Models of Computation II, Exercises 2: The Universal RM

1. Consider the register machine program P, given by the following code

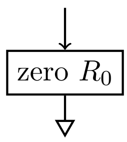
$$
\begin{aligned}
L_0 &: R_1^- \to L_1, L_6 \\
L_1 &: R_2^- \to L_2, L_4 \\
L_2 &: R_0^+ \to L_3 \\
L_3 &: R_3^+ \to L_1 \\
L_4 &: R_3^- \to L_5, L_0 \\
L_5 &: R_2^+ \to L_4 \\
L_6 &: HALT
\end{aligned}
$$

   which computes the function $f(x, y) = x \times y$. The code of P, written $\ulcorner P \urcorner$, has the form $\ulcorner [\ulcorner B_0 \urcorner, \ldots, \ulcorner B_6 \urcorner] \urcorner$ where B_i is the body of L_i for each i. Give the value of $\ulcorner B_i \urcorner$ for each i.

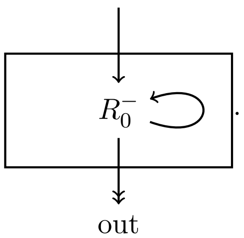
2. Consider the natural number $2^{216} \times 833$.

   (a) What register machine program is represented by this number?

   (b) What function of one argument is computed by this register machine?

3. We saw before how gadgets can be used to make complex register machines out of simpler components. For instance, you were given the gadget 

which was implemented as



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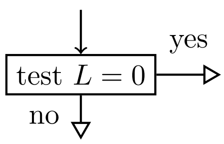
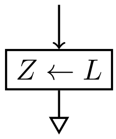
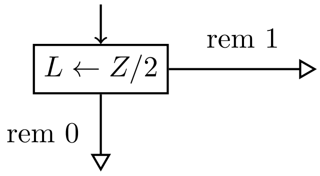
   (a) Define a gadget 

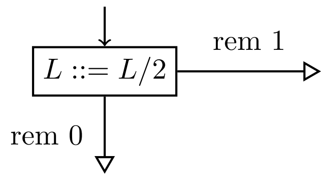
which determines whether the initial value of register L is 0, restoring L to its initial value. If L is initially 0, the gadget leaves L at 0 and takes the "yes" branch. If L is initially $l > 0$, the gadget leaves L at l and takes the "no" branch.

(b) Define a gadget 

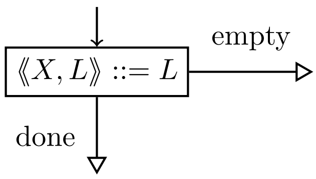
which, when initally $Z = 0$ and $L = l$, exits with $Z = l$ and $L = 0$. (It does not matter what the gadget does if $Z \neq 0$).

(c) Define a gadget 

which computes the quotient of Z by 2, taking the exit path corresponding to the remainder. If initially $Z = z$ and $L = 0$ then, when the gadget exits, $Z = 0$ and $L = \lfloor \frac{z}{2} \rfloor$. If z is even (the remainder is 0), the gadget exits on the "rem 0" branch (and $2L = z$); otherwise (the remainder is 1), the gadget exits on the "rem 1" branch (and $2L + 1 = z$).

(d) Using the gadgets you have already defined, define a gadget 

which computes the quotient of L by 2, taking the exit path corresponding to the remainder, but this time stores the result in L itself. (This gadget will need to use a scratch register, say Z, which is assumed to have initial value 0 and must be restored to having value 0 when the gadget exits.)

(e) Using previously-defined gadgets, define a gadget 

that

- if initially $X = x$ and $L = 0$ takes the "empty" exit with $X = L = 0$, and
- if initially $X = x$ and $L = \langle\langle y, z \rangle\rangle = 2^y(2z + 1)$ takes the "done" exit with $X = y$ and $L = z$.

(Hint: Note that, if $y > 0$ then

$$\frac{2^y(2z+1)}{2} = 2^{y-1}(2z+1) \text{ remainder } 0$$

and if $y = 0$ then

$$\frac{2^y(2z+1)}{2} = z \text{ remainder } 1.$$

Therefore we can compute y and z from $2^y(2z + 1)$ by repeatedly dividing by 2.)

(f) Give the full graph for the gadget defined in (e) by appropriately substituting each gadget used in the definition with its implementation. Does the result look familiar?

4. (**The three-register challenge**) The register machine in question 1 computes the function $f(x, y) = x \times y$. It uses four registers. Construct a register machine that computes the same function, but uses *only three registers*.

   This is a difficult problem. A few years ago a prize was offered and one student managed to produce a correct solution.