Quantum Computation (CO484)

Quantum Physics and Concepts

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Overview

Topics we will cover in this course will include:

1. Basic Quantum Physics
2. Mathematical Structure
3. Quantum Cryptography
4. Quantum Circuit Model
5. [MBQC, TQC, etc.]
6. Quantum Teleportation
7. Gover's Search Algorithm
8. Shor's Quantum Factorisation
9. [Quantum Error Correction]

Practicalities

Two Lecturers

Herbert Wiklicky

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Teaching $3\frac{1}{2}$ weeks until 30 October
Open-book coursework test 30 October

Mahdi Cheraghchi

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Teaching $3\frac{1}{2}$ weeks from 3 November
Open-book coursework test 24 November

Exam: Week 11, 11-15 December 2017, 2 hours (3 out of 4).

Different classes, different background, different applications.

Text Books

- Noson S. Yanofsky, Mirco A. Mannucci: Quantum Computing for Computer Scientists, Cambridge, 2008
- Michael A. Nielsen, Issac L. Chuang: Quantum Computation and Quantum Information, Cambridge, 2000
- Phillip Kaye, Raymond Laflamme, Michael Mosca: An Introduction to Quantum Computing, Oxford 2007
- N. David Mermin: Quantum Computer Science, Cambridge University Press, 2007
- A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi: Classical and Quantum Computation, AMS, 2002
- Eleanor Rieffel, Wolfgang Polak: Quantum Computing, A Gentle Introduction. MIT Press, 2014
- Richard J. Lipton, Kenneth W. Regan: Quantum Algorithms via Linear Algebra. MIT Press, 2014

Electronic Resources

Introductory Texts

- E.Rieffel, W.Polak: An introduction to quantum computing for non-physicists. ACM Computing Surveys, 2000 `doi:10.1145/367701.367709`
- N.S.Yanofsky: An Introduction to Quantum Computing `http://arxiv.org/abs/0708.0261`

Preprint Repository `http://arxiv.org`

Physics Background

- Chris J. Isham: Quantum Theory – Mathematical and Structural Foundations, Imperial College Press 1995
- Richard P. Feynman, Robert B. Leighton, Matthew Sands: The Feynman Lectures on Physics, Addison-Wesley 1965

Quantum Money (Stephen Wiesner 1960s)

Quantum Postulates: (i) It is impossible to clone a quantum states, (ii) in general, an inspection of a quantum state is irreversible and destructive.

Bank of Quantum issue bank notes with a unique quantum code.

Quantum Forger tries to make a copy of quantum money, however

- she can't copy/clone a banknote directly, and
- when she inspects it, she destroys the code.

Bank of Quantum can inspect the quantum code on a banknote

- to confirm it is authentic, and then
- issue a replacement quantum banknote.

Simon Singh: *Code Book*, Forth Estate, 1999.

Quantum History

Quantum Mechanics was ‘born’ or, better, proposed by M.Plank on

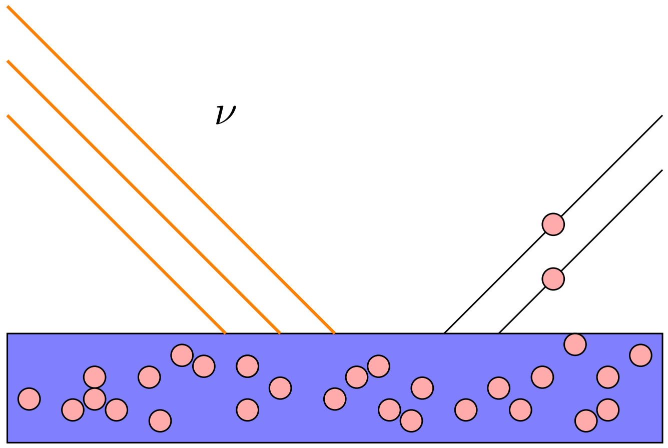
14 December 1900, 5:15pm (Berlin)

1900	Max Plank: Black Body Radiation
1905	Albert Einstein: Photoelectric Effect
1925	Werner Heisenberg: Matrix Mechanics
1926	Erwin Schrödinger: Wave Mechanics
1932	John von Neumann: Quantum Mechanics

Manjit Kumar: *Quantum – Einstein, Bohr and Their Great Debate about the Nature of Reality*, Icon Books 2009

Photoelectric Effect – Millikan Experiment

Experimental Setup:



Observed: The velocity, and thus kinetic energy, of the emitted electrons depends not on the intensity of the incoming light but only on its “colour”, i.e. frequency ν.

Radiation Law

Observed relationship:

$$W_k = h\nu - W_e$$

W_k ...Kinetic Energy of Electron

W_e ...Escape Energy of Material

ν ...Frequency of Light

h ...Plank's Constant

$$\begin{aligned} h &= 6.62559 \cdot 10^{-34} Js \\ \hbar &= \frac{h}{2\pi} = 1.05449 \cdot 10^{-34} Js \end{aligned}$$

Quantum Physical Problems

Around 1900 there were a number of experiments and observations which could not be explained using classical physics/mechanics, among them:

Spectra of Elements
: Emission/absorption only at particular "colours".

Stern-Gerlach Experiment
: Interference in double slit experiment.

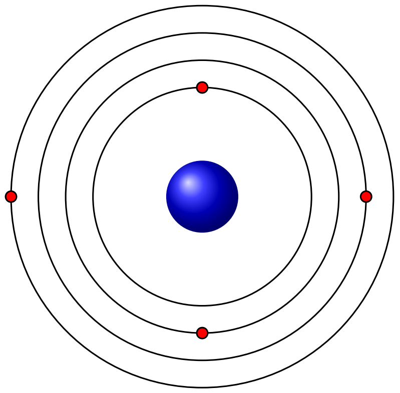
Black Body Radiation
: Radiation law involves "quantised" energy levels.

Photo-Electric Effect
: Einstein's explanation got him the Nobel prize.

These were the perhaps most exciting years in the history of theoretical physics, at the same time there were also breakthroughs in special and general relativity, etc.

Einstein's Explanation

Albert Einstein 1905: Not all **energy levels** are possible, they only come in quantised portions. In Bohr's (incomplete) "model" of the atom this corresponds to allowing only particular "orbits".



In this way one can also explain the spectral emissions (and absorption) of various elements, e.g. to analyse the material composition of stars (and to make great fireworks).

Quantum Paradoxes and Myths

There are a number of physical problems which require quantum mechanical explanations. Unfortunately, QM is not 'really intuitive'. This leads to various *Gedanken* experiments which point to a contradiction with so-called *common sense*.

- Black Body Radiation
- Double Slit Experiment
- Spectral Emissions
- Schrödinger's Cat
- Einstein-Podolsky-Rosen
- Quantum Teleportation

7. Whereof one cannot speak, thereof one must be silent.
Ludwig Wittgenstein: *Tractatus Logico-Philosophicus*, 1921

From Quantum Physics to Computation

There are a number of disciplines which play an important role in trying to understand *quantum mechanics* and in particular **quantum computation**.

Philosophy: What is the nature and meaning reality?

Logic: How can one reason about events, objects etc.?

Mathematics: How does the formal model look like?

Physics: Why does it work and what does it imply?

Computation: What can be computed and how?

Engineering: How can it all be implemented?

Each area has its own language which however often applies only to classical entities – for the quantum world we often have simply the wrong vocabulary.

Natural Philosophy

Arguably, **physics** is ultimately about explaining experiments and forecasting measurement results.

Observable: Entities which are (actually) measured when an experiment is conducted on a system.

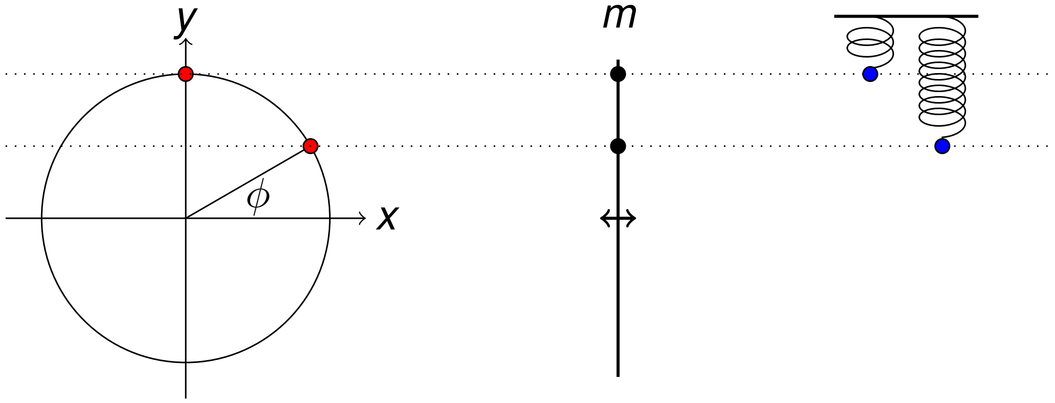
State: Entities which completely describe (or model) the system we are interested in.

Measurement brings together/establishes a relation between states and observables of a given system. Dynamics describes how observables and/or the state changes over time.

Related Questions: What is our knowledge of what? How do we obtain this information? What is a description on how the system changes?

Harmonic Oscillator or just a “Shadow”

One can observe the same “behaviour” of the shadow of a rotating object or an object on a spring.



Observable: **Shadow** m

State: **Position** (x, y) or: **Phase** ϕ

Measurement: $m((x, y)) = y$, or: $m(\phi) = \sin(\phi)$

Dynamics: $(x, y)(t) = (\cos(t), \sin(t))$ or also: $\phi(t) = t$

Postulates for Quantum Mechanics [$*$]

- Observables and states of a system are represented by *hermitian* (i.e. self-adjoint) elements a of a C*-algebra $\mathcal{A}$ and by *states* w (i.e. normalised linear functionals) over this algebra.

- Possible results of measurements of an observable a are given by the *spectrum* $\mathrm{Sp}(a)$ of an observable. Their probability distribution in a certain state w is given by the probability measure $\mu(w)$ induced by the state w on $\mathrm{Sp}(a)$.

Walter Thirring: *Quantum Mathematical Physics*, Springer 2002

Key Notions: A quantum systems is (may be) in a certain state, but physicists have to decide which properties they want to observe before a measurement is made (which instrument?).

Postulates for Quantum Mechanics (ca. 1950) [$*$]

- The quantum state of a (free) particle is described by a (normalised) complex valued [wave] function:

$$\vec{\psi} \in L^2 \ \text{ i.e. } \ \int |\vec{\psi}(x)|^2 dx = 1$$

- Two quantum states can be **superimposed**, i.e.

$$\psi = \alpha_1 \vec{\psi}_1 + \alpha_2 \vec{\psi}_2 \text{ with } |\alpha_1|^2 + |\alpha_2|^2 = 1$$

- Any observable A is represented by a linear, self-adjoint operator **A** on L^2.
- **Possible** measurement results are (only) the eigen-values λ_i of **A** corresponding to eigen-vectors/states $\vec{\phi}_i \in L^2$ with

$$\mathbf{A}\vec{\phi}_i = \lambda_i \vec{\phi}_i$$

- **Probability** to measure (the possible eigenvalue) λ_n if the system is in the state $\vec{\psi} = \sum_i \psi_i \vec{\phi}_i$ is

$$Pr(A = \lambda_n \mid \vec{\psi}) = |\psi_n|^2$$

Mathematical Framework

Quantum mechanics has a well-established and precise mathematical formulation (though its ‘common sense’ interpretation might be non-intuitive, probabilistic, etc.).

The (standard) mathematical model of quantum system uses:

- Complex Numbers $\mathbb{C}$,
- Vector Spaces, e.g. $\mathbb{C}^n$,
- Hilbert Spaces, i.e. inner products $\langle . | . \rangle$,
- Unitary and Self-Adjoint Matrices/Operators,
- Tensor Products $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$.

There are additional mathematical details in order to deal with “real” quantum physics, e.g. systems an infinite degree of freedom; for quantum computation it is however enough to study finite-dimensional Hilbert spaces.

Quantum Postulates I – States and Observables

The standard mathematical model of (closed) quantum systems is relatively simple and just requires some basic notions in **(complex) linear algebra**.

- The information describing the state of an (isolated) quantum mechanical system is represented mathematically by a (normalised) vector in a complex Hilbert space $\mathcal{H}$.
- An observable is represented mathematically by a self-adjoint matrix (operator) **A** acting on the Hilbert space $\mathcal{H}$.

Two states can be combined to form a new state $\alpha |x\rangle + \beta |y\rangle$ as long as $|\alpha|^2 + |\beta|^2 = 1$, by **superposition**.

Consequence: We can compute with many inputs in parallel.

Quantum States and Notation

The state of a quantum mechanical system is usually denoted by $|x\rangle \in \mathcal{H}$ (rather than maybe $\vec{x} \in \mathcal{H}$). This notation is 'inherited' from the inner product $\langle x|y\rangle$ of vectors x and y in a Hilbert space – which can be seen as describing the "*geometric angle*" between the two vectors in $\mathcal{H}$.

P.A.M. Dirac "invented" the bra-ket notation (most likely inspired by the limitations of old mechanical type-writers); Simply "take the inner product apart" to denote vectors in $\mathcal{H}$:

$$\text{inner product } \langle x|y\rangle \quad = \quad \text{product } \langle x| \cdot |y\rangle$$

For indexed sets of vectors $\{\mathbf{x}_i\}$ (maybe because typographic "typing" was problematic) different notations are used:

$$\mathbf{x}_i = \vec{x}_i = \mathfrak{x}_i = |i\rangle$$

Quantum States and Vectors

Finite quantum states can be described by vectors in $\mathbb{C}^n$, e.g.

$$\vec{\psi} = |\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \langle\phi| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Observables are defined by matrices **A** in $\mathcal{M}(\mathbb{C}^n) = \mathbb{C}^{n \times n}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{with eigenvalues } \lambda_0 = 1, \lambda_1 = 2$$

Note: There are sometimes two types of indices

- for enumerating, for example, all eigenvectors of an operator like **A** with $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- to enumerate coordinates of one vector, e.g. $\vec{\psi}_1 = 1/\sqrt{2}$, or better perhaps: $|0\rangle_1 = 0$.

Quantum Postulates II – Measurement

- The expected result (average) when measuring observable **A** of a system in state $|x\rangle \in \mathcal{H}$ is given by:

$$\langle A \rangle_x = \langle x| \mathbf{A} |x\rangle = \langle x| |\mathbf{A}x\rangle$$

- The only possible results are eigen-values λ_i of **A**.
- The probability of measuring λ_n in state $|x\rangle$ is

$$Pr(A = \lambda_n | x) = \langle x| \mathbf{P}_n |x\rangle$$

with $\mathbf{P}_n$ the orthogonal projection onto the n-th eigen-space of **A** generated by eigen-vector $|\lambda_n\rangle$

$$\mathbf{P}_n = |\lambda_n\rangle \langle\lambda_n|$$

then we have: $\mathbf{A} = \sum_i \lambda_i \mathbf{P}_i$ (Spectral Theorem).

Heisenberg's Uncertainty Relation

Theorem

For two observables $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *we have:*

$$(\Delta_{|x\rangle}\mathbf{A}_1)(\Delta_{|x\rangle}\mathbf{A}_2) \geq \frac{1}{2}\,|(\langle x|\ [\mathbf{A}_1,\mathbf{A}_2]\ |x\rangle)|$$

where the uncertainty (classically: variance) is defined by

$$(\Delta_{|x\rangle}\mathbf{A})^2 = \langle x|\ \mathbf{A}^2\ |x\rangle - \langle x|\ \mathbf{A}\ |x\rangle^2$$

and the commutator is defined as:

$$[\mathbf{A}_1,\mathbf{A}_2] = \mathbf{A}_1\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_1$$

see e.g. Isham: *Quantum Theory*, ICP 1995, Section 7.3.3.

Classical vs Quantum Mechanics

The usual interpretation of Heisenberg's uncertainty relation is this: When one tries to measures two observables $\mathbf{A}_1$ and $\mathbf{A}_2$ then – if the commutator $[\mathbf{A}_1,\mathbf{A}_2]$ is non-zero – a small $\Delta_{|x\rangle}\mathbf{A}_1$ implies a large $\Delta_{|x\rangle}\mathbf{A}_2$, and vice versa.

A standard example of so-called *incomensurable* observables are position $\mathbf{A}_1 = x$ and momentum $\mathbf{A}_2 = p$ (on an infinite-dimensional Hilbert Space $\mathcal{H}$) for which $[x,p] = i\hbar$ and thus:

$$\Delta x \Delta p \geq \hbar/2.$$

In **classical** physics observables always commute, are *comensurable*, i.e. $[\mathbf{A}_1,\mathbf{A}_2] = 0$. In **quantum** physics for most observables $[\mathbf{A}_1,\mathbf{A}_2] \neq 0$, i.e. the observable algebra is typically non-commutative or non-abelian (cf. multiplication of (complex) numbers vs multiplication of matrices).

Quantum Dynamics

- The **dynamics** of a (closed) system is described by the Schrödinger Equation:

$$i\hbar \frac{d\,|x\rangle}{dt} = \mathbf{H}\,|x\rangle$$

for the (self-adjoint) Hamiltonian operator **H** (energy).

- The **solution** is a unitary operator $\mathbf{U}_t$ (e.g. Isham 6.4)

$$\mathbf{U}_t = \exp(-\frac{i}{\hbar}t\mathbf{H})$$

Theorem

For any self-adjoint operator **A** *the operator*

$$\exp(i\mathbf{A}) = e^{i\mathbf{A}} = \sum_{n=0}^{\infty} \frac{(i\mathbf{A})^n}{n!}$$

is a unitary operator.

Irreversible vs Reversible

There are a number of immediate consequence of the postulates.

1. The state develops reversibly, i.e. $|x_t\rangle = \mathbf{U}_t\,|x_0\rangle$ for some unitary matrix (operator).
   Consequence: No cloning theorem, i.e. no duplication of information.
2. Measurement is partial (Heisenberg Uncertainty Relation).
   Consequence: The full state of a quantum computer is not observable.
3. Measurement is irreversible.
   Consequence: The state of a quantum system is irrevocably destroyed if we inspect it.

The mathematical structure has also consequences for any **Quantum Logic**, e.g. De Morgan fails, ‘Tertium non datur’ is not guaranteed, etc.

Quantum Physics vs Quantum Computation

Quantum Physics
Given a quantum system (device).
What is its dynamics?

- Heisenberg Picture:

$$\mathbf{A} \mapsto \mathbf{A}_t = \mathbf{A}(t) = e^{it\mathbf{H}}\mathbf{A}e^{-it\mathbf{H}}$$

- Schrödinger Picture:

$$|x\rangle \mapsto |x\rangle_t = |x(t)\rangle = e^{-it\mathbf{H}}|x\rangle$$

Quantum Computation
Given a desired computation (dynamics).
What quantum device (e.g. circuit) is needed to obtain this?

Quantum Computation

Quantum computation tries to utilise quantum systems/devices in order to perform computational tasks or to implement (secure) quantum communication protocols.

1973	C. Bennett: Reversible Computation
1980	P.A. Benioff: Quantum Turing Machine
1982	R. Feynman: Quantum Simulation
1985	D. Deutsch: Universal QTM
1994	P. Shor: Factorisations
1996	L. Grover: Database Search
2008	Harrow, Hassidim, Lloyd: Linear Equations

When will (cheap) quantum computers be available? What will be a **killer application** for quantum computation? When will we reach quantum supremacy?