *F* for calculating the factorial of *n*:

```
var
  m : {0..2};
  n : {0..2};

begin
m := 1;
while (n>1) do
  m := m*n;
  n := n-1;
od;
stop; # looping
end
```

Control Flow and LOS for F

$$\mathit{flow}(F) = \{(1,1,2),(2,1,3),(3,1,4),(4,1,2),(2,1,5),(5,1,5)\}$$

Control Flow and LOS for F

$$flow(F) = \{(1,1,2),(2,1,3),(3,1,4),(4,1,2),(2,1,5),(5,1,5)\}$$

$$\begin{aligned}
\mathbf{T}(F) = \; & \mathbf{U}(m \leftarrow 1) \otimes \mathbf{E}(1,2) + \\
& \mathbf{P}((n > 1)) \otimes \mathbf{E}(2,3) + \\
& \mathbf{U}(m \leftarrow (m * n)) \otimes \mathbf{E}(3,4) + \\
& \mathbf{U}(n \leftarrow (n - 1)) \otimes \mathbf{E}(4,2) + \\
& \mathbf{P}((n <= 1)) \otimes \mathbf{E}(2,5) + \\
& \mathbf{I} \otimes \mathbf{E}(5,5)
\end{aligned}$$

Introducing PAI

The matrix $\mathbf{T}(F)$ is very big already for small n.

n	$\dim(\mathbf{T}(F))$
2	45×45
3	140×140
4	625×625
5	3630×3630
6	25235×25235
7	201640×201640
8	1814445×1814445
9	18144050×18144050

We will show how we can drastically reduce the dimension of the LOS by using Probabilistic Abstract Interpretation.

Galois Connections

Definition

Let $\mathcal{C} = (\mathcal{C}, \leq_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ be two partially ordered sets with two order-preserving functions $\alpha : \mathcal{C} \mapsto \mathcal{D}$ and $\gamma : \mathcal{D} \mapsto \mathcal{C}$. Then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection iff

(i) $\alpha \circ \gamma$ is reductive i.e. $\forall d \in \mathcal{D}$, $\alpha \circ \gamma(d) \leq_{\mathcal{D}} d$,

(ii) $\gamma \circ \alpha$ is extensive i.e. $\forall c \in \mathcal{C}$, $c \leq_{\mathcal{C}} \gamma \circ \alpha(c)$.

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Proposition

Let $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ be a Galois connection. Then α and γ are quasi-inverse, i.e.

(i) $\alpha \circ \gamma \circ \alpha = \alpha$ *and* *(ii)* $\gamma \circ \alpha \circ \gamma = \gamma$

General Construction

The general construction of correct (and optimal) abstractions $f^\#$ of concrete function f is as follows:

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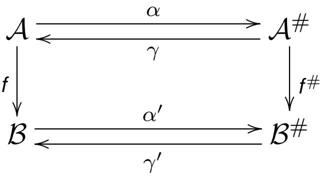

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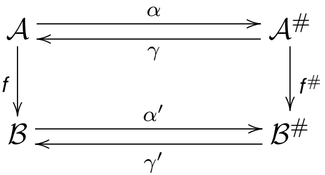

Correct approximation:

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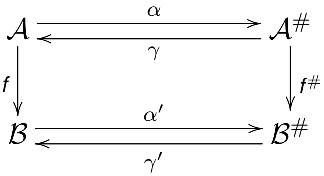

Correct approximation:

$$\alpha' \circ f \leq_{\#} f^{\#} \circ \alpha.$$

Induced semantics:

$$f^{\#} = \alpha' \circ f \circ \gamma.$$

Probabilistic Abstraction Domains

A probabilistic domain is essentially a vector space which represents the distributions $\textbf{Dist}(\textbf{State}) \subseteq \mathcal{V}(\textbf{State})$ on the state space **State** of a probabilistic transition system, i.e. for finite state spaces

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The notion of norm (distance) is essential for our treatment; we will consider normed vector spaces.

Moore-Penrose Generalised Inverse

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two (finite-dimensional) vector (Hilbert) spaces and $\mathbf{A} : \mathcal{C} \to \mathcal{D}$ a linear map. Then the linear map $\mathbf{A}^\dagger = \mathbf{G} : \mathcal{D} \to \mathcal{C}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$ iff

(i) $\mathbf{A} \circ \mathbf{G} = \mathbf{P}_A$,

(ii) $\mathbf{G} \circ \mathbf{A} = \mathbf{P}_G$,

where $\mathbf{P}_A$ and $\mathbf{P}_G$ denote orthogonal projections onto the ranges of $\mathbf{A}$ and $\mathbf{G}$.

(Orthogonal) Projections – Idempotents [Not for Exam]

On finite dimensional vector (Hilbert) spaces we have an inner product $\langle ., . \rangle$, standard

$$\langle x, y \rangle = \langle (x_i)_i, (y_i)_i \rangle = \sum_i x_i y_i$$

This measures some kind of similarity of vectors but also allows to define a norm:

$$\|x\|_2 = \sqrt{\langle x, x \rangle}$$

It also allows us to define an adjoint via:

$$\langle \mathbf{A}(x), y \rangle = \langle x, \mathbf{A}^*(y) \rangle$$

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- An operator $\mathbf{A}$ is self-adjoint if $\mathbf{A} = \mathbf{A}^*$.

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$$\langle \mathbf{A}(x), y \rangle = \langle x, \mathbf{A}^*(y) \rangle$$

- An operator $\mathbf{A}$ is self-adjoint if $\mathbf{A} = \mathbf{A}^*$.
- An (orthogonal) projection is a self-adjoint $\mathbf{E}$ with $\mathbf{EE} = \mathbf{E}$.

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A norm on a vector space $\mathcal{V}$ is a map $\|.\| : \mathcal{V} \mapsto \mathbb{R}$ such that for all $v, w \in \mathcal{V}$ and $c \in \mathbb{C}$:

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Note: The structural similarities between distances and partial orders can be made precise (cf. Category Theory).

Least Squares Solutions

Corollary

Let $\mathbf{P}$ *be a orthogonal projection on a finite dimensional vector space* $\mathcal{V}$*. Then for any* $\mathbf{x} \in \mathcal{V}$, $\mathbf{P}(\mathbf{x}) = \mathbf{x}\mathbf{P}$ *is the unique closest vector in* $\mathcal{V}$ *to* $\mathbf{x}$ *wrt to the Euclidean norm* $\|.\|_2$*.*

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Definition

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{u} \in \mathbb{R}^n$ is called a least squares solution to $\mathbf{Ax} = \mathbf{b}$ if

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Theorem

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ *and* $\mathbf{b} \in \mathbb{R}^m$*. Then* $\mathbf{A}^\dagger\mathbf{b}$ *is the minimal least squares solution to* $\mathbf{Ax} = \mathbf{b}$.

Vector Space Lifting

Free vector space construction on a set S:

$$\mathcal{V}(S) = \{\sum x_s s \mid x_s \in \mathbb{R}, s \in S\}$$

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Vector Space lifting: $\vec{\alpha} : \mathcal{V}(\mathcal{C}) \to \mathcal{V}(\mathcal{D})$

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Support Set: **supp** : $\mathcal{V}(\mathcal{C}) \rightarrow \mathcal{P}(\mathcal{C})$

$$\mathbf{supp}(\vec{x}) = \left\{c_i \mid \langle c_i, p_i \rangle \in \vec{x} \text{ and } p_i \neq 0\right\}$$

Relation with Classical Abstractions

Lemma

Let $\vec{\alpha}$ be a probabilistic abstraction function and let $\vec{\gamma}$ be its Moore-Penrose pseudo-inverse.

Then $\vec{\gamma} \circ \vec{\alpha}$ is extensive with respect to the inclusion on the support sets of vectors in $\mathcal{V}(\mathcal{C})$, i.e. $\forall \vec{x} \in \mathcal{V}(\mathcal{C})$,

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Proposition

$(\vec{\alpha}, \vec{\gamma})$ form a Galois connection wrt the support sets of $\mathcal{V}(\mathcal{C})$ and $\mathcal{V}(\mathcal{D})$, ordered by inclusion.

Examples of Lifted Abstractions

Parity Abstraction operator on $\mathcal{V}(\{1, \ldots, n\})$ (with n even):

$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}$$

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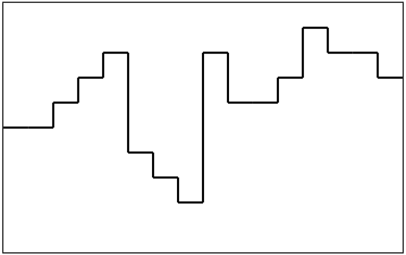

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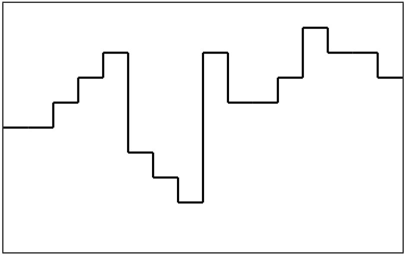

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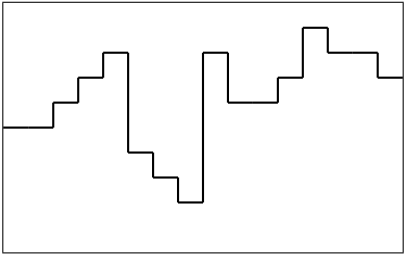

Each step function in $\mathcal{T}_n$ corresponds to a vector in $\mathbb{R}^n$, e.g.

$$\begin{pmatrix} 5 & 5 & 6 & 7 & 8 & 4 & 3 & 2 & 8 & 6 & 6 & 7 & 9 & 8 & 8 & 7 \end{pmatrix}$$

Example: Abstraction Matrices

$$\mathbf{A}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: Abstraction Matrices

$$\mathbf{G}_8 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Approximation Estimates

Compute the *least square error* as

$$\|f - f\mathbf{AG}\|.$$

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$$\begin{aligned}
\|f - f\mathbf{A}_8\mathbf{G}_8\| &= 3.5355 \\
\|f - f\mathbf{A}_4\mathbf{G}_4\| &= 5.3151 \\
\|f - f\mathbf{A}_2\mathbf{G}_2\| &= 5.9896 \\
\|f - f\mathbf{A}_1\mathbf{G}_1\| &= 7.6444
\end{aligned}$$

Tensor Product Properties

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Abstract Semantics

Moore-Penrose Pseudo-Inverse of a Tensor Product is:

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Via linearity we can construct $\mathbf{T}^\#$ in the same way as $\mathbf{T}$, i.e

$$\mathbf{T}^\#(P) = \sum_{\langle i,p_{ij},j\rangle \in \mathcal{F}(P)} p_{ij} \cdot \mathbf{T}^\#(\ell_i, \ell_j)$$

with local abstraction of individual variables:

$$\mathbf{T}^\#(\ell_i, \ell_j) = (\mathbf{A}_1^\dagger \mathbf{N}_{i1} \mathbf{A}_1) \otimes (\mathbf{A}_2^\dagger \mathbf{N}_{i2} \mathbf{A}_2) \otimes \ldots \otimes (\mathbf{A}_v^\dagger \mathbf{N}_{iv} \mathbf{A}_v) \otimes \mathbf{M}_{ij}$$

Argument [Not for Exam]

$$\mathbf{T}^{\#} \quad = \quad \mathbf{A}^{\dagger}\mathbf{T}\mathbf{A}$$

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$$\begin{aligned}
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Parity Analysis

Determine at each program point whether a variable is *even* or *odd*.

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$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A}^\dagger = \begin{pmatrix} \frac{2}{n} & 0 & \frac{2}{n} & 0 & \ldots & 0 \\ 0 & \frac{2}{n} & 0 & \frac{2}{n} & \ldots & \frac{2}{n} \end{pmatrix}$$

Example

$[m \leftarrow i]^1$;
while $[n > 1]^2$ **do**
  $[m \leftarrow m \times n]^3$;
  $[n \leftarrow n - 1]^4$
end while
$[\textbf{stop}]^5$

Example

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&+ \mathbf{U}(\mathtt{m} \leftarrow m \times n) \otimes \mathbf{E}(3,4) \\
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&+ \mathbf{I} \otimes \mathbf{E}(5,5)
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+ \quad & \mathbf{I}^\# \otimes \mathbf{E}(5,5)
\end{aligned}
$$

Abstract Semantics

Abstraction: $\mathbf{A} = \mathbf{A}_p \otimes \mathbf{I}$, i.e. m abstract (parity) but n concrete.

$$
\begin{aligned}
\mathbf{T}^\# \quad = \quad & \mathbf{U}^\#(m \leftarrow 1) \otimes \mathbf{E}(1,2) \\
+ \quad & \mathbf{P}^\#(n > 1) \otimes \mathbf{E}(2,3) \\
+ \quad & \mathbf{P}^\#(n \leq 1) \otimes \mathbf{E}(2,5) \\
+ \quad & \mathbf{U}^\#(m \leftarrow m \times n) \otimes \mathbf{E}(3,4) \\
+ \quad & \mathbf{U}^\#(n \leftarrow n - 1) \otimes \mathbf{E}(4,2) \\
+ \quad & \mathbf{I}^\# \otimes \mathbf{E}(5,5)
\end{aligned}
$$

Abstract Semantics

$$\mathbf{U}^{\#}(m \leftarrow 1) =$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \dots & 1 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{U}^{\#}(n \leftarrow n-1) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{P}^{\#}(n > 1) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{P}^{\#}(n \leq 1) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{U}^{\#}(m \leftarrow m \times n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \ddots \end{pmatrix}$$

Implementation

Implementation of concrete and abstract semantics of Factorial using **octave.** Ranges: $n \in \{1, \ldots, d\}$ and $m \in \{1, \ldots, d!\}$.

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d	$\dim(\mathbf{T}(F))$	$\dim(\mathbf{T}^{\#}(F))$
2	45	30
3	140	40
4	625	50
5	3630	60
6	25235	70
7	201640	80
8	1814445	90
9	18144050	100

Using uniform initial distributions $\mathbf{d_0}$ for n and m.

Scalablity

The abstract probabilities for m being **even** or **odd** when we execute the abstract program for various d values are:

d	**even**	**odd**
10	0.81818	0.18182
100	0.98019	0.019802
1000	0.99800	0.0019980
10000	0.99980	0.00019998

Ortholattice of Projection Operators [Not for Exam]

Define a partial order on self-adjoint operators and projections as follows: $\mathbf{H} \sqsubseteq \mathbf{K}$ iff $\mathbf{K} - \mathbf{H}$ is positive, i.e. there exists a $\mathbf{B}$ such that $\mathbf{K} - \mathbf{H} = \mathbf{B}^*\mathbf{B}$.

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Alternatively, order projections by inclusion of their image spaces, i.e. $\mathbf{E} \sqsubseteq \mathbf{F}$ iff $Y_{\mathbf{E}} \subseteq Y_{\mathbf{F}}$.

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Alternatively, order projections by inclusion of their image spaces, i.e. $\mathbf{E} \sqsubseteq \mathbf{F}$ iff $Y_\mathbf{E} \subseteq Y_\mathbf{F}$.

The orthogonal projections form a complete (ortho)lattice.

The range of the intersection $\mathbf{E} \sqcap \mathbf{F}$ is to the closure of the intersection of the image spaces of $\mathbf{E}$ and $\mathbf{F}$.

The union $\mathbf{E} \sqcup \mathbf{F}$ corresponds to the union of the images.

Computing Intersections/Unions [Not for Exam]

Associate to every Probabilistic Abstract Interpretation $(\mathbf{A}, \mathbf{G})$ a projection, similar to so-called "upper closure operators" (uco):

$$\mathbf{E} = \mathbf{AG} = \mathbf{AA}^{\dagger}.$$

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A general way to construct $\mathbf{E} \sqcap \mathbf{F}$ and (by exploiting de Morgan's law) also $\mathbf{E} \sqcup \mathbf{F} = (\mathbf{E}^{\perp} \sqcap \mathbf{F}^{\perp})^{\perp}$ is via an infinite approximation sequence and has been suggested by Halmos:

$$\mathbf{E} \sqcap \mathbf{F} = \lim_{n \to \infty} (\mathbf{EFE})^{n}.$$

Commutative Case [Not for Exam]

The concrete construction of $\mathbf{E} \sqcup \mathbf{F}$ and $\mathbf{E} \sqcap \mathbf{F}$ is in general not trivial. Only for commuting projections we have:

$$\mathbf{E} \sqcup \mathbf{F} = \mathbf{E} + \mathbf{F} - \mathbf{EF} \text{ and } \mathbf{E} \sqcap \mathbf{F} = \mathbf{EF}.$$

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Example

Consider a finite set Ω with a probability structure. For any (measurable) subset A of Ω define the characteristic function χ_A with $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.

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Example

Consider a finite set Ω with a probability structure. For any (measurable) subset A of Ω define the characteristic function χ_A with $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. The characteristic functions are (commutative) projections on random variables using pointwise multiplication, i.e. $X\chi_A\chi_A = X\chi_A$. We have $\chi_{A\cap B} = \chi_A\chi_B$ and $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B$.

Non-Commutative Case [Not for Exam]

The Moore-Penrose pseudo-inverse is also useful for computing the $\mathbf{E} \sqcap \mathbf{F}$ and $\mathbf{E} \sqcup \mathbf{F}$ of general, non-commuting projections via the parallel sum

$$\mathbf{A} : \mathbf{B} = \mathbf{A}(\mathbf{A} + \mathbf{B})^{\dagger}\mathbf{B}$$

The intersection of projections is given by:

$$\mathbf{E} \sqcap \mathbf{F} = 2(\mathbf{E} : \mathbf{F}) = \mathbf{E}(\mathbf{E} + \mathbf{F})^{\dagger}\mathbf{F} + \mathbf{F}(\mathbf{E} + \mathbf{F})^{\dagger}\mathbf{E}$$

Israel, Greville: *Gereralized Inverses, Theory and Applications*, Springer 2003

Variable Probabilities: Duel at High Noon

Consider a "duel" between two cowboys:

- Cowboy A – hitting probability a
- Cowboy B – hitting probability b

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2. Repeat until winner is known:
   - If it is *A*'s turn he will hit/shoot *B* with probability *a*;
     If *B* is shot then *A* is the winner, otherwise it's *B*'s turn.
   - If it is *B*'s turn he will hit/shoot *A* with probability *b*;
     If *A* is shot then *B* is the winner, otherwise it's *A*'s turn.

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   - If it is B's turn he will hit/shoot A with probability b;
     If A is shot then B is the winner, otherwise it's A's turn.

Question: What is the life expectancy of A or B?

Introduced by McIver and Morgan (2005).
Discussed in detail by Gretz, Katoen, McIver (2012/14)

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   - If it is B's turn he will hit/shoot A with probability b; If A is shot then B is the winner, otherwise it's A's turn.

Question: What is the life expectancy of A or B?
Question: What happens if A is learning to shoot better during the duel? How can we model dynamic probabilities?

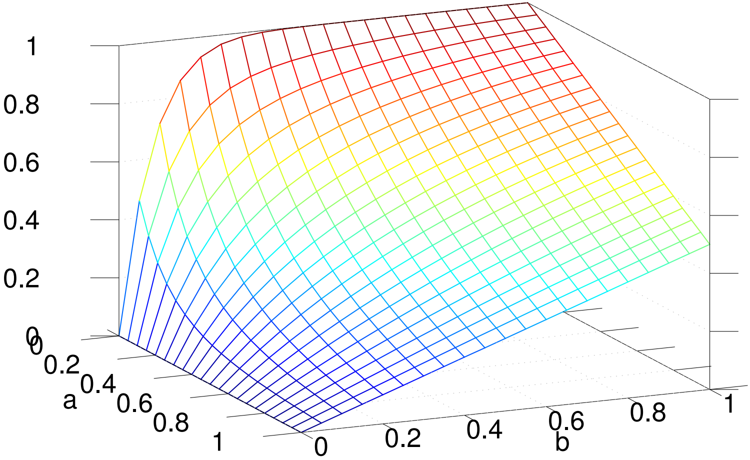
Introduced by McIver and Morgan (2005).
Discussed in detail by Gretz, Katoen, McIver (2012/14)

Example: Duelling Cowboys

```
begin
# who's first turn
choose 1:{t:=0} or 1:{t:=1} ro;
# continue until ...
c := 1;
while c == 1 do
if (t==0) then
  choose ak:{c:=0} or am:{t:=1} ro
else
  choose bk:{c:=0} or bm:{t:=0} ro
fi;
od;
stop; # terminal loop
end
```

Example: Duelling Cowboys [Not for Exam]

The survival chances, i.e. winning probability, for *A*.



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