Exercises

Program Analysis (CO70020)

Sheet 2

Exercise 1 *Consider the following* **while** *program:*

$$
\begin{array}{l}
[\texttt{x := 1}]^1; \\
\textbf{while } [\texttt{y>0}]^2 \textbf{ do } [\texttt{x := x-1}]^3; \\
[\texttt{x := 2}]^4;
\end{array}
$$

Perform a Live Variables Analysis for this program (state equations and construct solutions).

Solution The auxiliary functions $kill_{\mathsf{LV}}$ and gen_{LV} are given as:

$$
\begin{array}{lcl}
kill_{\mathsf{LV}}(1) & = & \{x\} \\
kill_{\mathsf{LV}}(2) & = & \emptyset \\
kill_{\mathsf{LV}}(3) & = & \{x\} \\
kill_{\mathsf{LV}}(4) & = & \{x\} \\
 & & \\
gen_{\mathsf{LV}}(1) & = & \emptyset \\
gen_{\mathsf{LV}}(2) & = & \{y\} \\
gen_{\mathsf{LV}}(3) & = & \{x\} \\
gen_{\mathsf{LV}}(4) & = & \emptyset
\end{array}
$$

The corresponding equations are therefore:

$$
\begin{array}{lcl}
\mathsf{LV}_{exit}(4) & = & \emptyset \\
\mathsf{LV}_{exit}(3) & = & \mathsf{LV}_{entry}(2) \\
\mathsf{LV}_{exit}(2) & = & \mathsf{LV}_{entry}(3) \cup \mathsf{LV}_{entry}(4) \\
\mathsf{LV}_{exit}(1) & = & \mathsf{LV}_{entry}(2) \\
 & & \\
\mathsf{LV}_{entry}(4) & = & \emptyset \\
\mathsf{LV}_{entry}(3) & = & (\mathsf{LV}_{exit}(3) \backslash \{x\}) \cup \{x\} \\
\mathsf{LV}_{entry}(2) & = & \mathsf{LV}_{exit}(2) \cup \{y\} \\
\mathsf{LV}_{entry}(1) & = & \mathsf{LV}_{exit}(1) \backslash \{x\}
\end{array}
$$

Thus the solution is the final column in the following table:

$\mathsf{LV}_{exit}(4) = \emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\ldots$
$\mathsf{LV}_{exit}(3) = \emptyset$	$\emptyset$	$\{x, y\}$	$\{x, y\}$	$\ldots$
$\mathsf{LV}_{exit}(2) = \emptyset$	$\{x\}$	$\{x, y\}$	$\{x, y\}$	$\ldots$
$\mathsf{LV}_{exit}(1) = \emptyset$	$\{x, y\}$	$\{x, y\}$	$\{x, y\}$	$\ldots$
$\mathsf{LV}_{entry}(4) = \emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\ldots$
$\mathsf{LV}_{entry}(3) = \emptyset$	$\{x\}$	$\{x, y\}$	$\{x, y\}$	$\ldots$
$\mathsf{LV}_{entry}(2) = \emptyset$	$\{x, y\}$	$\{x, y\}$	$\{x, y\}$	$\ldots$
$\mathsf{LV}_{entry}(1) = \emptyset$	$\{y\}$	$\{y\}$	$\{y\}$	$\ldots$

Exercise 2 *Construct/specify all elements of $\mathcal{P}(\{\mathtt{x}, \mathtt{y}, \mathtt{z}\})$, i.e. the power set of $\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$. Describe the sub-set relation on $\mathcal{P}(\{\mathtt{x}, \mathtt{y}, \mathtt{z}\})$, i.e. which sub-set is a sub-set of another sub-set. What is the maximum number of sub-sets a sub-set of $\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ can be included in and/or the hight of $\mathcal{P}(\{\mathtt{x}, \mathtt{y}, \mathtt{z}\})$?*

Solution This gives a "partially ordered set", more precisely a (prototypical) complete lattice; more details in the lectures on Lattice Theory. This set set has $s^3 = 8$ elements. A maximal inclusion "chain" is for example $\emptyset \subseteq \{\mathtt{x}\} \subseteq \{\mathtt{x}, \mathtt{y}\} \subseteq \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ which is of length 3.

Exercise 3 *Construct/specify all elements of $\mathcal{P}(\{\mathtt{x}, \mathtt{y}\} \times \{1, 2, 3\})$, i.e. the power set of the cartesian product $\{\mathtt{x}, \mathtt{y}\} \times \{1, 2, 3\}$. Describe the sub-set relation on $\mathcal{P}(\{\mathtt{x}, \mathtt{y}\} \times \{1, 2, 3\})$. What is the maximum number of sub-sets any sub-set of $\mathcal{P}(\{\mathtt{x}, \mathtt{y}\} \times \{1, 2, 3\})$ can be included in?*

Solution This gives a "partially ordered set", more precisely a complete lattice; more details in the lectures on Lattice Theory. There are 6 elements in $\{\mathtt{x}, \mathtt{y}\} \times \{1, 2, 3\}$ Thus $\emptyset \subseteq \{(\mathtt{x}, 1)\} \subseteq \ldots \subseteq \{\mathtt{x}, \mathtt{y}\} \times \{1, 2, 3\}$ is a maximal "chain" of length 6.

Exercise 4 *Consider the following* **While** *program*

$$\textbf{while}\ \ (\mathtt{x>0})\ \textbf{do}\ \ \mathtt{y:=y-1}$$

Describe the possible RD *solutions at every program point. What is the size of this "property space" and how many possible solution are there for the* RD *analysis?*

Solution Based on the fact that the the number of functions between finite sets is given by $|X \to Y| = |Y|^{|X|}$ we also have $\mathcal{P}(X) \equiv (X \to \{0, 1\})$ and thus $|\mathcal{P}(X)| = 2^{|X|}$, where $|.|$ denotes the cardinality/size of sets. For the Cartesian product we have: $|X \times Y| = |X| \times |Y|$.

We can label the given program as:

$$\textbf{while}\ \ ([\mathtt{x>0}]^1)\ \textbf{do}\ \ [\mathtt{y:=y-1}]^2$$

Then the "property space" is $L = \mathcal{P}(\{?, 1, 2\} \times \{\mathtt{x}, \mathtt{y}\})$.

The program has no isolated entry and one could argue that one needs to 'extend' it before we can perform the analysis, i.e. consider

$$[\textbf{skip}]^0;\ \textbf{while}\ \ ([\mathtt{x>0}]^1)\ \textbf{do}\ \ [\mathtt{y:=y-1}]^2$$

with property space $L = \mathcal{P}(\{?, 0, 1, 2\} \times \{\mathtt{x}, \mathtt{y}\})$.

The size of the property space is $|L| = 2^{l \times v}$ with $l = |\mathbf{Lab}|$ and $v = |\mathbf{Var}|$. Thus, $|L| = 2^{3\times 2} = 2^6 = 64$ for the original program and $|L| = 2^{4\times 2} = 2^8 = 256$ for the extended program.

There are l program points, thus we have $2 \times l$ equations, i.e. functions $\mathsf{RD}_{entry} : \mathbf{Lab} \to L$ and $\mathsf{RD}_{exit} : \mathbf{Lab} \to L$. For the original program we therefore have (arguably) $2 \times 64^2 = 8192$ potential solutions, for the extended program it's maybe $2 \times 256^3 = 33554432$.

Exercise 5 *Consider the set of all sets of the form:*

$$\{*\}, \{*, \{*\}\}, \ldots, \{*, \{*, \{*, \ldots\}\}\}, \ldots$$

i.e. $S_1 = \{*\}$ *and* $S_n = \{*\} \cup \{S_{n-1}\}$ *where* $*$ *is some element/object. Describe the element relation on this set of sets. What is the maximum number of sets any set of can be included in (be element of)?*

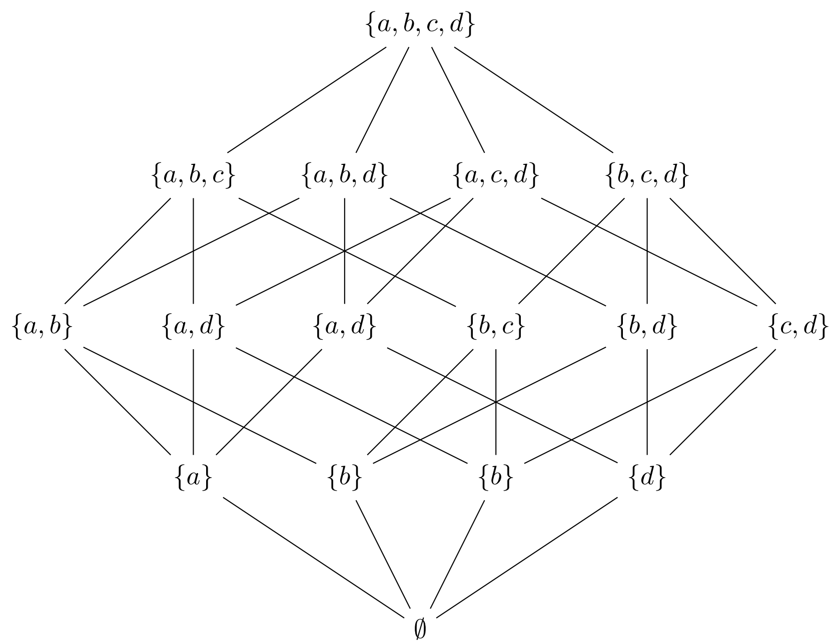
Solution This gives a "totally ordered" *non-complete* lattice, i.e. partial order, isomorphic to $\mathbb{N}$, more details in the lectures on Lattice Theory. There is no maximal "inclusion depth" (or you could say it is "∞").

Note that the inclusions "$\subseteq$" is not transitive on $\{S_i\}$, e.g. $\{*\} \subseteq \{*, \{*\}\}$ and $\{*, \{*\}\} \subseteq \{*, \{*, \{*\}\}\}$ but $\{*\} \not\subseteq \{*, \{*, \{*\}\}\}$. However, "$\subseteq$" can be extended in the obvious way (so-called *transitive closure*) to a partial order which is equivalent to "$\leq$".

Exercise 6 *Consider the power set* $\mathcal{P}(X)$ *of* $X = \{a, b, c, d\}$.

1. *Draw the Hasse diagram.*
2. *Give a monotone map from* $(\mathcal{P}(X), \subseteq)$ *into* $(\mathbb{Z}, \leq)$.
3. *Give a monotone map from* $(\mathbb{Z}, \leq)$ *into* $(\mathcal{P}(X), \subseteq)$.

Solution The Hasse diagram is similar to the diagram in the lecture.



Any constant map is, of course, a monotone map. A slightly more complicated example $s : \mathcal{P}(X) \mapsto \mathbb{Z}$ is given by:

$$s(S) = \text{number of elements in } S$$

and for $f : \mathbb{Z} \mapsto \mathcal{P}(X)$

$$f(x) = \begin{cases} \emptyset & \text{if } x \leq 0 \\ \{a\} & 1 \\ \{a, b\} & 2 \\ \{a, b, c\} & 3 \\ X & \text{if } 4 \leq x \end{cases}$$