Spatial logic of modal mu-calculus and tangled closure operators

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Abstract

There has been renewed interest in recent years in McKinsey and Tarski's interpretation of modal logic in topological spaces and their proof that S4 is the logic of any separable dense-in-itself metric space. Here we extend this work to the modal mu-calculus and to a logic of tangled closure operators that was developed by Fernández-Duque after these two languages had been shown by Dawar and Otto to have the same expressive power over finite transitive Kripke models. We prove that this equivalence remains true over topological spaces.

We establish the finite model property in Kripke semantics for various tangled closure logics with and without the universal modality $\forall$. We also extend the McKinsey–Tarski topological 'dissection lemma'. These results are used to construct a representation map (also called a d-p-morphism) from any dense-in-itself metric space X onto any finite connected locally connected serial transitive Kripke frame.

This yields completeness theorems over X for a number of languages: (i) the modal mu-calculus with the closure operator $\Diamond$; (ii) $\Diamond$ and the tangled closure operators $\langle t\rangle$; (iii) $\Diamond, \forall$; (iv) $\Diamond, \forall, \langle t\rangle$; (v) the derivative operator $\langle d\rangle$; (vi) $\langle d\rangle$ and the associated tangled closure operators $\langle dt\rangle$; (vii) $\langle d\rangle, \forall$; (viii) $\langle d\rangle, \forall, \langle dt\rangle$. Soundness also holds, if: (a) for languages with $\forall$, X is connected; (b) for languages with $\langle d\rangle$, X validates the well known axiom G_1. For countable languages without $\forall$, we prove strong completeness. We also show that in the presence of $\forall$, strong completeness fails if X is compact and locally connected.

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1 Introduction

Modal logic can be given semantics over topological spaces. In this setting, the modality $\Diamond$ can be interpreted in more than one way. The first and most obvious way is as *closure*. Writing $[\![\varphi]\!]$ for the set of points (in a topological model) at which a formula φ is true, $[\![\Diamond\varphi]\!]$ is defined to be the *closure* of $[\![\varphi]\!]$, so that $\Diamond\varphi$ holds at a point x if and only if every open

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neighbourhood of x contains a point y satisfying φ. Then, $\Box$ becomes the interior operator: $[\![\Box\varphi]\!]$ is the interior of $[\![\varphi]\!]$. Early studies of this semantics include [34, 35, 23, 24, 25].

In a seminal result, McKinsey and Tarski [24] proved that the logic of any given separable[1] dense-in-itself metric space in this semantics is S4: it can be axiomatised by the basic modal Hilbert system K augmented by the two axioms $\Box\varphi \to \varphi$ (T) and $\Box\varphi \to \Box\Box\varphi$ (4).

Motivated perhaps by the current wide interest in spatial logic, a wish to present simpler proofs in 'modern language', growing awareness of the work of particular groups such as Esakia's and Shehtman's, or involvement in new settings such as dynamic topology, interest in McKinsey and Tarski's result has revived in recent years. A number of new proofs of it have appeared, some for specific spaces or embodying other variants [26, 3, 1, 27, 33, 21, 14]. Very recently, strong completeness (every countably infinite S4-consistent set of modal formulas is satisfiable in every dense-in-itself metric space) was established by Kremer [17].

In this paper, we seek to extend McKinsey and Tarski's theorem to more powerful languages. We will extend the modal syntax in two separate ways: first, to the mu-calculus, which adds least and greatest fixed points to the basic modal language, and second, by adding an infinite sequence of new modalities $\Diamond_n$ of arity n ($n \geq 1$) introduced in the context of Kripke semantics by Dawar and Otto [6]. The semantics of $\Diamond_n$ is given by the mu-calculus formula

$$\Diamond_n(\varphi_1, \ldots, \varphi_n) \equiv \nu q \bigwedge_{1 \leq i \leq n} \Diamond(\varphi_i \wedge q),$$

for a new atom q not occurring in $\varphi_1, \ldots, \varphi_n$. The order and multiplicity of arguments to a $\Diamond_n$ is immaterial, so we will abbreviate $\Diamond_n(\gamma_1, \ldots, \gamma_n)$ to $\langle t\rangle\{\gamma_1, \ldots, \gamma_n\}$. Fernández-Duque used this to give the modalities topological semantics, dubbed them *tangled closure modalities* (this is why we use the notation $\langle t\rangle$), and studied them in [9, 10, 12, 11].

Dawar and Otto [6] showed that, somewhat surprisingly, the mu-calculus and the tangled modalities have exactly the same expressive power over finite Kripke models with transitive frames. We will prove that this remains true over topological spaces. So the tangled closure modalities offer a viable alternative to the mu-calculus in both these settings.

We go on to determine the logic of an arbitrary dense-in-itself metric space X in these languages. We will show that in the mu-calculus, the logic of X is axiomatised by a system called S4μ comprising Kozen's basic system for the mu-calculus augmented by the S4 axioms, and the tangled logic of X is axiomatised by a system called S4t similar to one in [10]. We will establish strong completeness for countable sets of formulas.

We will also consider the extension of the tangled language with the *universal modality,* '$\forall$'. (Earlier work on the universal modality in topological spaces includes [31, 22].) This language can express connectedness: there is a formula C valid in precisely the connected spaces. Adding this and some standard machinery for $\forall$ to the system S4t gives a system called 'S4t.UC'. We will show that every S4t.UC-consistent formula is satisfiable in every dense-in-itself metric space. Thus, the logic of an arbitrary connected dense-in-itself metric space is S4t.UC. We also show that strong completeness fails in general, even for the modal language plus the universal modality.

A second and more powerful spatial interpretation of $\Diamond$ is as the *derivative operator.* Following tradition, when considering this interpretation we will generally write the modal box and diamond as $[d]$ and $\langle d\rangle$. In this interpretation, $[\![\langle d\rangle\varphi]\!]$ is defined to be the set of *strict limit points* of $[\![\varphi]\!]$: so $\langle d\rangle\varphi$ holds at a point x precisely when every open neighbourhood

[1] The separability assumption was removed in [28].

of x contains a point $y \neq x$ satisfying φ. The original closure diamond is expressible by the derivative operator: $\Diamond\varphi$ is equivalent in any topological model to $\varphi \vee \langle d\rangle\varphi$, and $\Box\varphi$ to $\varphi \wedge [d]\varphi$. So in passing to $\langle d\rangle$, we have not reduced the power of the language.

Already in [24, Appendix I], McKinsey and Tarski discussed the derivative operator and asked a number of questions about it. It has since been studied by, among others, Esakia and his Tbilisi group ([8, 2], plus many other publications), Shehtman [30, 32], Lucero-Bryan [22], and Kudinov–Shehtman [20], section 3 of which contains a survey of results.

In the derivative semantics, determining the logic of a given dense-in-itself metric space is not a simple matter, for the logic can vary with the space. As McKinsey and Tarski observed, $\langle d\rangle\big((x \wedge \langle d\rangle\neg x) \vee (\neg x \wedge \langle d\rangle x)\big) \leftrightarrow \langle d\rangle x \wedge \langle d\rangle\neg x$ is valid in $\mathbb{R}^2$ but not in $\mathbb{R}$. This formula is valid in the same topological spaces as the formula G_1, where for each integer $n \geq 1$,

$$\mathrm{G}_n = \Big([d] \bigvee_{0\leq i\leq n} \Box Q_i\Big) \to \bigvee_{0\leq i\leq n} [d]\neg Q_i.$$

Here, $p_0, \ldots, p_n$ are pairwise distinct atoms, and for $i = 0, \ldots, n$,

$$Q_i = p_i \wedge \bigwedge_{i\neq j\leq n} \neg p_j.$$

In [30], Shehtman proved that the logic of $\mathbb{R}^n$ for finite $n \geq 2$ is $\mathrm{KD4G}_1$, axiomatised by the basic system K together with the axioms $\langle d\rangle\top$ (D), $[d]p \to [d][d]p$ (4), and G_1. The logic of $\mathbb{R}$ was shown by Shehtman [32] and Lucero-Bryan [22] to be $\mathrm{KD4G}_2$. The logic of every separable zero-dimensional dense-in-itself metric space (such as $\mathbb{Q}$ and the Cantor space) is just KD4 [30], the smallest possible logic of a dense-in-itself metric space in the derivative semantics. [4] proves that there are continuum-many logics of subspaces of the rationals in the language with $[d]$.

It is plain that $\mathrm{G}_1 \vdash \mathrm{G}_2 \vdash \mathrm{G}_3 \vdash \cdots$, so the logics $\mathrm{KD4G}_1 \supseteq \mathrm{KD4G}_2 \supseteq \cdots$ form a decreasing chain, and by [22, corollary 3.11], its intersection is KD4. Shehtman [30, problem 1] asked if $\mathrm{KD4G}_1$ is the largest possible logic of a dense-in-itself metric space in the derivative semantics.

In this paper, we answer Shehtman's question affirmatively: every $\mathrm{KD4G}_1$-consistent formula of the language with $\langle d\rangle$ is satisfiable in every dense-in-itself metric space. Thus, the logic of every dense-in-itself metric space that validates G_1 is exactly $\mathrm{KD4G}_1$. We also establish strong completeness for such spaces.

Adding the tangled closure operators, we prove similarly that the logic of every dense-in-itself metric space that validates G_1 is axiomatised by $\mathrm{KD4G}_1t$ (including the tangle axioms). We also prove strong completeness.

Further adding the universal modality, we show similarly that $\mathrm{KD4G}_1t.\mathrm{UC}$ (and $\mathrm{KD4G}_1.\mathrm{UC}$ if the tangle closure operators are dropped) axiomatises the logic of every connected dense-in-itself metric space that validates G_1. Strong completeness fails in general, as a consequence of the proof that it already fails for the weaker language with $\Box$ and $\forall$.

The reader can find a summary of our results in table 1 in section 10.

Our proof works in a fairly familiar way, similar in spirit to McKinsey and Tarski's original argument in [24] — indeed, we use some results from that paper. There are three main steps.

1. We establish the *finite model property* for the various logics, in Kripke semantics. This work may be of independent interest: earlier related results were proved in [30, 10].

2. We then prove a topological theorem that establishes Tarski's 'dissection lemma' [35, satz 3.10], [24, theorem 3.5] and a variant of it.

3. These topological results are used to construct a map from an arbitrary dense-in-itself metric space onto any finite connected $\mathrm{KD4G}_1$ Kripke frame, that preserves the required formulas.

Putting the three steps together proves completeness for all the languages, which is then lifted by a separate argument to strong completeness for languages without $\forall$.

It can be seen that our results concern the logic of each individual space within a large class of spaces (the dense-in-themselves metric spaces), rather than the logic of a large class of spaces, or of particular spaces such as $\mathbb{R}$. This is as in [24]. We do not assume separability, we consider languages that have not previously been much studied in the topological setting, and we obtain some results on strong completeness, a matter that has only recently been investigated in this setting.

2 Basic definitions

In this section, we lay out the main definitions, notation, and some basic results.

2.1 Notation for sets and binary relations

For a set X, we let $\wp(X)$ denote the power set (set of all subsets) of X. For $Y \in \wp(X)$ we write $X \setminus Y$ for $\{x \in X : x \notin Y\}$. Note that $(X \cap Y) \setminus Z = X \cap (Y \setminus Z)$, so we may omit the parentheses in such expressions. For a partial function $f : X \to Y$ we let $\operatorname{dom} f$ denote the domain of f, and $\operatorname{rng} f$ its range.

A *binary relation* on a set W is a subset of $W \times W$. Let R be a binary relation on W. We write any of $R(w_1, w_2)$, Rw_1w_2, and w_1Rw_2 to denote that $(w_1, w_2) \in R$. We say that R is *reflexive* if $R(w, w)$ for all $w \in W$, and *transitive* if $R(w_1, w_2)$ and $R(w_2, w_3)$ imply $R(w_1, w_3)$. We write R^* for the *reflexive transitive closure* of R: the smallest reflexive transitive binary relation that contains R. We also write

$$\begin{array}{rcllcl} R^{-1} & = & \{(w_2, w_1) \in W \times W : R(w_1, w_2)\}, & & & \\ R^{\circ} & = & \{(w_1, w_2) \in W \times W : R(w_1, w_2) \wedge R(w_2, w_1)\} & & = & R \cap R^{-1}, \\ R^{\bullet} & = & \{(w_1, w_2) \in W \times W : R(w_1, w_2) \wedge \neg R(w_2, w_1)\} & & = & R \setminus R^{-1}. \end{array}$$

For $w \in W$, we let $R(w)$ denote the set $\{w' \in W : R(w, w')\}$, sometimes called the set of *R-successors* or *R-alternatives* of w. For $W' \subseteq W$, we write $R \restriction W'$ for the binary relation $R \cap (W' \times W')$ on W'.

We write $\mathbb{R}$ for the set of real numbers, On for the class of ordinals, and ω for the first infinite ordinal.

2.2 Kripke frames

A *(Kripke) frame* is a pair $\mathcal{F} = (W, R)$, where W is a non-empty set of 'worlds' and R is a binary relation on W. We attribute properties to a frame by the usual extrapolation from the frame's components. So, we say that $\mathcal{F}$ is *finite* if W is finite, *reflexive* if R is reflexive, and *transitive* if R is transitive. Two frames are said to be *disjoint* if their respective sets of worlds are disjoint. And so on.

A *root of* $\mathcal{F}$ is an element $w \in W$ such that $W = R^*(w)$. Roots of a frame may not exist, nor be unique when they do. We say that $\mathcal{F}$ is *rooted* if it has a root. At the other end, an element $w \in W$ is said to be R*-maximal* if $R^\bullet(w) = \emptyset$. Such an element has no 'proper' R-successors, of which it is not itself an R-successor.

A *subframe* of $\mathcal{F}$ is a frame of the form $\mathcal{F}' = (W', R \restriction W')$, for non-empty $W' \subseteq W$. It is simply a substructure of $\mathcal{F}$ in the usual model-theoretic sense. We call $\mathcal{F}'$ the *subframe of* $\mathcal{F}$ *based on* W'. We say that $\mathcal{F}'$ is a *generated* or *inner subframe* of $\mathcal{F}$ if $R(w) \subseteq W'$ for every $w \in W'$ — equivalently, $R \restriction W' = R \cap (W' \times W)$. For $w \in W$, we write:

- $\mathcal{F}(w)$ for the subframe $(R(w), R \restriction R(w))$ of $\mathcal{F}$ based on $R(w)$,
- $\mathcal{F}^*(w)$ for the subframe $(R^*(w), R \restriction R^*(w))$ of $\mathcal{F}$ generated by w.

For an integer $n \geq 1$, we say that $\mathcal{F}$ is n*-connected* if it is not the union of $n+1$ disjoint generated subframes (recall that subframes are non-empty), *connected* if it is 1-connected, and *locally* n*-connected* if for each $w \in W$, the subframe $\mathcal{F}(w)$ is n-connected. Note that $\mathcal{F}$ is n-connected iff the equivalence relation $(R \cup R^{-1})^*$ on W has at most n equivalence classes. Every rooted frame is connected. Connectedness will be discussed in more detail in section 5.10.

2.3 Topological spaces

We will assume some familiarity with topology, but we take some time to reprise the main concepts and notation. A *topological space* is a pair (X, τ), where X is a set and $\tau \subseteq \wp(X)$ satisfies:

1. if $\mathcal{S} \subseteq \tau$ then $\bigcup \mathcal{S} \in \tau$,
2. if $\mathcal{S} \subseteq \tau$ is finite then $\bigcap \mathcal{S} \in \tau$, on the understanding that $\bigcap \emptyset = X$.

So τ is a set of subsets of X closed under unions and finite intersections. By taking $\mathcal{S} = \emptyset$, it follows that $\emptyset, X \in \tau$. The elements of τ are called *open subsets* of X, or just *open sets.* An *open neighbourhood* of a point $x \in X$ is an open set containing x. A subset $C \subseteq X$ is called *closed* if $X \setminus C$ is open. The set of closed subsets of X is closed under intersections and finite unions. If O is open and C closed then $O \setminus C$ is open and $C \setminus O$ is closed.

We use the signs int, cl, $\langle d \rangle$ to denote the *interior, closure,* and *derivative* operators, respectively. So for $S \subseteq X$,

- $\operatorname{int} S = \bigcup \{O \in \tau : O \subseteq S\}$ — the largest open set contained in S,
- $\operatorname{cl} S = \bigcap \{C \subseteq X : C \text{ closed}, S \subseteq C\}$ — the smallest closed set containing S; we have $\operatorname{cl} S = \{x \in X : S \cap O \neq \emptyset \text{ for every open neighbourhood } O \text{ of } x\}$,
- $\langle d \rangle S = \{x \in X : S \cap O \setminus \{x\} \neq \emptyset \text{ for every open neighbourhood } O \text{ of } x\}$.

Then $\operatorname{int} S \subseteq S \subseteq \operatorname{cl} S \supseteq \langle d \rangle S$. For all subsets A, B of X, we have

$$\begin{aligned} \operatorname{cl}(A \cup B) &= \operatorname{cl} A \cup \operatorname{cl} B, \\ \langle d \rangle (A \cup B) &= \langle d \rangle A \cup \langle d \rangle B, \\ \operatorname{int}(A \cap B) &= \operatorname{int} A \cap \operatorname{int} B. \end{aligned}$$

That is, *closure and* $\langle d \rangle$ *are additive and interior is multiplicative.*

We follow standard practice and identify (notationally) the space (X, τ) with X. The reader should note that we do allow empty topological spaces, where $X = \emptyset$. This is particularly useful when dealing with subspaces.

A *subspace* of X is a topological space of the form $(Y, \{O \cap Y : O \in \tau\})$, for (possibly empty) $Y \subseteq X$. It is a subset of X, made into a topological space by endowing it with what is called the *subspace topology.* It is said to be an *open subspace* if Y is an open subset of X. As with X, we identify (notationally) the subspace with its underlying set, Y. We write $\mathrm{int}_Y, \mathrm{cl}_Y$ for the operations of interior and closure in the subspace Y. It can be checked that for every $S \subseteq Y$ we have $\mathrm{cl}_Y\, S = Y \cap \mathrm{cl}\, S$, and if Y is an open subspace then $\mathrm{int}_Y\, S = \mathrm{int}\, S$.

We will be considering various properties that a topological space X may have. We leave most of them for later, but we mention now that X is said to be *T1* if every singleton subset $\{x\}$ is closed, *dense in itself* if no singleton subset is open, *connected* if it is not the union of two disjoint non-empty open sets, and *separable* if it has a countable subset D with $X = \mathrm{cl}\, D$.

2.4 Metric spaces

A *metric space* is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a 'distance function' (having nothing to do with the operator $\langle d \rangle$ above) satisfying, for all $x, y, z \in X$,

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ iff $x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, z) \leq d(x, y) + d(y, z)$ (the 'triangle inequality').

We assume some experience of working with this definition, in particular with the triangle inequality. Examples of metric spaces abound and include the real numbers $\mathbb{R}$ with the standard distance function $d(x, y) = |x - y|$, $\mathbb{R}^n$ with Pythagorean distance, etc. As usual, we often identify (notationally) (X, d) with X.

Let (X, d) be a metric space, and $x \in X$. For non-empty $S \subseteq X$, define

$$d(x, S) = \inf\{d(x, y) : y \in S\}.$$

We leave $d(x, \emptyset)$ undefined. For a real number $\varepsilon > 0$, we let $N_\varepsilon(x)$ denote the so-called 'open ball' $\{y \in X : d(x, y) < \varepsilon\}$. A metric space (X, d) gives rise to a topological space (X, τ_d) in which a subset $O \subseteq X$ is declared to be open (i.e., in τ_d) iff for every $x \in O$, there is some $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq O$. In other words, the open sets are the unions of open balls. We frequently regard a metric space (X, d) equally as a topological space (X, τ_d). So, we will say that a metric space has a given topological property (such as being dense in itself) if the associated topological space has the property. As an example, it can be checked that every metric space is T1.

A *subspace* of a metric space (X, d) is a pair of the form $(Y, d \restriction Y \times Y)$, where $Y \subseteq X$. It is plainly a metric space, and the topological space $(Y, \tau_{d \restriction Y \times Y})$ is a subspace of (X, τ_d).

2.5 Fixed points

Let X be a set and $f : \wp(X) \to \wp(X)$ be a map. We say that f is *monotonic* if $f(S) \subseteq f(S')$ whenever $S \subseteq S' \subseteq X$. By a well known theorem of Knaster and Tarski [36], actually formulated for complete lattices, every monotonic $f : \wp(X) \to \wp(X)$ has *least and greatest fixed points* — there is a unique $\subseteq$-minimal subset $L \subseteq X$ such that $f(L) = L$, and a unique $\subseteq$-maximal $G \subseteq X$ such that $f(G) = G$. We write $L = LFP(f)$ and $G = GFP(f)$.

There are a couple of useful ways to 'compute' these fixed points. First, define by recursion a subset $S_\alpha \subseteq X$ for each ordinal α, by $S_0 = \emptyset$, $S_{\alpha+1} = f(S_\alpha)$, and $S_\delta = \bigcup_{\alpha<\delta} S_\alpha$ for limit ordinals δ. The S_α form an increasing chain terminating in $LFP(f)$, so

$$LFP(f) = \bigcup_{\alpha \in \mathrm{On}} S_\alpha.$$

A similar expression can be given for $GFP(f)$. Second, a subset $S \subseteq X$ is said to be a *pre-fixed point* of f if $f(S) \subseteq S$, and a *post-fixed point* if $f(S) \supseteq S$. In [36] it is proved that $LFP(f)$ is the intersection of all pre-fixed points of f, and dually for $GFP(f)$:

$$\begin{aligned} LFP(f) &= \bigcap\{S \subseteq X : f(S) \subseteq S\}, \\ GFP(f) &= \bigcup\{S \subseteq X : f(S) \supseteq S\}. \end{aligned}$$

For $f : \wp(X) \to \wp(X)$, define $f' : \wp(X) \to \wp(X)$ by $f'(S) = X \setminus f(X \setminus S)$. It is an exercise to check that f is monotonic iff f' is, and in that case, $GFP(f) = X \setminus LFP(f')$.

Least fixed points are used in the semantics of the mu-calculus, coming up next.

2.6 Languages

We assume some familiarity with modal languages and the mu-calculus. We fix a set Var of *propositional variables,* or *atoms.* Sometimes we may make assumptions on Var — for example, that it is finite. We will be considering various logical languages. The biggest of them is denoted by $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$, which is a set of formulas defined as follows:

1. each $p \in \mathsf{Var}$ is a formula (of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$),
2. $\top$ is a formula,
3. if φ, ψ are formulas then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $\Box\varphi$, $[d]\varphi$, and $\forall\varphi$,
4. if Δ is a non-empty finite set of formulas then $\langle t\rangle\Delta$ and $\langle dt\rangle\Delta$ are formulas,
5. if $q \in \mathsf{Var}$ and φ is a formula that is *positive in* q (that is, every free occurrence of q as an atomic subformula of φ is in the scope of an even number of negations in φ; *free* means 'not in the scope of any μq in ψ'), then $\mu q\varphi$ is a formula, in which all occurrences of q are *bound.* Bound atoms arise only in this way.

For formulas φ, ψ, and $q \in \mathsf{Var}$, the expression $\varphi(\psi/q)$ denotes the result of replacing every free occurrence of q in φ by ψ, where the result is well-formed — that is, all of its subformulas of the form $\mu p\theta$ are such that θ is positive in p. For example, if $\varphi = \mu p\, q$ then $\varphi(\neg p/q) = \mu p\, \neg p$ is not well-formed.

We use standard abbreviations: $\bot$ denotes $\neg\top$, $(\varphi \vee \psi)$ denotes $\neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \to \psi)$ denotes $\neg(\varphi \wedge \neg\psi)$, $(\varphi \leftrightarrow \psi)$ denotes $(\varphi \to \psi) \wedge (\psi \to \varphi)$, $\Diamond\varphi$ denotes $\neg\Box\neg\varphi$, $\langle d\rangle\varphi$ denotes $\neg[d]\neg\varphi$, $\exists\varphi$ denotes $\neg\forall\neg\varphi$, and if φ is positive in q then $\nu q\varphi$ denotes $\neg\mu q\neg\varphi(\neg q/q)$ (this is well-formed). For a non-empty finite set $\Delta = \{\delta_1, \ldots, \delta_n\}$ of formulas, we let $\bigwedge\Delta$ denote $\delta_1 \wedge \ldots \wedge \delta_n$ and $\bigvee\Delta$ denote $\delta_1 \vee \ldots \vee \delta_n$ (the order and bracketing of the conjuncts will always be immaterial). We set $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. Parentheses will be omitted where possible, by the usual methods.

The connectives $\langle t \rangle, \langle dt \rangle$ are called *tangle connectives,* or (more fully) *tangled closure operators.*

We will be using various *sublanguages* of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$, and they will be denoted in the obvious way by omitting prohibited operators from the notation. So for example, $\mathcal{L}^{\mu}_{\Box\forall}$ denotes the language consisting of all $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formulas that do not involve $[d]$, $\langle t \rangle$, or $\langle dt \rangle$.

2.7 Kripke semantics

An *assignment* or *valuation* into a frame $\mathcal{F} = (W, R)$ is a map $h : \mathsf{Var} \to \wp(W)$. A *Kripke model* is a triple $\mathcal{M} = (W, R, h)$, where (W, R) is a frame and h an assignment into it. The *frame of* $\mathcal{M}$ is (W, R), and we say that $\mathcal{M}$ is finite, reflexive, transitive, etc., if its frame is.

For every Kripke model $\mathcal{M} = (W, R, h)$ and every world $w \in W$, we define the notion $\mathcal{M}, w \models \varphi$ of a formula φ of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ being *true at w in $\mathcal{M}$.* The definition is by induction on φ, as follows:

1. $\mathcal{M}, w \models p$ iff $w \in h(p)$, for $p \in \mathsf{Var}$.
2. $\mathcal{M}, w \models \top$.
3. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$.
4. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.
5. $\mathcal{M}, w \models \Box\varphi$ iff $\mathcal{M}, v \models \varphi$ for every $v \in R(w)$.
6. The truth condition for $[d]\varphi$ is exactly the same as for $\Box\varphi$.
7. $\mathcal{M}, w \models \forall\varphi$ iff $\mathcal{M}, v \models \varphi$ for every $v \in W$.
8. $\mathcal{M}, w \models \langle t \rangle\Delta$ iff there are worlds $w = w_0, w_1, \ldots \in W$ with $R(w_n, w_{n+1})$ for each $n < \omega$ and such that for each $\delta \in \Delta$ there are infinitely many $n < \omega$ with $\mathcal{M}, w_n \models \delta$.
9. The truth condition for $\langle dt \rangle\Delta$ is exactly the same as for $\langle t \rangle\Delta$.
10. The truth condition for $\mu q\varphi$ takes longer to explain. For an assignment $h : \mathsf{Var} \to \wp(W)$ and $S \subseteq W$, define a new assignment $h[S/q] : \mathsf{Var} \to \wp(W)$ by

    $$h[S/q](p) = \begin{cases} S, & \text{if } p = q, \\ h(p), & \text{otherwise,} \end{cases}$$

    for $p \in \mathsf{Var}$. Inductively, the set $[\![\varphi]\!]_h = \{w \in W : (W, R, h), w \models \varphi\}$ is well defined, for every assignment h into (W, R). Define a map $f : \wp(W) \to \wp(W)$ by

    $$f(S) = [\![\varphi]\!]_{h[S/q]} \quad \text{for } S \subseteq W.$$

    Since φ is positive in q, it can be shown that f is monotonic, so it has a least fixed point, $LFP(f)$ (see section 2.5). We define $\mathcal{M}, w \models \mu q\varphi$ iff $w \in LFP(f)$.

In the notation of the last clause, it can be checked that $\mathcal{M}, w \models \nu q\varphi$ iff $w \in GFP(f)$.

A word on the semantics of $\langle d\rangle$ and $\langle dt\rangle$. Let us temporarily write $\varphi \equiv \psi$ to mean that $\mathcal{M}, w \models \varphi \leftrightarrow \psi$ for every *transitive* Kripke model $\mathcal{M} = (W, R, h)$ and every $w \in W$. Then it can be checked that for every non-empty finite set Δ of formulas,

$$\begin{aligned} \langle t\rangle\Delta &\equiv \nu q \bigwedge_{\delta\in\Delta} \Diamond(\delta \wedge q), \\ \langle dt\rangle\Delta &\equiv \nu q \bigwedge_{\delta\in\Delta} \langle d\rangle(\delta \wedge q), \end{aligned} \tag{2.1}$$

if $q \in \mathsf{Var}$ is a 'new' atom that does not occur in any formula in Δ. For more details, see lemma 4.2. In a sense, (2.1) is the 'official' definition of the semantics of the tangle connectives, which boils down to clause 8 above in the case of transitive Kripke models.

2.8 Kripke semantics in generated submodels

Let $\mathcal{M} = (W, R, h)$ be a Kripke model. A *generated submodel* of $\mathcal{M}$ is a model of the form $\mathcal{M}' = (W', R', h')$, where (W', R') is a generated subframe of (W, R) and $h' : \mathsf{Var} \to \wp(W')$ is given by $h'(p) = h(p) \cap W'$ for $p \in \mathsf{Var}$. The following is an easy extension to $\mathcal{L}^{\mu\langle d\rangle\langle dt\rangle}_{\Box[d]}$ of a well known result in modal logic:

LEMMA 2.1. *Let $\mathcal{M}' = (W', R', h')$ be a generated submodel of $\mathcal{M} = (W, R, h)$. Then for each $\varphi \in \mathcal{L}^{\mu\langle d\rangle\langle dt\rangle}_{\Box[d]}$ and $w \in W'$, we have*

$$\mathcal{M}, w \models \varphi \iff \mathcal{M}', w \models \varphi.$$

There is no distinction between $\Box$ and $[d]$ or between $\langle t\rangle$ and $\langle dt\rangle$ in Kripke semantics. This is not so in topological semantics, our next topic.

2.9 Topological semantics

Given a topological space X, an *assignment* into X is simply a map $h : \mathsf{Var} \to \wp(X)$. A *topological model* is a pair (X, h), where X is a topological space and h an assignment into X. We will also be considering topological models where Var is replaced by some other set of atoms. Details will be given later.

As with Kripke models, we attribute a topological property to a topological model if the underlying topological space has the property.

For every topological model (X, h) and every point $x \in X$, we define $(X, h), x \models \varphi$, for a $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ, by induction on φ:

1. $(X, h), x \models p$ iff $x \in h(p)$, for $p \in \mathsf{Var}$.
2. $(X, h), x \models \top$.
3. $(X, h), x \models \neg\varphi$ iff $(X, h), x \not\models \varphi$.
4. $(X, h), x \models \varphi \wedge \psi$ iff $(X, h), x \models \varphi$ and $(X, h), x \models \psi$.
5. $(X, h), x \models \Box\varphi$ iff there is an open neighbourhood O of x with $(X, h), y \models \varphi$ for every $y \in O$.

6. $(X,h),x \models [d]\varphi$ iff there is an open neighbourhood O of x with $(X,h),y \models \varphi$ for every $y \in O \setminus \{x\}$. We do not require φ to hold at x itself.

7. $(X,h),x \models \forall\varphi$ iff $(X,h),y \models \varphi$ for every $y \in X$.

8. For a non-empty finite set Δ of formulas for which we have inductively defined semantics, write $[\![\delta]\!] = \{x \in X : (X,h),x \models \delta\}$, for each $\delta \in \Delta$. Then define:

   - $(X,h),x \models \langle t\rangle\Delta$ iff there is some $S \subseteq X$ such that $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$,
   - $(X,h),x \models \langle dt\rangle\Delta$ iff there is some $S \subseteq X$ such that $x \in S \subseteq \bigcap_{\delta\in\Delta} \langle d\rangle([\![\delta]\!] \cap S)$.

9. Suppose inductively that $[\![\varphi]\!]_h = \{x \in X : (X,h),x \models \varphi\}$ is well defined, for every assignment h into X. Define a map $f : \wp(X) \to \wp(X)$ by

$$f(S) = [\![\varphi]\!]_{h[S/q]} \quad \text{for } S \subseteq X,$$

   where $h[S/q]$ is defined as in Kripke semantics. Again, f is monotonic, and we define $(X,h),x \models \mu q\varphi$ iff $x \in LFP(f)$.

The definition makes sense but has no content if X is empty: there are no points $x \in X$ to evaluate at. Writing $[\![\varphi]\!]_h = \{x \in X : (X,h),x \models \varphi\}$, we have $[\![\Box\varphi]\!]_h = \mathrm{int}([\![\varphi]\!]_h)$, $[\![\Diamond\varphi]\!]_h = \mathrm{cl}([\![\varphi]\!]_h)$, and $[\![\langle d\rangle\varphi]\!]_h = \langle d\rangle([\![\varphi]\!]_h)$ for each φ, h. Again, $[\![\nu q\varphi]\!] = GFP(f)$, where φ, f are as in the last clause.

REMARK 2.2. Again we briefly discuss the semantics of $\langle t\rangle$ and $\langle dt\rangle$ (see clause 8 above). With $\varphi \equiv \psi$ redefined to mean that $(X,h),x \models \varphi \leftrightarrow \psi$ for every topological model (X,h) and $x \in X$, the equivalences in (2.1) above continue to hold, and indeed they motivate clause 8. However, there is a perhaps more intuitive meaning for $\langle t\rangle$ and $\langle dt\rangle$ in terms of *games,* which are used extensively in the mu-calculus. Let players $\forall$, $\exists$ play a game of length ω on X. Initially, the position is x. In each round, if the current position is $y \in X$, player $\forall$ chooses an open neighbourhood O of y and a formula $\delta \in \Delta$. Player $\exists$ must select a point $z \in O$ at which δ is true (and with $z \neq y$ in the case of $\langle dt\rangle$). If she cannot, player $\forall$ wins. That is the end of the round, and the next round commences from position z. Player $\exists$ wins if she survives every round. It can be checked that $(X,h),x \models \langle t\rangle\Delta$ (respectively, $(X,h),x \models \langle dt\rangle\Delta$) iff $\exists$ has a winning strategy in this game (respectively, the game where she must additionally choose $z \neq y$).

2.10 Topological semantics in open subspaces

Let X be a topological space and Y a subspace of X. Each assignment $h : \mathsf{Var} \to \wp(X)$ into X induces an assignment h_Y into Y, via $h_Y(p) = Y \cap h(p)$, for each $p \in \mathsf{Var}$. Thus, we can evaluate formulas at points in Y in both (X,h) and (Y,h_Y). Because the semantics of the connectives $\Box, [d], \langle t\rangle, \langle dt\rangle$ depend on only arbitrarily small open neighbourhoods of the evaluation point, it is easily seen that if Y is an *open* subspace of X, we get the same result for every formula not involving $\forall$. That is, the following analogue of lemma 2.1 holds:

LEMMA 2.3. *Whenever Y is an open subspace of X, we have $(X,h),y \models \varphi$ iff $(Y,h_Y),y \models \varphi$, for every $y \in Y$ and $\varphi \in \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]}$.*

(This holds vacuously if Y is empty.)

2.11 Hilbert systems

These are familiar, and we will be informal. A *Hilbert system* H in a given language $\mathcal{L} \subseteq \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ is a set of *axioms,* which are $\mathcal{L}$-formulas, and *inference rules,* which have the form

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}, \tag{2.2}$$

for $\mathcal{L}$-formulas $\varphi_1, \ldots, \varphi_n, \psi$. A *derivation in H (of length l)* is a sequence $\varphi_1, \ldots, \varphi_l$ of $\mathcal{L}$-formulas such that each φ_i $(1 \leq i \leq l)$ is either an H-axiom or is derived from earlier φ_j by an H-rule — that is, there are $1 \leq j_1, \ldots, j_n < i$ such that

$$\frac{\varphi_{j_1}, \ldots, \varphi_{j_n}}{\varphi_i}$$

is an instance of a rule of H.

A *theorem of H* is a formula that occurs in some derivation in H. An *H-logic* is a set of $\mathcal{L}$-formulas that contains all H-axioms and is closed under all H-rules. The set of theorems of H is the smallest H-logic. Sometimes we identify (notationally) H with this set, or present H implicitly by defining an H-logic.

A formula φ is *consistent* with H if $\neg\varphi$ is not a theorem of H. A set Γ of formulas is *consistent* with H if $\bigwedge \Gamma_0$ is consistent with H, for every finite $\Gamma_0 \subseteq \Gamma$.

Some familiar Hilbert systems used later are:

K: the axioms comprise (i) all instances of propositional tautologies (e.g., $\varphi \to (\psi \to \varphi)$, etc.) and (ii) all formulas of the form $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (the so-called 'normality' schema). The inference rules are:

- modus ponens: $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$
- $\Box$-generalisation: $\dfrac{\varphi}{\Box\varphi}$

K4: this is K plus all instances of the '4' schema: $\Box\varphi \to \Box\Box\varphi$.

S4: this is K plus all instances of the S4 schemata: $\Box\varphi \to \varphi$ and $\Box\varphi \to \Box\Box\varphi$.

The well known substitution rule $\dfrac{\varphi}{\varphi(\psi/q)}$ is not always sound in the mu-calculus and is not needed in other systems, so we omit it.

As usual, we denote particular Hilbert systems by sequences of letters and numbers indicating the axioms present. For example, S4.UC denotes the extension of S4 by the axioms generated by two schemes U and C to be seen later. The letter t will denote the schemata for the tangle operator given in section 5.3.

2.12 Satisfiability, validity, equivalence

Let $\mathcal{F} = (W, R)$ be a Kripke frame and X a topological space. A set Γ of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formulas is said to be *satisfiable in $\mathcal{F}$* if there exist an assignment h into $\mathcal{F}$ and a world $w \in W$ such that

$(W, R, h), w \models \gamma$ for every $\gamma \in \Gamma$. Similarly, Γ is said to be *satisfiable in* X if there exist an assignment h into X and a point $x \in X$ such that $(X, h), x \models \gamma$ for every $\gamma \in \Gamma$.

Let φ be an $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula. We say that φ is *satisfiable in* $\mathcal{F}$*, or in* X*,* if the set $\{\varphi\}$ is so satisfiable. We say that φ is *valid in* $\mathcal{F}$ *(respectively, in* X*)* if $\neg\varphi$ is not satisfiable in $\mathcal{F}$ (respectively, in X). We may also say in this case that $\mathcal{F}$ or X *validates* φ.

We also say that φ is *equivalent* to a formula ψ in $\mathcal{F}$ (respectively, X) if $\varphi \leftrightarrow \psi$ is valid in $\mathcal{F}$ (respectively, X).

2.13 Logics

Let $\mathcal{K}$ be a class of Kripke frames or topological spaces. In the context of a given language $\mathcal{L} \subseteq \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$, the *($\mathcal{L}$)-logic of* $\mathcal{K}$ is the set of all $\mathcal{L}$-formulas that are valid in every member of $\mathcal{K}$. A Hilbert system H for $\mathcal{L}$ whose set of theorems is T, say, is said to be

- *sound over* $\mathcal{K}$ if T is a subset of the logic of $\mathcal{K}$ (all H-theorems are valid in $\mathcal{K}$),
- *weakly complete*, or simply *complete, over* $\mathcal{K}$ if T contains the logic of $\mathcal{K}$ (all $\mathcal{K}$-valid formulas are H-theorems),
- *strongly complete over* $\mathcal{K}$ if every countable H-consistent set Γ of $\mathcal{L}$-formulas is satisfiable in some structure in $\mathcal{K}$.

The logic of a single frame $\mathcal{F}$ is defined to be the logic of the class $\{\mathcal{F}\}$; similar definitions are used for the other terms here.

We say that a Kripke frame $\mathcal{F}$ is an *H-frame,* or that $\mathcal{F}$ *validates H,* if H is sound over $\mathcal{F}$. To establish this, it is enough to check that each axiom of H is valid in $\mathcal{F}$, and that each rule of H preserves $\mathcal{F}$-validity (in the notation in (2.2) above, this means that if $\varphi_1, \ldots, \varphi_n$ are valid in $\mathcal{F}$ then so is ψ). We assume familiarity with basic results about modal validity: for example, that a frame is a K4-frame iff it is transitive, and an S4-frame iff it is reflexive and transitive.

It can be checked that H is weakly complete over $\mathcal{K}$ iff every *finite* H-consistent set of formulas is satisfiable in some structure in $\mathcal{K}$. Hence, every strongly complete Hilbert system is also weakly complete. The main aim of this paper is to provide Hilbert systems that are (where possible) sound and strongly complete over various topological spaces, with respect to various sublanguages of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$.

A system H is said to have the *finite model property over* $\mathcal{K}$ if each H-consistent formula is satisfiable in some *finite* member of $\mathcal{K}$. Equivalently, this means that H is weakly complete over the class of finite members of $\mathcal{K}$ (i.e. any formula valid in all finite members of $\mathcal{K}$ is an H-theorem).

3 Hilbert systems for mu-calculus

We now present a very brief diversion on a Hilbert system for the mu-calculus that is sound and complete over the class of finite reflexive transitive Kripke frames. It will be used to translate μ to $\langle t\rangle$ and to axiomatise the $\mathcal{L}^{\mu}_{\Box}$-logic of dense-in-themselves metric spaces. In this section, all formulas are $\mathcal{L}^{\mu}_{\Box}$-formulas, all Hilbert systems are for this language, and we assume that Var is infinite.

DEFINITION 3.1. Consider the two Hilbert systems:

Kμ: standard modal logic K with the axioms comprising all instances of propositional tautologies and of normality ($\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$), and the inference rules modus ponens, $\Box$-generalisation, plus the following for each formula φ positive in q:

- fixed point axiom: $\varphi(\mu q\varphi/q) \to \mu q\varphi$, provided that no free occurrence of an atom in $\mu q\varphi$ gets bound in $\varphi(\mu q\varphi/q)$ — consequently, $\varphi(\mu q\varphi/q)$ is well formed (the idea is roughly that $\mu q\varphi$ is a pre-fixed point of φ)
- fixed point rule: $\dfrac{\varphi(\psi/q) \to \psi}{\mu q\varphi \to \psi}$, provided that no free occurrence of an atom in ψ gets bound in $\varphi(\psi/q)$ — hence, $\varphi(\psi/q)$ is well formed (the idea this time is roughly that $\mu q\varphi$ is the least pre-fixed point of φ).

We write $\mathrm{K}\mu \vdash \varphi$ if φ is a theorem of this system. It is well known (see, e.g., [5, §6]) that the system is equivalent to the original equational system of Kozen [16].

S4μ: this is Kμ plus the S4 schemata $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$. We write $\mathrm{S4}\mu \vdash \varphi$ if φ is a theorem of this system.

The following combines some famous and difficult work in the mu-calculus.

FACT 3.2 ([16, 38, 15])**.** Kμ *is sound and complete over the class of all finite Kripke frames.*

We are going to extend it to show that S4μ is sound and complete over the class of finite reflexive transitive frames, and, later, over every dense-in-itself metric space. First, a form of the substitution rule can be established.

LEMMA 3.3. *Suppose φ, ψ are formulas such that for each atom s occurring free in ψ, there is no subformula of φ of the form $\mu s\theta$. If $\mathrm{S4}\mu \vdash \varphi$, then $\mathrm{S4}\mu \vdash \varphi(\psi/p)$ for any atom p.*

Proof (sketch). Let φ, ψ, p be as stipulated. For a formula α, write $\alpha^\dagger = \alpha(\psi/p)$. We show that $\mathrm{S4}\mu \vdash \alpha \Rightarrow \mathrm{S4}\mu \vdash \alpha^\dagger$ (when the stipulation holds) by induction on the length of a derivation of φ in S4μ.

Suppose that φ is an instance $\alpha(\mu q\alpha/q) \to \mu q\alpha$ of the fixed point axiom. Then $\varphi^\dagger$ is valid in all Kripke frames, so by fact 3.2, $\mathrm{K}\mu \vdash \varphi^\dagger$ and hence certainly $\mathrm{S4}\mu \vdash \varphi^\dagger$.

Suppose that φ is derived by the fixed point rule, so that $\varphi = \mu q\alpha \to \beta$ for some α, β, q meeting the condition of the rule, and $\alpha(\beta/q) \to \beta$ occurs earlier in the derivation. If s occurs free in ψ then there is no μs in $\mu q\alpha \to \beta$, so none in $\alpha(\beta/q) \to \beta$ either. So the inductive hypothesis applies, to give $\mathrm{S4}\mu \vdash (\alpha(\beta/q) \to \beta)^\dagger$. Let us evaluate this. If $p = q$, it is $\mathrm{S4}\mu \vdash \alpha(\beta^\dagger/q) \to \beta^\dagger$. By our stipulation, the fixed point rule applies, giving $\mathrm{S4}\mu \vdash \mu q\alpha \to \beta^\dagger$. But $(\mu q\alpha)^\dagger = \mu q\alpha$. So $\mathrm{S4}\mu \vdash \varphi^\dagger$ as required. If instead $p \neq q$, then it is $\mathrm{S4}\mu \vdash \alpha^\dagger(\beta^\dagger/q) \to \beta^\dagger$. Again, the rule applies, to give $\mathrm{S4}\mu \vdash \mu q\alpha^\dagger \to \beta^\dagger$. But this is exactly $\mathrm{S4}\mu \vdash \varphi^\dagger$.

All other cases of the induction are easy and left to the reader. $\Box$

DEFINITION 3.4. For a formula φ, define a new formula φ^* by induction:

- $p^* = p$ for $p \in \mathsf{Var}$;
- $-^*$ *commutes* with the boolean connectives and μ. That is, $\top^* = \top$, $(\neg\varphi)^* = \neg\varphi^*$, $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, and $(\mu q\varphi)^* = \mu q\varphi^*$.

- $(\Box\varphi)^* = \nu q(\varphi^* \wedge \Box q)$, where $q \in \mathsf{Var}$ is a 'new' atom not occurring in φ^*.

The formula φ^* is plainly well formed, for all $\varphi \in \mathcal{L}^\mu_\Box$.

LEMMA 3.5. *Let φ be any formula. Then for every Kripke model (W, R, h) and $w \in W$, we have $(W, R, h), w \models \varphi^*$ iff $(W, R^*, h), w \models \varphi$, where (recall) R^* is the reflexive transitive closure of R.*

Proof. The proof is by induction on φ. The atomic and boolean cases are easy. Assuming the result for φ, it is a well known exercise in the mu-calculus to check that $(W, R, h), w \models (\Box\varphi)^*$ iff $(W, R, h), u \models \varphi^*$ for every $u \in R^*(w)$. Inductively, this is iff $(W, R^*, h), u \models \varphi$ for every $u \in R^*(w)$, iff $(W, R^*, h), w \models \Box\varphi$ as required.

Finally assume that the result holds for φ, positive in q, for every Kripke model. For a formula ψ and Kripke model (W, R, h), write $[\![\psi]\!]_{(W,R,h)} = \{w \in W : (W, R, h), w \models \psi\}$. Then $(W, R, h), w \models (\mu q\varphi)^*$ iff $(W, R, h), w \models \mu q\varphi^*$, iff w is in the least fixed point of the map $f : \wp(W) \to \wp(W)$ given by $f(S) = [\![\varphi^*]\!]_{(W,R,h[S/q])}$. But inductively, $f(S) = [\![\varphi]\!]_{(W,R^*,h[S/q])}$. So this is iff $(W, R^*, h), w \models \mu q\varphi$ as required. $\Box$

LEMMA 3.6. $\mathrm{S4}\mu \vdash \varphi \leftrightarrow \varphi^*$ *for every* φ.

Proof. Again, the proof is by induction on φ. We write just '$\vdash$' for '$\mathrm{S4}\mu \vdash$' in the proof. We also write $\alpha \equiv \beta$ for $\vdash \alpha \leftrightarrow \beta$. First, replace all bound atoms in φ by fresh ones, to give a formula $\overline{\varphi}$. More formally, $\overline{\psi}$ is defined for each subformula ψ of φ by induction: $\overline{\mu q\psi} = \mu s(\overline{\psi}(s/q))$, where s is a new atom associated with ψ and not occurring in φ, and $\overline{\cdot}$ commutes with all other operators. By fact 3.2, $\overline{\varphi} \equiv \varphi$ and $(\overline{\varphi})^* \equiv \varphi^*$. So, replacing φ by $\overline{\varphi}$, we can suppose without loss of generality that for each atom q that occurs free in φ, there is no subformula of φ of the form $\mu q\theta$. The $-^*$ operator preserves this condition, so it holds for φ^* as well.

For atomic φ, the result is trivial since $\varphi^* = \varphi$, and booleans are fine.

Assume inductively that $\varphi \equiv \varphi^*$ and consider $\Box\varphi$. We need to show that $\Box\varphi \equiv \nu q(\varphi^* \wedge \Box q)$, for 'new' q — that is, $\Box\varphi \equiv \neg\mu q\neg(\varphi^* \wedge \Box\neg q)$. By a tautology, it is enough to show $\neg\Box\varphi \equiv \mu q\neg(\varphi^* \wedge \Box\neg q)$. By fact 3.2, $\neg\Box\varphi \equiv \Diamond\neg\varphi$ and $\mu q\neg(\varphi^* \wedge \Box\neg q) \equiv \mu q(\neg\varphi^* \vee \Diamond q)$. So, letting $\psi = \neg\varphi$, it is enough to prove

$$\Diamond\psi \equiv \mu q\chi, \text{ where } \chi = \psi^* \vee \Diamond q. \tag{3.1}$$

Note that the inductive hypothesis gives $\psi \equiv \psi^*$, and that $\chi(\theta/q)$ is well-formed for any well-formed θ. Let $\chi^0 = \bot$, and $\chi^{n+1} = \chi(\chi^n/q)$ for $n < \omega$. The following claim, needed only for $n = 2$, is an instance of a more general result.

Claim. $\vdash \chi^n \to \mu q\chi$ for each $n < \omega$.
Proof of claim. By induction on n. For $n = 0$, it is $\vdash \bot \to \mu q\chi$, a tautology. Assume inductively that $\vdash \chi^n \to \mu q\chi$. We desire $\vdash \psi^* \vee \Diamond\chi^n \to \mu q\chi$. By the fixed point axiom, it is enough to prove that $\vdash \psi^* \vee \Diamond\chi^n \to \chi(\mu q\chi/q)$ — that is, $\vdash \psi^* \vee \Diamond\chi^n \to \psi^* \vee \Diamond\mu q\chi$. But the inductive hypothesis plus standard uses of generalisation and normality yield $\vdash \Diamond\chi^n \to \Diamond\mu q\chi$, and the result follows using tautologies and modus ponens. This proves the claim.

Towards (3.1), we first show that $\vdash \Diamond\psi \to \mu q\chi$. Observe that inductively, $\chi^1 = \psi^* \vee \Diamond\bot \equiv \psi$ and $\chi^2 = \psi^* \vee \Diamond\chi^1 \equiv \psi \vee \Diamond\psi$. By the claim for $n = 2$, and tautologies, $\vdash \psi \vee \Diamond\psi \to \mu q\chi$ and applying more tautologies yields $\vdash \Diamond\psi \to \mu q\chi$.

Now we show $\vdash \mu q\chi \to \Diamond\psi$. By the fixed point rule, it is enough to show $\vdash \chi(\Diamond\psi/q) \to \Diamond\psi$. That is, $\vdash \psi^* \vee \Diamond\Diamond\psi \to \Diamond\psi$. But given the inductive hypothesis, this is just what the S4 axioms say. This proves (3.1) and completes the case of $\Box\varphi$.

Finally assume the result for φ positive in q, and consider the case $\mu q\varphi$. All formulas below meet all necessary conditions because of our initial assumption on φ. By the inductive hypothesis and lemma 3.3 we get $\vdash \varphi(\mu q\varphi^*/q) \to \varphi^*(\mu q\varphi^*/q)$. The fixed point axiom gives $\vdash \varphi^*(\mu q\varphi^*/q) \to \mu q\varphi^*$. Putting the two together gives $\vdash \varphi(\mu q\varphi^*/q) \to \mu q\varphi^*$. This says that $\mu q\varphi^*$ is a pre-fixed point of φ, so the fixed point rule gives $\vdash \mu q\varphi \to \mu q\varphi^*$. The converse, $\vdash \mu q\varphi^* \to \mu q\varphi$, is similar. □

THEOREM 3.7. *The system* S4μ *is sound and complete over the class of finite reflexive transitive Kripke frames (finite S4 frames).*

Proof. Soundness is easily checked. Conversely, assume that φ is consistent with S4μ. By lemma 3.6, φ^* is consistent with S4μ and hence with Kμ as well. By fact 3.2, there is a finite Kripke model $\mathcal{M} = (W, R, h)$ in which φ^* is satisfied at w, say. We do not know that (W, R) is reflexive or transitive. However, by lemma 3.5 we have $(W, R^*, h), w \models \varphi$ as well, and R^* is reflexive and transitive. □

4 Translations

The language $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ has some redundancy. We can express $\Box$ with $[d]$, and $\langle t\rangle$ with $\langle dt\rangle$ (but not vice versa). We can also express $\langle t\rangle, \langle dt\rangle$ with μ — and often vice versa, using results of Dawar and Otto [6].

Later, we will need translations that work in both topological spaces and (possibly restricted) Kripke models. In this section, we will explore translations — but only to the extent needed for later work. We will again assume that Var is infinite.

4.1 Translating $\langle d\rangle$ and $\langle dt\rangle$ to μ

This is the simplest case. We have already seen the idea, in the equivalence of $\langle t\rangle$- and $\langle t\rangle$-formulas to ν-formulas given in (2.1).

DEFINITION 4.1. For each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ, we define a $\mathcal{L}^{\mu}_{\Box[d]\forall}$-formula φ^μ as follows:

1. $p^\mu = p$ for $p \in \mathsf{Var}$.
2. $-^\mu$ commutes with the boolean connectives, $\Box$, $[d]$, $\forall$, and μ (cf. definition 3.4).
3. $(\langle t\rangle\Delta)^\mu = \nu q \bigwedge_{\delta\in\Delta} \Diamond(\delta^\mu \wedge q)$, where $q \in \mathsf{Var}$ does not occur in any δ^μ ($\delta \in \Delta$).
4. $(\langle dt\rangle\Delta)^\mu = \nu q \bigwedge_{\delta\in\Delta} \langle d\rangle(\delta^\mu \wedge q)$, where $q \in \mathsf{Var}$ does not occur in any δ^μ ($\delta \in \Delta$).

These formulas can be checked to be well formed. The translation simply replaces $\langle t\rangle$ by an expression using μ and $\Box$, and similarly for $\langle dt\rangle$. So if $\varphi \in \mathcal{L}^{\langle t\rangle}_{\Box}$ then $\varphi^\mu \in \mathcal{L}^{\mu}_{\Box}$, if $\varphi \in \mathcal{L}^{\langle dt\rangle}_{[d]}$ then $\varphi^\mu \in \mathcal{L}^{\mu}_{[d]}$, etc.

This translation is faithful in all relevant semantics:

LEMMA 4.2. *Let φ be any $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula. Then φ is equivalent to φ^μ in every transitive Kripke frame and in every topological space. (See section 2.12 for the definition of equivalence.)*

Proof. An easy induction on φ. We consider only the case $\langle t\rangle\Delta$ (for finite $\Delta \neq \emptyset$), in Kripke semantics (the case $\langle dt\rangle\Delta$ is of course identical). Assume the lemma for each $\delta \in \Delta$. Take any transitive Kripke model $\mathcal{M} = (W, R, h)$ and any $w \in W$. Inductively, $\mathcal{M}, w \models (\langle t\rangle\Delta)^\mu$ iff $\mathcal{M}, w \models \nu q \bigwedge_{\delta\in\Delta} \Diamond(\delta \wedge q)$. By the post-fixed point characterisation of greatest fixed points, this holds iff $(*)$ there is $S \subseteq W$ with $w \in S$ and such that for every $s \in S$ and $\delta \in \Delta$, there is $t \in S$ with sRt and $\mathcal{M}, t \models \delta$.

Assuming $(*)$, it is easy to choose a sequence $w = s_0Rs_1Rs_2\ldots$ in S by induction so that $\{n < \omega : \mathcal{M}, s_n \models \delta\}$ is infinite for every $\delta \in \Delta$. It follows that $\mathcal{M}, w \models \langle t\rangle\Delta$. Conversely, if $\mathcal{M}, w \models \langle t\rangle\Delta$ then there are worlds $w = w_0Rw_1Rw_2\ldots$ in W with $\{n < \omega : \mathcal{M}, w_n \models \delta\}$ infinite for every $\delta \in \Delta$. Let $S = \{w_n : n < \omega\}$. Then $w \in S$, and for each $w_n \in S$ and $\delta \in \Delta$, there is $m > n$ with $\mathcal{M}, w_m \models \delta$. Then $w_m \in S$, and by transitivity of R we have w_nRw_m. So $(*)$ holds. $\square$

4.2 Translating $\Box$ to $[d]$ and $\langle t\rangle$ to $\langle dt\rangle$

Just replacing $\Box$ by $[d]$ and $\langle t\rangle$ by $\langle dt\rangle$ in a formula $\varphi \in \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ yields an $\mathcal{L}^{\mu\langle dt\rangle}_{[d]\forall}$-formula equivalent to φ in all Kripke frames. But the two are not equivalent in topological spaces, so we seek a better translation that works in both semantics.

DEFINITION 4.3. For each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ, we define a $\mathcal{L}^{\mu\langle dt\rangle}_{[d]\forall}$-formula φ^d as follows:

1. $p^d = p$ for $p \in \mathsf{Var}$.
2. $-^d$ commutes with the boolean connectives, $[d]$, $\langle dt\rangle$, $\forall$, and μ.
3. $(\Box\varphi)^d = \varphi^d \wedge [d]\varphi^d$.
4. $(\langle t\rangle\Delta)^d = (\bigwedge \Delta^d) \vee \langle d\rangle(\bigwedge \Delta^d) \vee (\langle dt\rangle\Delta^d)$, where $\Delta^d = \{\delta^d : \delta \in \Delta\}$.

Again, φ^d is always well formed. The translation $-^d$ is pretty good:

LEMMA 4.4. *Each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ is equivalent to φ^d in every reflexive Kripke frame.*

Proof. An easy induction on φ. To show, e.g., that $\Box\varphi$ implies $(\Box\varphi)^d$, we need reflexivity. We also note that $\bigwedge\Delta$ and $\langle d\rangle\bigwedge\Delta$ both imply $\langle t\rangle\Delta$ in reflexive Kripke models. $\square$

LEMMA 4.5. *Each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ is equivalent to φ^d in every T1 topological space.*

Proof. Let X be a T1 topological space. We prove by induction on φ that each $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$-formula φ is equivalent to φ^d in X. We consider only two cases: $\Box\varphi$ and $\langle t\rangle\Delta$. Inductively assume the result for φ and each formula in the finite set Δ of formulas, let h be an assignment into X, and let $x \in X$. In the proof, we write '$x \models$ ' as short for '$(X, h), x \models$', and for a formula φ, we write $[\![\varphi]\!] = \{y \in X : y \models \varphi\}$.

We prove that $x \models \Box\varphi \leftrightarrow (\Box\varphi)^d$. We have $x \models \Box\varphi$ iff for some open neighbourhood O of x, we have $(X, h), y \models \varphi$ for every $y \in O$. This is plainly iff $x \models \varphi \wedge [d]\varphi$. Inductively, this is iff $x \models \varphi^d \wedge [d]\varphi^d$ — i.e., iff $x \models (\Box\varphi)^d$.

Now we prove that $x \models \langle t\rangle\Delta \leftrightarrow (\langle t\rangle\Delta)^d$. Recall that

$$(\langle t\rangle\Delta)^d = (\bigwedge \Delta^d) \vee \langle d\rangle(\bigwedge \Delta^d) \vee (\langle dt\rangle\Delta^d).$$

First we prove that $x \models (\langle t\rangle\Delta)^d \to \langle t\rangle\Delta$. Suppose that $x \models (\langle t\rangle\Delta)^d$. To show that $x \models \langle t\rangle\Delta$, we need to find $S \subseteq X$ with $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$. If $x \models \bigwedge \Delta^d$, take $S = \{x\}$. If $x \models \langle d\rangle \bigwedge \Delta^d$, take $S = \{x\} \cup [\![\bigwedge \Delta^d]\!]$. And if $x \models \langle dt\rangle\Delta^d$, there is $S \subseteq X$ with $x \in S \subseteq \bigcap_{\delta\in\Delta} \langle d\rangle([\![\delta]\!] \cap S)$; then $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$ as required.

It remains to prove that $x \models \langle t\rangle\Delta \to (\langle t\rangle\Delta)^d$. So suppose that $x \models \langle t\rangle\Delta$. If $x \models (\bigwedge \Delta^d) \vee \langle d\rangle(\bigwedge \Delta^d)$, we are done.

So suppose not. Thus, there is an open neighbourhood U of x with $y \models \neg \bigwedge \Delta^d$ for every $y \in U$. So for every $y \in U$, there is $\delta_y \in \Delta$ with $y \models \neg\delta_y^d$.

We prove that $x \models \langle dt\rangle\Delta^d$.

Since $x \models \langle t\rangle\Delta$, there is $S \subseteq X$ with $x \in S \subseteq \bigcap_{\delta\in\Delta} \mathrm{cl}([\![\delta]\!] \cap S)$.

Claim. Put $S' = U \cap S$. Then $x \in S' \subseteq \bigcap_{\delta\in\Delta} \langle d\rangle([\![\delta^d]\!] \cap S')$.

Proof of claim. Plainly, $x \in S'$. For the other half, let $y \in S'$ and $\delta \in \Delta$ be arbitrary; we show that $y \in \langle d\rangle([\![\delta^d]\!] \cap S')$. So let O be any open neighbourhood of y. Then $O \cap U$ is an open neighbourhood of y too. As $y \in S' \subseteq S \subseteq \mathrm{cl}([\![\delta_y]\!] \cap S)$, there is some $z \in O \cap U \cap S$ with $z \models \delta_y$. But $y \models \neg\delta_y^d$, so inductively, $y \models \neg\delta_y$. It follows that $z \neq y$.

As X is T1, $\{y\}$ is closed, so $O \cap U \setminus \{y\}$ is an open neighbourhood of z. Since $z \in S \subseteq \mathrm{cl}([\![\delta]\!] \cap S)$, there is $t \in O \cap U \cap S \setminus \{y\} = O \cap S' \setminus \{y\}$ with $t \models \delta$. Since O was arbitrary, this shows that $y \in \langle d\rangle([\![\delta]\!] \cap S')$. Since inductively, $[\![\delta]\!] = [\![\delta^d]\!]$, this proves the claim.

By definition of the semantics, the claim immediately yields $x \models \langle dt\rangle\Delta^d$ as required. This completes the induction and the proof. □

The reader may like to construct an alternative proof using the games described in remark 2.2. To see that the 'T1' hypothesis cannot be dropped, consider the 'indiscrete' space X with two points, x, y, with $\tau = \{\emptyset, X\}$. Let $p \in \mathsf{Var}$ and $h : \mathsf{Var} \to \wp X$ satisfy $h(p) = \{x\}$. Then $(X, h), x \models \langle t\rangle\{p, \neg p\}$, but $(X, h), x \not\models (p \wedge \neg p) \vee \langle d\rangle(p \wedge \neg p) \vee \langle dt\rangle\{p, \neg p\}$.

4.3 Translating μ to $\langle t\rangle$

We use this translation only to prove strong completeness for $\mathcal{L}^\mu_\Box$ in theorem 9.3(2). Fortunately, most of the hard work involved has already been done by others. We will need only the fact below, but its proof was a major enterprise.

FACT 4.6 (Dawar–Otto, [6, theorem 4.57(5)])**.** *For each formula φ of $\mathcal{L}^\mu_\Box$, there is a formula φ^t of $\mathcal{L}^{\langle t\rangle}_\Box$ that is equivalent to φ in every finite transitive Kripke frame.*

To lift this to topological spaces, we will use the proof theory from section 3.

COROLLARY 4.7. *Each $\mathcal{L}^\mu_\Box$-formula φ is equivalent to φ^t in every topological space.*

Proof. By fact 4.6 and lemma 4.2, $\varphi \leftrightarrow (\varphi^t)^\mu$ is an $\mathcal{L}^\mu_\Box$-formula valid in every finite transitive Kripke frame. By theorem 3.7, S4$\mu \vdash \varphi \leftrightarrow (\varphi^t)^\mu$.

Now it is easy to check that S4μ is sound over every topological space. (The S4 axioms are sound by definition of the topological semantics of $\Box$, and the fixed point axiom and rule are sound by the semantics of μ.) Hence, $\varphi \leftrightarrow (\varphi^t)^\mu$ is valid in every topological space. But by lemma 4.2, $(\varphi^t)^\mu$ is equivalent to φ^t in every topological space. We conclude that φ is equivalent to φ^t in every topological space, as required. □

By the corollary and lemma 4.2, $\mathcal{L}^{\mu}_{\Box}$ and $\mathcal{L}^{\langle t\rangle}_{\Box}$ uniformly have the same expressive power in every topological space.

Since $\Box$, $[d]$ and $\langle t\rangle$, $\langle dt\rangle$ are indistinguishable in Kripke semantics, a similar analysis would give a translation from $\mathcal{L}^{\mu}_{[d]}$ to $\mathcal{L}^{\langle dt\rangle}_{[d]}$ valid in every topological space. (For this purpose, the T axiom $\Box\varphi \to \varphi$ would be dropped in section 3, and the translation in definition 3.4 adapted to represent transitive closure.) The translation would show that $\mathcal{L}^{\mu}_{[d]}$ and $\mathcal{L}^{\langle dt\rangle}_{[d]}$ are equally expressive over all topological spaces. We could use it to lift weak completeness for $\mathcal{L}^{\mu}_{[d]}$ to strong completeness. Unfortunately, we do not have a weak completeness result for $\mathcal{L}^{\mu}_{[d]}$ to lift.

5 Finite model property

The main work of our paper starts here. In this section, we establish a number of finite model property results for sublanguages of $\mathcal{L}^{\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$, by modifying a filtration approach pioneered in the context of $\mathcal{L}_{[d]}$ by Shehtman [30] and used later by Lucero-Bryan for $\mathcal{L}_{[d]\forall}$ [22]. The finite model property for the systems $\mathrm{KD4G}_n$ (and others) was proved by Zakharyaschev [39], using canonical formulas. The finite model property for an S4-like tangle system was proved by Fernández-Duque in [10], by a different method, and the scheme **Fix** and a variant of **Ind** in section 5.3 below appear in [10, §3].

5.1 Clusters in Transitive Frames

We work within models on *K4 frames* (W, R), i.e. R is a transitive binary relation on W. If xRy, we may say that y comes *R-after* x, or is *R-later than* x, or is an *R-successor* of x. If $xR^{\bullet}y$, i.e. xRy but not yRx, then y is *strictly* after/later, or is a *proper* R-successor. A point x is *reflexive* if xRx, and *irreflexive* otherwise. R is (ir)reflexive on a set $X \subseteq W$ if every member of X is (ir)reflexive.

An *R-cluster* is a subset C of W that is an equivalence class under the equivalence relation

$$\{(x, y) : x = y \text{ or } xRyRx\}.$$

A cluster is *degenerate* if it is a singleton $\{x\}$ with x irreflexive. Note that a cluster C can only contain an irreflexive point if it is a singleton. For, if C has more than one element, then for each $x \in C$ there is some $y \in C$ with $x \neq y$, so $xRyRx$ and thus xRx by transitivity. On a non-degenerate cluster R is universal. For C to be non-degenerate it suffices that there exist $x, y \in C$ with xRy, regardless of whether $x = y$ or not.

Write C_x for the R-cluster containing x. Thus $C_x = \{x\} \cup \{y : xRyRx\}$. The relation R lifts to a well-defined *partial* ordering of clusters by putting C_xRC_y iff xRy. A cluster C is *R-maximal* when there is no cluster that comes strictly R-after it, i.e. when CRC' implies $C = C'$. A point $x \in W$ is *R-maximal,* or just *maximal* if R is understood, if C_x is a maximal cluster, or equivalently if xRy implies yRx.

An *R-chain* is a sequence $C_1, C_2, \ldots$ of pairwise distinct clusters with $C_1RC_2R\cdots$. In a finite frame, such a chain is of finite length. Hence we can define a notion of *rank* in a finite frame by declaring the rank of a cluster C to be the number of clusters in the longest chain of clusters starting with C. So the rank is always ≥ 1, and a rank-1 cluster is maximal. The rank of a point x is defined to be the rank of C_x. The key property of this notion is that if $xR^{\bullet}y$, equivalently if C_y comes strictly R-after C_x, then y has smaller rank than x.

An *endless R-path* is a sequence $\{x_n : n < \omega\}$ such that $x_n R x_{n+1}$ for all n. Such a path *starts at/from* x_0. The terms of the sequence need not be distinct: for instance, any reflexive point x gives rise to the endless R-path $xRxRxR\ldots$. In a finite frame, an endless path must eventually enter some *non-degenerate* cluster C and stay there, i.e. there is some n such that $x_m \in C$ for all $m \geq n$.

Recall that $R(x) = \{y \in W : xRy\}$ is the set of R-successors of x, and that (W', R') is an *inner* subframe of (W, R) if (W', R') is a subframe of (W, R) that is *R-closed.* This means that R' is the restriction of R to $W' \subseteq W$, and $x \in W'$ implies $R(x) \subseteq W'$. In this situation every R'-cluster is an R-cluster, and every R-cluster that intersects W' is a subset of W' and is an R'-cluster.

5.2 Syntax and Semantics

We will work initially in the language $\mathcal{L}_{\Box}^{\langle t \rangle}$. Recall that we assume a set Var of propositional variables, which may be finite or infinite. Formulas are constructed from these variables by the standard Boolean connectives, the unary modality $\Box$ (with dual $\Diamond$) and the *tangle* connective $\langle t \rangle$ which assigns a formula $\langle t \rangle\Gamma$ to each finite set Γ of formulas.

Later we will want to add additional connectives, such as the universal modality $\forall$ and its dual $\exists$.

We use the standard notion from section 2.7 of a Kripke model $\mathcal{M} = (W, R, h)$ on a (transitive) frame as given by a valuation function $h : \mathsf{Var} \to \wp W$, giving rise to a truth/satisfaction relation $\mathcal{M}, x \models \varphi$ with $\mathcal{M}, x \models p$ iff $x \in h(p)$ for all $p \in \mathsf{Var}$ and $x \in W$. The modality $\Diamond$ is modelled by R in the usual Kripkean way:

$$\mathcal{M}, x \models \Diamond\varphi \text{ iff there is a } y \text{ with } xRy \text{ and } y \models \varphi. \tag{5.1}$$

The condition for $\mathcal{M}, x \models \langle t \rangle\Gamma$ is that

> there exists an endless R-path $\{x_n : n < \omega\}$ with $x = x_0$ along which each member γ of Γ is true infinitely often, i.e. $\{n < \omega : \mathcal{M}, x_n \models \gamma\}$ is infinite.

A set Γ of formulas is *satisfied by the cluster C* if each member of Γ is true in $\mathcal{M}$ at some point of C. So Γ fails to be satisfied by C if some member of Γ is false at every point of C. In a *finite* model, since an endless path must eventually enter some non-degenerate cluster and stay there, we get that

$$x \models \langle t \rangle\Gamma \text{ iff there is a } y \text{ with } xRy \textbf{ and } \boldsymbol{yRy} \text{ and } \Gamma \text{ is satisfied by } C_y \tag{5.2}$$

To put this another way, $x \models \langle t \rangle\Gamma$ iff Γ is satisfied by some *non-degenerate* cluster following C_x.

Write $\langle t \rangle\varphi$ for the formula $\langle t \rangle\{\varphi\}$. Then $\langle t \rangle\varphi$ is true at x iff there is an endless path starting at x along which φ is true infinitely often. For finite models we have

$$x \models \langle t \rangle\varphi \text{ iff there is a } y \text{ with } xRy \text{ and } yRy \text{ and } y \models \varphi,$$

i.e. the meaning of $\langle t \rangle\varphi$ is that there is a *reflexive* alternative at which φ is true. Thus for finite *reflexive* models (i.e. S4 models) this reduces to the standard Kripkean interpretation (5.1) of $\Diamond$. More strongly, it is evident that $\langle t \rangle\varphi \leftrightarrow \Diamond\varphi$ is valid in all S4 frames (and $\langle t \rangle\varphi \to \Diamond\varphi$ is valid in all K4 frames).

Write $\Diamond^*\varphi$ for the formula $\varphi \vee \Diamond\varphi$, and $\Box^*\varphi$ for $\varphi \wedge \Box\varphi$. In any transitive frame, define $R^* = R \cup \{(x,x) : x \in W\}$. Then R^* is the reflexive-transitive closure of R, and in any model on the frame we have

$$\mathcal{M}, x \models \Box^*\varphi \text{ iff for all } y, \text{ if } xR^*y \text{ then } \mathcal{M}, y \models \varphi.$$

and

$$\mathcal{M}, x \models \Diamond^*\varphi \text{ iff for some } y,\ xR^*y \text{ and } \mathcal{M}, y \models \varphi.$$

Note that if $C_x = C_y$, then xR^*y. For each x let $R^*(x) = \{y \in W : xR^*y\}$. Then $R^*(x) = \{x\} \cup R(x)$.

5.3 Tangle Systems and Logics

A *tangle system* is any Hilbert system whose axioms include all tautologies and all instances of the schemes

K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

4: $\Diamond\Diamond\varphi \to \Diamond\varphi$

Fix: $\langle t\rangle\Gamma \to \Diamond(\gamma \wedge \langle t\rangle\Gamma)$, all $\gamma \in \Gamma$.

Ind: $\Box^*(\varphi \to \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge \varphi)) \to (\varphi \to \langle t\rangle\Gamma)$.

and whose rules include modus ponens and $\Box$-generalisation. The smallest tangle system will be denoted K4t.

A *tangle logic* (or just *logic* in this section) is a set L of formulas that is a K4t-logic. Any logic includes the following:

$\langle t\rangle\varphi \to \Diamond\varphi$

4$_*$: $\Diamond\Diamond^*\varphi \to \Diamond\varphi$

4$_t$: $\Diamond\langle t\rangle\Gamma \to \langle t\rangle\Gamma$

4_t will be explicitly needed in our finite model property proof, in relation to a condition called (r4). Here is a derivation of 4_t, in which the justification "Bool" means by principles of Boolean logic, "Reg" is the rule *from* $\varphi \to \psi$ *infer* $\Diamond\varphi \to \Diamond\psi$, and "Nec" is the rule *from* φ *infer* $\Box^*\varphi$.

For each $\gamma \in \Gamma$ we derive

1.	$\langle t\rangle\Gamma \to \Diamond(\gamma \wedge \langle t\rangle\Gamma)$	Fix
2.	$\Diamond(\Gamma \wedge \langle t\rangle\Gamma) \to \Diamond\langle t\rangle\Gamma$	K-theorem (Bool + Reg)
3.	$\langle t\rangle\Gamma \to \Diamond\langle t\rangle\Gamma$	1, 2 Bool
4.	$\gamma \wedge \langle t\rangle\Gamma \to \gamma \wedge \Diamond\langle t\rangle\Gamma$	3, Bool
5.	$\Diamond(\gamma \wedge \langle t\rangle\Gamma) \to \Diamond(\gamma \wedge \Diamond\langle t\rangle\Gamma)$	4, Reg
6.	$\langle t\rangle\Gamma \to \Diamond(\gamma \wedge \Diamond\langle t\rangle\Gamma)$	1, 5 Bool
7.	$\Diamond\langle t\rangle\Gamma \to \Diamond\Diamond(\gamma \wedge \Diamond\langle t\rangle\Gamma)$	6, Reg
8.	$\Diamond\langle t\rangle\Gamma \to \Diamond(\gamma \wedge \Diamond\langle t\rangle\Gamma)$	7, **Axiom 4**, Bool

Since this holds for every $\gamma \in \Gamma$ we can continue with

9.	$\Diamond\langle t\rangle\Gamma \to \bigwedge_{\gamma\in\Gamma}\Diamond(\gamma \wedge \Diamond\langle t\rangle\Gamma)$	8 for all $\gamma\in\Gamma$, Bool
10.	$\Box^*(\Diamond\langle t\rangle\Gamma \to \bigwedge_{\gamma\in\Gamma}\Diamond(\gamma \wedge \Diamond\langle t\rangle\Gamma))$	9, Nec
11.	$\Box^*(\Diamond\langle t\rangle\Gamma \to \bigwedge_{\gamma\in\Gamma}\Diamond(\gamma \wedge \Diamond\langle t\rangle\Gamma)) \to (\Diamond\langle t\rangle\Gamma \to \langle t\rangle\Gamma)$	Ind with $\varphi = \Diamond\langle t\rangle\Gamma$
12.	$\Diamond\langle t\rangle\Gamma \to \langle t\rangle\Gamma$	10, 11 Bool

5.4 Canonical Frame

For a tangle logic L, the canonical frame is $\mathcal{F}_L = (W_L, R_L)$, with W_L the set of maximally L-consistent sets of formulas, and xR_Ly iff $\{\Diamond\varphi : \varphi \in y\} \subseteq x$ iff $\{\varphi : \Box\varphi \in x\} \subseteq y$. R_L is transitive, by the K4 axiom 4.

Suppose $\mathcal{F} = (W, R)$ is an inner subframe of $\mathcal{F}_L$, i.e. W is an R_L-closed subset of W_L, and R is the restriction of R_L to W.

By standard canonical frame theory, we have that for all formulas φ and all $x \in W$:

$$\Diamond\varphi \in x \quad \text{iff} \quad \text{for some } y \in W,\ xRy \text{ and } \varphi \in y. \tag{5.3}$$

$$\Diamond^*\varphi \in x \quad \text{iff} \quad \text{for some } y \in W,\ xR^*y \text{ and } \varphi \in y. \tag{5.4}$$

$$\Box\varphi \in x \quad \text{iff} \quad \text{for all } y \in W,\ xRy \text{ implies } \varphi \in y. \tag{5.5}$$

$$\Box^*\varphi \in x \quad \text{iff} \quad \text{for all } y \in W,\ xR^*y \text{ implies } \varphi \in y. \tag{5.6}$$

We will say that a sequence $\{x_n : n < \omega\}$ in $\mathcal{F}$ *fulfils* the formula $\langle t\rangle\Gamma$ if each member of Γ belongs to x_n for infinitely many n. The role of the axiom Fix is to provide such sequences:

LEMMA 5.1. *In $\mathcal{F}$, if $\langle t\rangle\Gamma \in x$ then there is an endless R-path starting from x that fulfils $\langle t\rangle\Gamma$. Moreover, $\langle t\rangle\Gamma$ belongs to every member of this path.*

Proof. Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$. Put $x_0 = x$. From $\langle t\rangle\Gamma \in x_0$ by axiom Fix we get $\Diamond(\gamma_1 \wedge \langle t\rangle\Gamma) \in x_0$, so by (5.3) there exists $x_1 \in W$ with x_0Rx_1 and $\gamma_1, \langle t\rangle\Gamma \in x_1$. Since $\langle t\rangle\Gamma \in x_1$, by Fix again there exists $x_2 \in W$ with x_1Rx_2 and $\gamma_2, \langle t\rangle\Gamma \in x_2$. Continuing in this way ad infinitum cycling through the list $\gamma_1, \dots, \gamma_k$ we generate a sequence fulfilling $\langle t\rangle\Gamma$, with $\gamma_i \in x_n$ whenever $n \equiv i \mod k$, and $\langle t\rangle\Gamma \in x_n$ for all $n < \omega$. $\Box$

The canonical model $\mathcal{M}_L$ on $\mathcal{F}_L$ has $\mathcal{M}_L, x \models \varphi$ iff $\varphi \in x$, *provided that φ is $\langle t\rangle$-free.* But this 'Truth Lemma' can fail for formulas containing the tangle connective, even though all instances of the tangle axioms belong to every member of W_L. For this reason we will work directly with the structure of $\mathcal{F}_L$ and the relation $\varphi \in x$, rather than with truth in $\mathcal{M}_L$.

For an example of failure of the Truth Lemma, consider the set

$$\Sigma = \{p_0, q, \Box(p_{2n} \to \Diamond(p_{2n+1} \wedge \neg q)), \Box(p_{2n+1} \to \Diamond(p_{2n+2} \wedge q)) : n < \omega\},$$

where q and the p_n's are distinct variables. Each finite subset of $\Sigma \cup \{\neg\langle t\rangle\{q, \neg q\}\}$ is satisfiable in a transitive frame, and so is $L_{\mathrm{K4}t}$-consistent where $L_{\mathrm{K4}t}$ is the smallest logic. Explanation: if Γ is a finite subset, $\mathcal{M}$ a model with transitive frame, and $\mathcal{M}, x \models \Gamma$, then $\{\varphi : \mathcal{M}, y \models \varphi$ for all worlds y of $\mathcal{M}\}$ is a logic that excludes $\neg\bigwedge\Gamma$, so $\neg\bigwedge\Gamma \notin L_{\mathrm{K4}t}$.

Since the proof theory is finitary, it follows that $\Sigma \cup \{\neg\langle t\rangle\{q, \neg q\}\}$ is $L_{\mathrm{K4}t}$-consistent, so is included in some member x of $W_{L_{\mathrm{K4}t}}$. Using the fact that $\Sigma \subseteq x$, together with (5.3) and (5.5), we can construct an endless $R_{L_{\mathrm{K4}t}}$-path starting from x that fulfills $\{q, \neg q\}$, hence satisfies each of q and $\neg q$ infinitely often in $\mathcal{M}_{L_{\mathrm{K4}t}}$. Thus $\mathcal{M}_{L_{\mathrm{K4}t}}, x \models \langle t\rangle\{q, \neg q\}$. But $\langle t\rangle\{q, \neg q\} \notin x$, since $\neg\langle t\rangle\{q, \neg q\} \in x$ and x is $L_{\mathrm{K4}t}$-consistent.

5.5 Definable Reductions

Fix a finite set Φ of formulas closed under subformulas. Let Φ^t be the set of all formulas in Φ of the form $\langle t\rangle\Gamma$, and $\Phi^\Diamond$ be the set of all formulas in Φ of the form $\Diamond\varphi$.

Let $\mathcal{F} = (W, R)$ be an inner subframe of $\mathcal{F}_L$. Then by a *definable reduction of $\mathcal{F}$ via Φ* we mean a pair $(\mathcal{M}_\Phi, f)$, where $\mathcal{M}_\Phi = (W_\Phi, R_\Phi, h_\Phi)$ is a model on a finite transitive frame, and $f : W \to W_\Phi$ is a surjective function, such that the following hold for all $x, y \in W$:

(r1): $p \in x$ iff $f(x) \in h_\Phi(p)$, for all $p \in \mathsf{Var} \cap \Phi$.

(r2): $f(x) = f(y)$ implies $x \cap \Phi = y \cap \Phi$.

(r3): xRy implies $f(x)R_\Phi f(y)$.

(r4): $f(x)R_\Phi f(y)$ implies $y \cap \Phi^t \subseteq x \cap \Phi^t$ and $\{\Diamond\varphi \in \Phi : \Diamond^*\varphi \in y\} \subseteq x$.

(r5): For each subset C of W_Φ there is a formula φ that defines $f^{-1}(C)$ in W, i.e. $\varphi \in y$ iff $f(y) \in C$.

We will make crucial use of the following consequence of this definition.

LEMMA 5.2. *If $f(x)$ and $f(y)$ belong to the same R_Φ-cluster, then $x \cap \Phi^t = y \cap \Phi^t$ and $x \cap \Phi^\Diamond = y \cap \Phi^\Diamond$.*

Proof. If $f(x) = f(y)$, then $x \cap \Phi = y \cap \Phi$ by (r2) and so $x \cap \Phi^t = y \cap \Phi^t$ and $x \cap \Phi^\Diamond = y \cap \Phi^\Diamond$. But if $f(x) \neq f(y)$, then $f(x)R_\Phi f(y)R_\Phi f(x)$, and so $y \cap \Phi^t \subseteq x \cap \Phi^t \subseteq y \cap \Phi^t$ by (r4). Also if $\Diamond\varphi \in y \cap \Phi$ then $\Diamond^*\varphi = \varphi \vee \Diamond\varphi \in y$, and so $\Diamond\varphi \in x$ by (r4), and likewise $\Diamond\varphi \in x \cap \Phi$ implies $\Diamond\varphi \in y$. $\square$

Note that the second conclusion of (r4) is a concise way of expressing that both

$$\{\Diamond\varphi \in \Phi : \varphi \in y\} \subseteq x \quad \text{and} \quad \{\Diamond\varphi \in \Phi : \Diamond\varphi \in y\} \subseteq x.$$

Given a definable reduction $(\mathcal{M}_\Phi, f)$ of $\mathcal{F}$, we will replace R_Φ by a weaker relation R_t, producing a new model $\mathcal{M}_t = (W_\Phi, R_t, h_\Phi)$, the *untangling* of $\mathcal{M}_\Phi$, with the property that satisfaction in $\mathcal{M}_t$ of any formula $\varphi \in \Phi$ corresponds exactly via f to membership of φ in points of $\mathcal{F}$. In other words, $\varphi \in x$ iff $\mathcal{M}_t, f(x) \models \varphi$, a result we refer to as the *Reduction Lemma.* The definition of R_t will cause each R_Φ-cluster to be decomposed into a partially ordered set of smaller R_t-clusters.

In what follows we will write $|x|$ for $f(x)$. Then as f is surjective, each member of W_Φ is equal to $|x|$ for some $x \in W$. In later applications the set W_Φ will be a set of equivalence classes $|x|$ of points $x \in W$, under a suitable equivalence relation, and f will be the natural map $x \mapsto |x|$.

Our first step makes the key use of the axiom Ind:

LEMMA 5.3. *Let $\langle t\rangle\Gamma \in \Phi$. Suppose that $\langle t\rangle\Gamma \notin x$, where $x \in W$, and let $|x| \in C \subseteq W_\Phi$. Then there is a formula $\gamma \in \Gamma$ and some $y \in W$ such that xR^*y, $|y| \in C$ and*

$$\textit{if } yRz \textit{ and } |z| \in C, \textit{ then } \gamma \notin z. \tag{5.7}$$

Proof. By (r5) there is a formula φ that defines $\{y \in W : |y| \in C\}$, i.e. $\varphi \in y$ iff $|y| \in C$. Then $\varphi \in x$ and $\langle t\rangle\Gamma \notin x$, so by the axiom Ind, $\Box^*(\varphi \to \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge \varphi)) \notin x$. Hence by (5.6) there is a y with xR^*y and $(\varphi \to \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge \varphi)) \notin y$. Then $\varphi \in y$, so $|y| \in C$, and for some $\gamma \in \Gamma$ we have $\Diamond(\gamma \wedge \varphi) \notin y$. Hence by (5.3), if yRz and $|z| \in C$, then $\gamma \wedge \varphi \notin z$ and $\varphi \in z$, so $\gamma \notin z$, which gives (5.7). □

LEMMA 5.4. *Let formulas $\langle t\rangle\Gamma_1, \dots, \langle t\rangle\Gamma_k$ belong to Φ but not to x. Suppose that $|x| \in C \subseteq W_\Phi$. Then there are formulas $\gamma_1 \in \Gamma_1, \dots, \gamma_k \in \Gamma_k$ and some $y \in W$ such that xR^*y, $|y| \in C$ and*

$$\textit{if } yRz \textit{ and } |z| \in C, \textit{ then } \{\gamma_1, \dots, \gamma_k\} \cap z = \emptyset. \tag{5.8}$$

Proof. If $k = 0$, take $y = x$; we are done. Now assume $k > 0$. By Lemma 5.3, there exists $\gamma_1 \in \Gamma_1$ and $y_1 \in W$ such that xR^*y_1, $|y_1| \in C$ and

$$\text{if } y_1Rz \text{ and } |z| \in C, \text{ then } \gamma_1 \notin z. \tag{5.9}$$

Now $\langle t\rangle\Gamma_2 \notin x$, so $\Diamond\langle t\rangle\Gamma_2 \notin x$ by scheme 4_t. Hence $\Diamond^*\langle t\rangle\Gamma_2 = \langle t\rangle\Gamma_2 \vee \Diamond\langle t\rangle\Gamma_2 \notin x$. As xR^*y_1, this implies $\langle t\rangle\Gamma_2 \notin y_1$ by (5.4). So by Lemma 5.3 again, with y_1 in place of x, there exists $\gamma_2 \in \Gamma_2$ and $y_2 \in W$ such that $y_1R^*y_2$, $|y_2| \in C$ and

$$\text{if } y_2Rz \text{ and } |z| \in C, \text{ then } \gamma_2 \notin z. \tag{5.10}$$

Now by transitivity of R^* we have xR^*y_2. Also if y_2Rz and $|z| \in C$, then from $y_1R^*y_2Rz$ we get y_1Rz, and so $\gamma_1 \notin z$ by (5.9). Together with (5.10) this shows that $\{\gamma_1, \gamma_2\} \cap z = \emptyset$.

If $k = 2$ this proves (5.8) with $y = y_2$. Otherwise we repeat, applying Lemma 5.3 again with y_2 in place of x and so on, eventually obtaining the desired y as y_k. □

Define a formula $\varphi \in \Phi$ to be *realised* at a member $|z|$ of W_Φ iff $\varphi \in z$. Note that this definition does not depend on how the member is named, for if $|z| = |z'|$, then $z \cap \Phi = z' \cap \Phi$ by (r2), and so $\varphi \in z$ iff $\varphi \in z'$.

LEMMA 5.5. *Let C be any R_Φ-cluster. Then there is some $y \in W$ with $|y| \in C$, such that for any formula $\langle t\rangle\Gamma \in \Phi^t - y$ there is a formula in Γ that is not realised at any $|z| \in C$ such that yRz.*

Proof. Take any $|x| \in C$, and put $\Phi^t - x = \{\langle t\rangle\Gamma_1, \dots, \langle t\rangle\Gamma_k\}$. By Lemma 5.4 there is some y with xR^*y and $|y| \in C$, and formulas $\gamma_i \in \Gamma_i$ for $1 \le i \le k$ such that if yRz and $|z| \in C$, then $\gamma_i \notin z$, hence γ_i is not realised at $|z|$.

Now $|x|$ and $|y|$ belong to the same R_Φ-cluster C, so $y \cap \Phi^t = x \cap \Phi^t$ by Lemma 5.2. Hence $\Phi^t - y = \Phi^t - x$. So if $\langle t\rangle\Gamma \in \Phi^t - y$, then $\Gamma = \Gamma_i$ for some i, and then γ_i is a member of Γ not realised at any $|z| \in C$ such that yRz. □

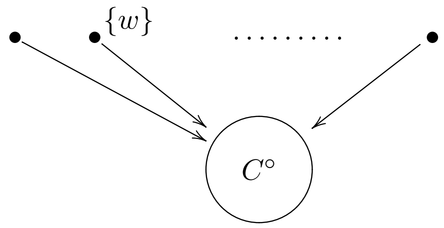
Now for each R_Φ-cluster C, choose and fix a point y as given by Lemma 5.5. Call y the *critical point for* C, and put

$$C^\circ = \{|z| \in C : yRz\}.$$

Lemma 5.5 states that if $\langle t\rangle\Gamma \in \Phi^t - y$, then there is a formula in Γ that is not realised at any point of C°.

We call C° the *nucleus* of the cluster C. If yRy then $|y| \in C^\circ$, but in general $|y|$ need not belong to C°. Indeed the nucleus could be empty. For instance, it must be empty when C is a degenerate cluster. To show this, suppose that $C^\circ \neq \emptyset$. Then there is some $|z| \in C$ with yRz, hence $|y|R_\Phi|z|$ by (r3), so as $|y| \in C$ this shows that C is *non*-degenerate. Consequently, if the nucleus is non-empty then the relation R_Φ is universal on it.

We introduce the subrelation R_t of R_Φ to refine the structure of C by decomposing it into the nucleus C° as an R_t-cluster together with a singleton *degenerate* R_t-cluster $\{w\}$ for each $w \in C - C^\circ$. These degenerate clusters all have C° as an R_t-successor but are incomparable with each other. So the structure replacing C looks like



with the black dots being the degenerate clusters determined by the points of $C - C^\circ$. Doing this to each cluster of (W_Φ, R_Φ) produces a new transitive frame $\mathcal{F}_t = (W_\Phi, R_t)$ with $R_t \subseteq R_\Phi$.

R_t can be more formally defined on W_Φ simply by specifying, for each $w, v \in W_\Phi$, that wR_tv iff $wR_\Phi v$ and either

- w and v belong to different R_Φ-clusters; or
- w and v belong to the same R_Φ-cluster C, and $v \in C^\circ$.

This ensures that each member of C is R_t-related to every member of the nucleus of C. The restriction of R_t to C is equal to $C \times C^\circ$, so we could also define R_t as the union of the relations $C \times C^\circ$ for all R_Φ-clusters C, plus all inter-cluster instances of R_Φ.

If the nucleus is empty, then so is the relation R_t on C, and C decomposes into a set of pairwise incomparable degenerate clusters. If $C = C^\circ$, then R_t is universal on C, identical to the restriction of R_Φ to C.

LEMMA 5.6 (Reduction lemma)**.** *Every formula in Φ is true in $\mathcal{M}_t = (W_\Phi, R_t, h_\Phi)$ precisely at the points at which it is realised, i.e. for all $\varphi \in \Phi$ and all $x \in W$,*

$$\mathcal{M}_t, |x| \models \varphi \text{ iff } \varphi \in x. \tag{5.11}$$

Proof. This is by induction on the formation of formulas. For the base case of a variable $p \in \Phi$, we have $\mathcal{M}_t, |x| \models p$ iff $|x| \in h_\Phi(p)$, which holds iff $p \in x$ by (r1). The inductive cases of the Boolean connectives are standard.

Next, take the case of a formula $\Diamond\varphi \in \Phi$, under the induction hypothesis that (5.11) holds for all $x \in W$. Suppose first that $\mathcal{M}_t, |x| \models \Diamond\varphi$. Then there is some $y \in W$ with $|x|R_t|y|$ and $\mathcal{M}_t, |y| \models \varphi$, hence $\varphi \in y$ by the induction hypothesis on φ. Then $\Diamond^*\varphi \in y$. But $R_t \subseteq R_\Phi$, so $|x|R_\Phi|y|$, implying that $\Diamond\varphi \in x$, as required, by (r4). Conversely, suppose that $\Diamond\varphi \in x$. Let C be the R_Φ-cluster of $|x|$, and y the critical point for C. Then $\Diamond\varphi \in y$ by Lemma 5.2, so there is some z with yRz and $\varphi \in z$, hence $\mathcal{M}_t, |z| \models \varphi$ by induction hypothesis. Now if $|z| \in C$, then $|z|$ belongs to the nucleus of C and hence $|x|R_t|z|$. But if $|z| \notin C$, then as $|y|R_\Phi|z|$ by (r3), and hence $|x|R_\Phi|z|$, the R_Φ-cluster of $|z|$ is strictly R_Φ-later than C, and again $|x|R_t|z|$. So in any case we have $|x|R_t|z|$ and $\mathcal{M}_t, |z| \models \varphi$, giving $\mathcal{M}_t, |x| \models \Diamond\varphi$. That completes this inductive case of $\Diamond\varphi$.

Finally we have the most intricate case of a formula $\langle t\rangle\Gamma \in \Phi$, under the induction hypothesis that (5.11) holds for every member of Γ for all $x \in W$. Then we have to show that for all $z \in W$,

$$\mathcal{M}_t, |z| \models \langle t\rangle\Gamma \text{ iff } \langle t\rangle\Gamma \in z. \tag{5.12}$$

The proof proceeds by strong induction on the *rank* of $|z|$. Take $x \in W$ and suppose that (5.12) holds for every z for which the rank of $|z|$ is *less than* the rank of $|x|$. We show that $\mathcal{M}_t, |x| \models \langle t\rangle\Gamma$ iff $\langle t\rangle\Gamma \in x$. Let C be the R_Φ-cluster of $|x|$, and y the critical point for C.

Assume first that $\langle t\rangle\Gamma \in x$. Then $\langle t\rangle\Gamma \in y$ by Lemma 5.2. By Lemma 5.1, there is an endless R-path $\{y_n : n < \omega\}$ starting from $y = y_0$ that fulfills $\langle t\rangle\Gamma$ and has $\langle t\rangle\Gamma$ belonging to each point. Then by (r3) the sequence $\{|y_n| : n < \omega\}$ is an endless R_Φ-path in W_Φ starting at $|y| \in C$.

Suppose that $|y_n| \in C$ for all n. Then for all $n > 0$, since yRy_n we get $|y_n| \in C^\circ$. So there is the endless R_t-path $\pi = |x|R_t|y_1|R_t|y_2|R_t\cdots$ starting at $|x|$. As $\{y_n : n < \omega\}$ fulfills $\langle t\rangle\Gamma$, for each $\gamma \in \Gamma$ there are infinitely many n for which $\gamma \in y_n$ and so $\mathcal{M}_t, |y_n| \models \gamma$ by the induction hypothesis on members of Γ. Thus each member of Γ is true infinitely often along π, implying that $\mathcal{M}_t, |x| \models \langle t\rangle\Gamma$.

If however there is an $n > 0$ with $|y_n| \notin C$, then the R_Φ-cluster of $|y_n|$ is strictly R_Φ-later than C, so $|x|R_t|y_n|$ and $|y_n|$ has smaller rank than $|x|$. Since $\langle t\rangle\Gamma \in y_n$, the induction hypothesis (5.12) on rank then implies that $\mathcal{M}_t, |y_n| \models \langle t\rangle\Gamma$. So there is an endless R_t-path π from $|y_n|$ along which each member of Γ is true infinitely often. Since $|x|R_t|y_n|$, we can append $|x|$ to the front of π to obtain such an R_t-path starting from $|x|$, showing that $\mathcal{M}_t, |x| \models \langle t\rangle\Gamma$ (this last part is an argument for soundness of 4_t). So in both cases we get $\mathcal{M}_t, |x| \models \langle t\rangle\Gamma$. That proves the forward implication of (5.11) for $\langle t\rangle\Gamma$.

For the converse implication, suppose $\mathcal{M}_t, |x| \models \langle t\rangle\Gamma$. Since W_Φ is finite, it follows by (5.2) that there exists a $z \in W$ with $|x|R_t|z|$ and $|z|R_t|z|$ and the R_t-cluster of $|z|$ satisfies Γ. By the induction hypothesis (5.11) on members of Γ, every formula in Γ is realised at some point of this cluster. Suppose first there is such a z for which the rank of $|z|$ is less than that of $|x|$. Then as the R_t-cluster of $|z|$ is non-degenerate and satisfies Γ, we have $\mathcal{M}_t, |z| \models \langle t\rangle\Gamma$. Induction hypothesis (5.12) then implies that $\langle t\rangle\Gamma \in z$. But $|x|R_\Phi|z|$, as $|x|R_t|z|$, so by (r4) we get the required conclusion that $\langle t\rangle\Gamma \in x$.

If however there is no such z with $|z|$ of lower rank than $|x|$, then the $|z|$ that does exist must have the same rank as $|x|$, so it belongs to C. Hence as $|x|R_t|z|$, the definition of R_t implies that $|z| \in C^\circ$. Thus the R_t-cluster of $|z|$ is C°. Therefore every formula in Γ is realised at some point of C°, i.e. at some $|z'| \in C$ with yRz'. But Lemma 5.5 states that if $\langle t\rangle\Gamma \notin y$, then some member of Γ is not realised in C°. Therefore we must have $\langle t\rangle\Gamma \in y$. Then $\langle t\rangle\Gamma \in x$ as required, by Lemma 5.2. That finishes the inductive proof that $\mathcal{M}_t$ satisfies the Reduction Lemma. □

5.6 Adding Seriality

Suppose the logic L contains the D-axiom $\Diamond\top$. Then R_L is *serial*: $\forall x\exists y(xR_Ly)$. Hence the relation R of the inner subframe $\mathcal{F}$ is serial. From this we can show that R_t is serial. The key point is that any maximal R_Φ-cluster C must have a *non-empty* nucleus. For, if y is the critical point for C, then there is a z with yRz, as R is serial. But then $|y|R_\Phi|z|$ by (r3) and so $|z| \in C$ as C is maximal. Hence $|z| \in C^\circ$, making the nucleus non-empty. Now every member of C is R_t-related to any member of C° so altogether this implies that R_t is serial on the rank 1 cluster C. But any point of rank > 1 will be R_t-related to points of lower rank, and indeed to points in the nucleus of some rank 1 cluster. Since R_t is reflexive on a nucleus, this shows that R_t satisfies the stronger condition that $\forall w \exists v(wR_tvR_tv)$ — "every world sees a reflexive world".

5.7 Adding Reflexivity

Suppose that L contains the scheme

T: $\varphi \to \Diamond\varphi$.

Then it contains

T$_t$**:** $\bigwedge\Gamma \to \langle t\rangle\Gamma$.

To see this, let $\varphi = \bigwedge\Gamma$. Then $\varphi \to \bigwedge_{\gamma\in\Gamma}(\gamma \wedge \varphi)$ is a tautology, hence derivable. From that we derive

$$\Box^*(\varphi \to \bigwedge\nolimits_{\gamma\in\Gamma}\Diamond(\gamma \wedge \varphi)) \tag{5.13}$$

using the instances $(\gamma\wedge\varphi) \to \Diamond(\gamma\wedge\varphi)$ of axiom T and K-principles. But (5.13) is an antecedent of axiom Ind, so we apply it to derive $\varphi \to \langle t\rangle\Gamma$, which is T_t in this case.

Axiom T ensures that the canonical frame relation R_L is reflexive, and hence so is R_Φ by (r3). Thus no R_Φ-cluster is degenerate. We modify the definition of R_t to make it reflexive as well. The change occurs in the case of an R_Φ-cluster C having $C \neq C^\circ$. Then instead of making the singletons $\{w\}$ for $w \in C - C^\circ$ be degenerate, we make them all into *non-R_t-degenerate* clusters by requiring that wR_tw. Formally this is done by adding to the definition of wR_tv the third possibility that

- w and v belong to the same R_Φ-cluster C, and $w = v \in C - C^\circ$.

Equivalently, the restriction of R_t to C is equal to $(C \times C^\circ) \cup \{(w, w) : w \in C - C^\circ\}$.

The proof of the Reduction Lemma for the resulting reflexive and transitive model $\mathcal{M}_t$ now requires an adjustment in one place, in its last paragraph, where $|x|R_t|z| \in C$. In the original proof above, this implied that the R_t-cluster of $|z|$ is C°. But now we have the new possibility that $|x| = |z| \in C - C^\circ$. Then the R_t-cluster of $|z|$ is $\{|z|\}$, so every formula of $\mathcal{F}$ is realised at $|z|$, implying $\bigwedge\Gamma \in z$. The scheme T_t now ensures that $\langle t\rangle\Gamma \in z$, so by Lemma 5.2 we still get the required result that $\langle t\rangle\Gamma \in x$, and the Reduction Lemma still holds for this modified reflexive version of $\mathcal{M}_t$.

5.8 Finite model property over K4, KD4 and S4

Given a logic L and a finite set Φ of formulas closed under subformulas, we can construct a definable reduction of any inner subframe $\mathcal{F} = (W, R)$ of $\mathcal{F}_L$ by filtration through Φ. An equivalence relation $\sim$ on W is given by putting $x \sim y$ iff $x \cap \Phi = y \cap \Phi$. Then with $|x| = \{y \in W : x \sim y\}$ we put $W_\Phi = \{|x| : x \in W\}$.

Letting $R_\lambda = \{(|x|, |y|) : xRy\}$ (the least filtration of R through Φ), we define $R_\Phi \subseteq W_\Phi \times W_\Phi$ to be the *transitive closure* of R_λ. Thus $wR_\Phi v$ iff there exist $w_1, \dots, w_n \in W_\Phi$, for some $n > 1$, such that $w = w_1R_\lambda \cdots R_\lambda w_n = v$. The definition of $\mathcal{M}_\Phi$ is completed by putting $h_\Phi(p) = \{|x| : p \in x\}$ for $p \in \Phi$, and $h_\Phi(p) = \emptyset$ (or anything) otherwise. We call $\mathcal{M}_\Phi$ the *standard transitive filtration through* Φ.

The surjective function $f : W \to W_\Phi$ is given by $f(x) = |x|$. The conditions (r1) and (r2) for a definable reduction are then immediate, and the definability condition (r5) is standard. For (r3) observe that xRy implies $|x|R_\lambda|y|$ and hence $|x|R_\Phi|y|$.

(r4) takes more work, but is also standard for the case of $\Diamond$, and similar for $\langle t\rangle$. To prove it, let $|x|R_\Phi|y|$. Then by definition of R_Φ as the transitive closure of R_λ, there are finitely many elements $x_1, y_1, \dots, x_n, y_n$ of W (for some $n \geq 1$) such that

$$x \sim x_1Ry_1 \sim x_2Ry_2 \sim \cdots \sim x_nRy_n \sim y.$$

Then $\langle t\rangle\Gamma \in y \cap \Phi^t$ implies $\langle t\rangle\Gamma \in y_n$ as $y_n \sim y$, hence $\Diamond\langle t\rangle\Gamma \in x_n$ as x_nRy, which implies $\langle t\rangle\Gamma \in x_n$ by the scheme 4_t. If $n = 1$ we then get $\langle t\rangle\Gamma \in x$ because $x \sim x_1$. But if $n > 1$, we repeat this argument back along the above chain of relations, leading to $\langle t\rangle\Gamma \in x_{n-1}$, $\dots$,$\langle t\rangle\Gamma \in x_1$, and then $\langle t\rangle\Gamma \in x$ as required to conclude that $y \cap \Phi^t \subseteq x \cap \Phi^t$.

To show that $\{\Diamond\varphi \in \Phi : \Diamond^*\varphi \in y\} \subseteq x$, note that if $\Diamond^*\varphi \in y$, then either $\varphi \in y$ or $\Diamond\varphi \in y$. If $\varphi \in y$, then $\varphi \in y_n$ as $y_n \sim y$ and $\varphi \in \Phi$, hence $\Diamond\varphi \in x_n$ as x_nRy_n. But if $\Diamond\varphi \in y$ then $\Diamond\varphi \in y_n$, hence $\Diamond\Diamond\varphi \in x_n$, and so again $\Diamond\varphi \in x_n$, this time by scheme 4. Repeating this back along the chain leads to $\Diamond\varphi \in x$ as required.

Thus $(\mathcal{M}_\Phi, f)$ as defined is a definable reduction of $\mathcal{F}$.

From this we can obtain a proof that the the smallest tangle system K4t has the finite model property over transitive frames. If $L_{\mathrm{K4}t}$ is its set of theorems, put $\mathcal{F} = \mathcal{F}_{L_{\mathrm{K4}t}}$. If φ is a K4t-consistent formula then $\varphi \in x$ for some point x of $\mathcal{F}$. Let Φ be the set of subformulas of φ, and $\mathcal{M}_t$ the model derived from the model $\mathcal{M}_\Phi$ just defined. Then $\mathcal{M}_t, |x| \models \varphi$ by the Reduction Lemma. But the finite frame $\mathcal{F}_t = (W_\Phi, R_t)$ is transitive, so K4t has the finite model property over transitive frames, i.e. K4 frames.

If we replace K4t here by the smallest tangle system KD4t containing $\Diamond\top$, then the frame $\mathcal{F}_t$ of the last paragraph is serial, so $\{\psi : \mathcal{F}_t \models \psi\}$ is then a logic that contains $\Diamond\top$, hence includes $L_{\mathrm{KD4}t}$. Thus KD4t has the finite model property over serial transitive (i.e. KD4) frames.

Similarly, since $\mathcal{M}_t$ is reflexive when L contains the scheme T, we get that the smallest tangle system S4t containing T has the finite model property over reflexive transitive (i.e. S4) frames.

5.9 Universal Modality

Extend the syntax to include the universal modality $\forall$ with semantics $\mathcal{M}, x \models \forall\varphi$ iff for all y, $\mathcal{M}, y \models \varphi$. Let K4$t$.U be the smallest tangle system that includes the S5 axioms and rules for $\forall$, and the scheme

U: $\forall\varphi \to \Box\varphi$,

equivalently $\Diamond\varphi \to \exists\varphi$, where $\exists = \neg\forall\neg$ is the dual modality to $\forall$.

Let L be any K4t.U-logic. Define a relation S_L on W_L by: xS_Ly iff $\{\varphi : \forall\varphi \in x\} \subseteq y$ iff $\{\exists\varphi : \varphi \in y\} \subseteq x$. Then S_L is an equivalence relation with $R_L \subseteq S_L$. Also

$$\forall\varphi \in x \text{ iff for all } y \in W_L,\ xS_Ly \text{ implies } \varphi \in y.$$

For any fixed $x \in W_L$, let W^x be the equivalence class $S_L(x) = \{y \in W_L : xS_Ly\}$. Then for $z \in W^x$,

$$\forall\varphi \in z \text{ iff for all } y \in W^x,\ \varphi \in y. \tag{5.14}$$

Let R^x be the restriction of R_L to W^x. Since $R_L \subseteq S_L$ it follows that $\mathcal{F}^x = (W^x, R^x)$ is an inner subframe of (W_L, R_L). If $\mathcal{M}_\Phi$ is a definable reduction of $\mathcal{F}^x$, and $\mathcal{M}_t$ its untangling, then using (5.14) it can be shown that if a formula $\varphi \in \Phi$ satisfies the Reduction Lemma

$$\mathcal{M}_t, |z| \models \varphi \text{ iff } \varphi \in z$$

for all z in $\mathcal{M}_t$, then so does $\forall\varphi$. So the Reduction Lemma holds for all members of Φ.

Now the standard transitive filtration can be applied to $\mathcal{F}^x$ to produce a definable reduction of it. Consequently, if φ is an L-consistent formula, x is a point of W_L with $\varphi \in x$, and Φ is the set of all subformulas of φ, then $\mathcal{M}_t, |x| \models \varphi$ where $\mathcal{M}_t$ is the untangling of the standard transitive filtration of $\mathcal{F}^x$ through Φ. That establishes the finite model property for K4t.U over transitive frames.

This construction preserves seriality and reflexiveness in passing from R_L to R^x and then R_t. The outcome is that the finite model property continues to hold for the tangle systems KD4t.U and S4t.U over the KD4 and S4 frames, respectively.

5.10 Path Connectedness

A *connecting path between w and v* in a frame (W, R) is a finite sequence $w = w_0, \dots, w_n = v$, for some $n \geq 0$, such that for all $i < n$, either $w_i R w_{i+1}$ or $w_{i+1} R w_i$. We say that such a path has *length n*. The points w and v of W are *path connected* if there exists a connecting path between them of some finite length. Note that any point w is connected to itself by a path of length 0 (put $n = 0$ and $w = w_0$). The relation "w and v are path connected" is an equivalence relation whose equivalence classes are the *path components* of the frame. The frame is *path connected* if it has a single path component, i.e. any two points have a connecting path between them. This is iff the frame is connected in the sense of section 2.2.

Later we will make use of the fact that a path component P is R-closed. For if $x \in P$ and xRy, then x and y are path connected, so $y \in P$. It follows that any R-cluster C that intersects P must be included in P, for if $x \in P \cap C$ and $y \in C$, then xR^*y and so $y \in P$, showing that $C \subseteq P$.

We now wish to show that in passing from the frame $\mathcal{F}_\Phi = (W_\Phi, R_\Phi)$ to its untangling $\mathcal{F}_t$, there is no loss of path connectivity. The two frames have the same path connectedness relation and so have the same path components. The idea is that the relations that are broken by the untangling only occur between elements of the same R_Φ-cluster, so it suffices to show that such elements are still path connected in $\mathcal{F}_t$. *For this we need to make the assumption that Φ contains the formula $\Diamond\top$.* This is harmless as we can always add it and its subformula $\top$, preserving finiteness of Φ.

LEMMA 5.7. *Let $\Diamond\top \in \Phi$. If w, w' are points in W_Φ with $wR_\Phi w'$ or $w'R_\Phi w$, but neither $wR_t w'$ or $w'R_t w$, then there exist a v with $wR_t v$ and $w'R_t v$.*

Proof. If $wR_\Phi w'$, then since not $wR_t w'$ we must have w and w' in the same cluster. The same follows if $w'R_\Phi w$, since not $w'R_t w$.

Thus there is an R_Φ-cluster C with $w, w' \in C$, so both $wR_\Phi w'$ and $w'R_\Phi w$. If C is not R_Φ-maximal, then there is an R_Φ-cluster C' with $CR_\Phi C'$ and $C \neq C'$. Taking any $v \in C'$ we then get $wR_t v$ and $w'R_t v$.

The alternative is that C is R_Φ-maximal. Then we show that the nucleus C° is non-empty. Let $w = |u|$ and $w' = |t|$. Since $|u|R_\Phi|t|$ and $\top \in t$, and $\Diamond\top \in \Phi$, property (r4) implies that $\Diamond\top \in u$. Now if y is the critical point for C, then $\Diamond\top \in y$ by Lemma 5.2. Hence there is a z with yRz. So $|y|R_\Phi|z|$ by (r3). Maximality of C then ensures that $|z| \in C$, so this implies that $|z| \in C^\circ$. Then by definition of R_t, since $w, w' \in C$ we have $wR_t|z|$ and $w'R_t|z|$. □

LEMMA 5.8. *If $\Diamond\top \in \Phi$, then two members of W_Φ are path connected in $\mathcal{F}_\Phi$ if, and only if, they are path connected in $\mathcal{F}_t$. Hence the two frames have the same path components.*

Proof. Since $R_t \subseteq R_\Phi$, a connecting path in $\mathcal{F}_t$ is a connecting path in $\mathcal{F}_\Phi$, so points that are path connected in $\mathcal{F}_t$ are path connected in $\mathcal{F}_\Phi$.

Conversely, let $\pi = w_0, \dots, w_n$ be a connecting path in $\mathcal{F}_\Phi$. If, for all $i < n$, either $w_i R_t w_{i+1}$ or $w_{i+1} R_t w_i$, then π is a connecting path in $\mathcal{F}_t$. If not, then for each i for which this fails, by Lemma 5.7 there exists some v_i with $w_i R_t v_i$ and $w_{i+1} R_t v_i$. Insert v_i between w_i and w_{i+1} in the path. Doing this for all "defective" $i < n$, creates a new sequence that is now a connecting path in $\mathcal{F}_t$ between the same endpoints. □

Now let K4t.UC be the smallest extension of system K4t.U in the language with $\forall$ that includes the scheme

C: $\forall(\Box^*\varphi \vee \Box^*\neg\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$,

or equivalently $\exists\varphi \wedge \exists\neg\varphi \to \exists(\Diamond^*\varphi \wedge \Diamond^*\neg\varphi)$.

Let L be any K4t.UC-logic. Let $\mathcal{F}^x$ be a point-generated subframe of (W_L, R_L) as above, and $\mathcal{M}_\Phi$ its standard transitive filtration through Φ. Then the frame $\mathcal{F}_\Phi = (W_\Phi, R_\Phi)$ of $\mathcal{M}_\Phi$ is path connected, as shown by Shehtman [31] as follows. If P is the path component of $|x|$ in $\mathcal{M}_\Phi$, take a formula φ that defines $f^{-1}(P)$ in W^x, i.e. $\varphi \in y$ iff $|y| \in P$, for all $y \in W^x$. Suppose, for the sake of contradiction, that $P \neq W_\Phi$. Then there is some $z \in W^x$ with $|z| \notin P$, hence $\neg\varphi \in z$. Since $\varphi \in x$, this gives $\exists\varphi \wedge \exists\neg\varphi \in x$. By the scheme C it follows that for some $y \in W^x$, $\Diamond^*\varphi \wedge \Diamond^*\neg\varphi \in y$. Hence there are $z, w \in W^x$ with yR^*z, $\varphi \in z$, yR^*w and $\neg\varphi \in w$.

From this we get $|y|{R_\Phi}^*|z|$ and $|y|{R_\Phi}^*|w|$ so the sequence $|z|, |y|, |w|$ is a connecting path between $|z|$ and $|w|$ in $\mathcal{F}_\Phi$. But $|z| \in P$ as $\varphi \in z$, so this implies $|w| \in P$. Hence $\varphi \in w$, contradicting the fact that $\neg\varphi \in w$. The contradiction forces us to conclude that $P = W_\Phi$, and hence that $\mathcal{F}_\Phi$ is path connected.

From Lemma 5.8 it now follows that the untangling $\mathcal{F}_t$ of $\mathcal{F}_\Phi$ is also path connected when L includes scheme C and $\Diamond\top \in \Phi$. Hence the finite model property holds for K4t.UC over path-connected transitive frames.

The arguments for the preservation of seriality and reflexiveness by $\mathcal{F}_t$ continue to hold here. This gives us proofs of the finite model property for the systems, KD4t.UC and S4t.UC over path-connected KD4 and S4 frames, respectively.

Note that for the $\mathcal{L}_{\Box\forall}$-fragments of these logics (i.e. their restrictions to the language without $\langle t\rangle$), our analysis reconstructs the finite model property proof of [31] by using $\mathcal{M}_\Phi$ instead of $\mathcal{M}_t$. For, restricting to this language, if $\mathcal{M}_\Phi$ is a standard transitive filtration of an inner subframe of $\mathcal{F}_L$, then any $\langle t\rangle$-free formula is true in $\mathcal{M}_\Phi$ precisely at the points at which it is realised (for $\mathcal{L}_\Box$ this is a classical result first formulated and proved in [29]). Thus a finite satisfying model for a consistent $\mathcal{L}_{\Box\forall}$-formula can be obtained as a model of this form $\mathcal{M}_\Phi$. Since seriality and reflexivity are preserved in passing from R_L to R_Φ, and $\mathcal{F}_\Phi$ is path connected in the presence of axiom C, it follows that the finite model property holds for each of the systems K4.UC, KD4.UC and S4.UC in the language $\mathcal{L}_{\Box\forall}$.

5.11 The Schemes G_n

Fix $n \geq 1$ and take $n+1$ variables $p_0, \ldots, p_n$. For each $i \leq n$, define the formula

$$Q_i = p_i \wedge \bigwedge_{i \neq j \leq n} \neg p_j. \tag{5.15}$$

G_n is the scheme consisting of all uniform substitution instances of the formula

$$\bigwedge_{i \leq n} \Diamond Q_i \to \Diamond(\bigwedge_{i \leq n} \Diamond^* \neg Q_i). \tag{5.16}$$

This is equivalent in any logic to

$$\Box(\bigvee_{i \leq n} \Box^* Q_i) \to \bigvee_{i \leq n} \Box \neg Q_i,$$

the form in which the G_n's were introduced in [30]. When $n = 1$, (5.16) is

$$\Diamond(p_0 \wedge \neg p_1) \wedge \Diamond(p_1 \wedge \neg p_0) \to \Diamond(\Diamond^* \neg(p_0 \wedge \neg p_1) \wedge \Diamond^* \neg(p_1 \wedge \neg p_0)). \tag{5.17}$$

As an axiom, (5.17) is equivalent to

$$\Diamond p \wedge \Diamond \neg p \to \Diamond(\Diamond^* p \wedge \Diamond^* \neg p), \tag{5.18}$$

or in dual form $\Box(\Box^* p \vee \Box^* \neg p) \to \Box p \vee \Box \neg p$, which is the form in which G_1 was first defined in [30]. To derive (5.18) from (5.17), substitute p for p_0 and $\neg p$ for p_1 in (5.17). Conversely, substituting $p_0 \wedge \neg p_1$ for p in (5.18) leads to a derivation of (5.17).

For the semantics of G_n, we use the set $R(x) = \{y \in W : xRy\}$ of R-successors of x in a frame (W, R). We can view $R(x)$ as a frame in its own right, under the restriction of R to $R(x)$, and consider whether it is path connected, or how many path components it has etc. (W, R) is called *locally n-connected* if, for all $x \in W$, the frame $\mathcal{F}(x) = (R(x), R{\restriction}R(x))$ has at most n path components. This is equivalent to the definition in section 2.2. Note that path components in $\mathcal{F}(x)$ are defined by connecting paths in (W, R) that lie entirely within $R(x)$.

FACT 5.9. *A K4 frame validates G_n iff it is locally n-connected.*

For a proof of this see [22, Theorem 3.7].

5.12 Weak Models

We now assume that the set Var of variables is *finite*. The adjective "weak" is sometimes applied to languages with finitely many variables, as well as to models for weak languages and to canonical frames built from them. Weak models may enjoy special properties. For instance, a proof is given in [30, Lemma 8] that in a weak *distinguished*[2] model on a transitive frame, there are only finitely many maximal clusters. This was used to show that a weak canonical model for the $\mathcal{L}_\Box$-system $\mathrm{K4DG}_1$ is locally 1-connected, and from this to obtain the finite model property for that system. The corresponding versions of these results for $\mathrm{K4DG}_n$ with $n \geq 2$ are worked out in [22].

[2]A model is distinguished if for any two of its distinct points there is a formula that is true in the model at one of the points and not the other.

We wish to lift these results to the language $\mathcal{L}_{\Box}^{\langle t\rangle}$ with tangle. One issue is that the property of a canonical model being distinguished depends on it satisfying the Truth Lemma: $\mathcal{M}_L, x \models \varphi$ iff $\varphi \in x$. As we have seen, this fails for tangle logics. Therefore we must continue to work directly with the relation of membership of formulas in points of W_L, rather than with their truth in $\mathcal{M}_L$. We will see that it is still possible to recover Shehtman's analysis of maximal clusters in $\mathcal{F}_L$, with the aid of both tangle axioms.

Another issue is that we want to work over $\mathrm{K4G}_n$ without assuming the seriality axiom. This requires further adjustments, and care with the distinction between R and R^*.

Let L be any tangle logic in our weak language. Put $\mathsf{At} = \mathsf{Var} \cup \{\Diamond\top\}$. For each $s \subseteq \mathsf{At}$ define the formula

$$\chi(s) = \bigwedge_{\varphi \in s} \varphi \wedge \bigwedge_{\varphi \in \mathsf{At} \setminus s} \neg\varphi.$$

For each point x of W_L define $\tau(x) = x \cap \mathsf{At}$. Think of At as a set of "atoms" and $\tau(x)$ as the "atomic type" of x. It is evident that for any $x \in W_L$ and $s \subseteq \mathsf{At}$ we have

$$\chi(s) \in x \text{ iff } s = \tau(x). \tag{5.19}$$

Writing $\chi(x)$ for the formula $\chi(\tau(x))$, we see from (5.19) that $\chi(x) \in x$, and in general $\chi(y) \in x$ iff $\tau(y) = \tau(x)$.

Now fix an inner subframe $\mathcal{F} = (W, R)$ of $\mathcal{F}_L$. If C is an R-cluster in $\mathcal{F}$, let

$$\delta C = \{\tau(x) : x \in C\}$$

be the set of atomic types of members of C. We are going to show that maximal clusters in $\mathcal{F}$ are determined by their atomic types. They key to this is:

LEMMA 5.10. *Let C and C' be maximal clusters in $\mathcal{F}$ with $\delta C = \delta C'$. Then for all formulas φ, if $x \in C$ and $x' \in C'$ have $\tau(x) = \tau(x')$, then $\varphi \in x$ iff $\varphi \in x'$. Thus, $x = x'$.*

Proof. Suppose C and C' are maximal with $\delta C = \delta C'$. The key property of maximality that is used is that if $x \in C$ and xRy, then $y \in C$, and likewise for C'.

The proof proceeds by induction on the formation of φ. The base case, when $\varphi \in \mathsf{Var}$, is immediate from the fact that then $\varphi \in x$ iff $\varphi \in \tau(x)$. The induction cases for the Boolean connectives are straightforward from properties of maximally consistent sets.

Now take the case of a formula $\Diamond\varphi$ under the induction hypothesis that the result holds for φ, i.e. $\varphi \in x$ iff $\varphi \in x'$ for any $x \in C$ and $x' \in C'$ such that $\tau(x) = \tau(x')$. Take such x and x', and assume $\Diamond\varphi \in x$. Then $\varphi \in y$ for some y such that xRy. Then $y \in C$ as C is maximal. Hence $\tau(y) \in \delta C = \delta C'$, so $\tau(y) = \tau(y')$ for some $y' \in C'$. Therefore $\varphi \in y'$ by the induction hypothesis on φ. But $\Diamond\top \in x$ (as xRy), so $\Diamond\top \in \tau(x) = \tau(x')$. This gives $\Diamond\top \in x'$ which ensures that $x'Rz$ for some z, with $z \in C'$ as C' is maximal, hence C' is a non-degenerate cluster.[3] It follows that $x'Ry'$, so $\Diamond\varphi \in x'$ as required. Likewise $\Diamond\varphi \in x'$ implies $\Diamond\varphi \in x$, and the Lemma holds for $\Diamond\varphi$.

Finally we have the case of a formula $\langle t\rangle\Gamma$ under the induction hypothesis that the result holds for every $\gamma \in \Gamma$. Suppose $x \in C$ and $\tau(x) = \tau(x')$ for some $x' \in C'$. Let $\langle t\rangle\Gamma \in x$. Then by axiom Fix, for each $\gamma \in \Gamma$ we have $\Diamond(\gamma \wedge \langle t\rangle\Gamma) \in x$, implying that $\Diamond\gamma \in x$. Then applying to $\Diamond\gamma$ the analysis of $\Diamond\varphi$ in the previous paragraph, we conclude that C' is non-degenerate

[3]That is the reason for including $\Diamond\top$ in At.

and there is some $y_\gamma \in C'$ with $\gamma \in y_\gamma$. Now if $x'R^*z$, then $z \in C'$ so for each $\gamma \in \Gamma$ we have zRy_γ, implying that $\Diamond\gamma \in z$. This proves that $\Box^*(\bigwedge_{\gamma\in\Gamma}\Diamond\gamma) \in x'$. But putting $\varphi = \top$ in axiom Ind shows that the formula

$$\Box^*(\top \to \bigwedge_{\gamma\in\Gamma}\Diamond(\gamma\wedge\top)) \to (\top \to \langle t\rangle\Gamma)$$

is an L-theorem, From this we can derive that $\Box^*(\bigwedge_{\gamma\in\Gamma}\Diamond\gamma) \to \langle t\rangle\Gamma$ is an L-theorem, and hence belongs to x'. Therefore $\langle t\rangle\Gamma \in x'$ as required. Likewise $\langle t\rangle\Gamma \in x'$ implies $\langle t\rangle\Gamma \in x$, and so the Lemma holds for $\langle t\rangle\Gamma$. □

COROLLARY 5.11. *If C and C' are maximal clusters in $\mathcal{F}$ with $\delta C = \delta C'$, then $C = C'$.*

Proof. If $x \in C$, then $\tau(x) \in \delta C = \delta C'$, so there exists $x' \in C'$ with $\tau(x) = \tau(x')$. Lemma 5.10 then implies that $x = x' \in C'$, showing $C \subseteq C'$. Likewise $C' \subseteq C$. □

COROLLARY 5.12. *The set M of all maximal clusters of $\mathcal{F}$ is finite.*

Proof. The map $C \mapsto \delta C$ is an injection of M into the double power set $\wp\wp\mathsf{At}$ of the finite set At. This gives an upper bound of $2^{2^{n+1}}$ on the number of maximal clusters, where n is the size of Var. □

Given subsets X, Y of W with $X \subseteq Y$, we say that X is *definable within Y in $\mathcal{F}$* if there is a formula φ such that for all $y \in Y$, $y \in X$ iff $\varphi \in y$. We now work towards showing that within each inner subframe $R(x)$ in $\mathcal{F}$, each path component is definable. For each cluster C, define the formula

$$\alpha(C) = \bigwedge_{s\in\delta C}\Diamond^*\chi(s) \wedge \bigwedge_{s\in\wp\mathsf{At}\setminus\delta C}\neg\Diamond^*\chi(s).$$

The next result shows that a maximal cluster is definable within the set of all maximal elements of $\mathcal{F}$.

LEMMA 5.13. *If C is a maximal cluster and x is any maximal element of $\mathcal{F}$, then $x \in C$ iff $\alpha(C) \in x$.*

Proof. Let $x \in C$. If $s \in \delta C$, then $s = \tau(y)$ for some y such that $y \in C$, hence xR^*y, and $\chi(s) = \chi(y) \in y$, showing that $\Diamond^*\chi(s) \in x$. The converse of this also holds: if $\Diamond^*\chi(s) \in x$, then for some y, xR^*y and $\chi(s) \in y$. Hence $y \in C$ by maximality of C, and $s = \tau(y)$ by (5.19), so $s \in \delta C$. Contrapositively then, if $s \notin \delta C$, then $\Diamond^*\chi(s) \notin x$, so $\neg\Diamond^*\chi(s) \in x$. Altogether this shows that all conjuncts of $\alpha(C)$ are in x, so $\alpha(C) \in x$.

In the opposite direction, suppose $\alpha(C) \in x$. Let C' be the cluster of x. Then we want $C = C'$ to conclude that $x \in C$. Since x is maximal, i.e. C' is maximal, it is enough by Corollary 5.11 to show that $\delta C = \delta C'$.

Now if $s \in \delta C$, then $s = \tau(y)$ for some $y \in C$. But $\Diamond^*\chi(s)$ is a conjunct of $\alpha(C) \in x$, so $\Diamond^*\chi(s) \in x$. Hence there exists z with xR^*z and $\chi(s) \in z$. Then $z \in C'$ by maximality of C', and by (5.19) $s = \chi(z) \in \delta C'$.

Conversely, if $s \in \delta C'$, with $s = \tau(y)$ for some $y \in C'$, then xR^*y as $x \in C'$, and so $\Diamond^*\chi(s) \in x$ as $\chi(s) = \chi(y) \in y$. Hence $\neg\Diamond^*\chi(s) \notin x$. But then we must have $s \in \delta C$, for otherwise $\neg\Diamond^*\chi(s)$ would be a conjunction of $\alpha(C)$ and so would belong to x. □

It is shown in [30] that any transitive canonical frame (weak or not) has the *Zorn property*:

$$\forall x\, \exists y(xR^*y \text{ and } y \text{ is } R\text{-maximal}).$$

Note the use of R^*: the statement is that either x is R-maximal, or it has an R-maximal successor. The essence of the proof is that the relation $\{(x,y) : xR^\bullet y \text{ or } x = y\}$ is a partial ordering for which every chain has an upper bound, so by Zorn's Lemma $R(x)$ has a maximal element provided that it is non-empty.

The Zorn property is preserved under inner substructures, so it holds for our frame $\mathcal{F}$. One interesting consequence is:

LEMMA 5.14. *For each $x \in W$, the frame $\mathcal{F}(x) = (R(x), R{\restriction}R(x))$ has finitely many path components, as does $\mathcal{F}$ itself.*

Proof. The following argument works for both $\mathcal{F}$ and $\mathcal{F}(x)$, noting that the $R{\restriction}R(x)$-cluster of an element of $\mathcal{F}(x)$ is the same as its R-cluster in $\mathcal{F}$, and that all maximal clusters of $\mathcal{F}(x)$ are maximal in $\mathcal{F}$.

Let P be a path component and $y \in P$. By the Zorn property there is an R-maximal z with yR^*z. Then $z \in P$ as P is R^*-closed. So the R-cluster of z is a subset of P. Since this cluster is maximal, that proves that every path component contains a maximal cluster.

Now distinct path components are disjoint and so cannot contain the same maximal cluster. Since there are finitely many maximal clusters (Corollary 5.12), there can only be finitely many path components. $\square$

LEMMA 5.15. *Let C be a maximal cluster in $\mathcal{F}$. Then for all $x \in W$:*

(1) *$C \subseteq R(x)$ iff $\Diamond\Box^*\alpha(C) \in x$.*

(2) *$C \subseteq R^*(x)$ iff $\Diamond^*\Box^*\alpha(C) \in x$.*

Proof. For (1), first let $C \subseteq R(x)$. Take any $y \in C$. Then if yR^*z we have $z \in C$ as C is maximal, therefore $\alpha(C) \in z$ by Lemma 5.13. Thus $\Box^*\alpha(C) \in y$. But $y \in R(x)$, so then $\Diamond\Box^*\alpha(C) \in x$.

Conversely, if $\Diamond\Box^*\alpha(C) \in x$ then for some y, xRy and $\Box^*\alpha(C) \in y$. By the Zorn property, take a maximal z with yR^*z. Then $\alpha(C) \in z$, so $z \in C$ by Lemma 5.13. From $xRyR^*z$ we get xRz, so $z \in R(x) \cap C$. Since $R(x)$ is R^*-closed, this is enough to force $C \subseteq R(x)$.

The proof of (2) is similar to (1), replacing R by R^* where required. $\square$

For a given $x \in W$, let P be a path component of the frame $\mathcal{F}(x) = (R(x), R{\restriction}R(x))$. Let $M(P)$ be the set of all maximal R-clusters C that have $C \subseteq P$. Then $M(P) \subseteq M$, where M is the set of all maximal clusters of $\mathcal{F}$, so $M(P)$ is finite by Corollary 5.12. Define the formula

$$\alpha(P) = \bigvee\{\Diamond^*\Box^*\alpha(C) : C \in M(P)\}.$$

Then $\alpha(P)$ defines P within $R(x)$:

LEMMA 5.16. *For all $y \in R(x)$, $y \in P$ iff $\alpha(P) \in y$.*

Proof. Let $y \in R(x)$. If $y \in P$, take an R-maximal z with yR^*z, by the Zorn property. Then $z \in R(x)$, and z is path connected to $y \in P$, so $z \in P$. The cluster C_z of z is then included in P (if $w \in C_z$ then zR^*w so $w \in P$), and C_z is maximal, so $C_z \in M(P)$. The maximality of C_z together with Lemma 5.13 then ensure that $\Box^*\alpha(C_z) \in z$. Hence $\Diamond^*\Box^*\alpha(C_z) \in y$. But $\Diamond^*\Box^*\alpha(C_z)$ is a disjunct of $\alpha(P)$, so $\alpha(P) \in y$.

Conversely, if $\alpha(P) \in y$, then $\Diamond^*\Box^*\alpha(C) \in y$ for some $C \in M(P)$. By Lemma 5.15(2), $C \subseteq R^*(y)$. Taking any $z \in C$, since also $C \subseteq P$ we have $yR^*z \in P$, hence $y \in P$. □

THEOREM 5.17. *Suppose that L includes the scheme G_n. Then every inner subframe $\mathcal{F}$ of $\mathcal{F}_L$ is locally n-connected.*

Proof. Let $x \in W$. We have to show that $R(x)$ has at most n path components. If it has fewer than n there is nothing to do, so suppose $R(x)$ has at least n path components $P_0, \dots, P_{n-1}$. Put $P_n = R(x) \setminus (P_0 \cup \cdots \cup P_{n-1})$. We will prove that $P_n = \emptyset$, confirming that there can be no more components.

For each $i < n$, let φ_i be the formula $\alpha(P_i)$ that defines P_i within $R(x)$ according to Lemma 5.16. Let φ_n be $\neg\bigvee\{\alpha(P_i) : 0 \leq i < n\}$, so φ_n defines P_n within $R(x)$. Now for all $i \leq n$ let ψ_i be the formula obtained by uniform substitution of $\varphi_0, \dots, \varphi_n$ for $p_0, \dots, p_n$ in the formula Q_i of (5.15). Observe that since the $n+1$ sets $P_0, \dots, P_n$ form a partition of $R(x)$, each $y \in R(x)$ contains ψ_i for exactly one $i \leq n$, and indeed ψ_i defines the same subset of $R(x)$ as φ_i.

Now suppose, for the sake of contradiction, that $P_n \neq \emptyset$.[4] Then for each $i \leq n$ we can choose an element $y_i \in P_i$. Then xRy_i and $\psi_i \in y_i$. It follows that $\bigwedge_{i \leq n} \Diamond\psi_i \in x$. Since all instances of G_n are in x, we then get $\Diamond(\bigwedge_{i \leq n} \Diamond^*\neg\psi_i) \in x$. So there is some $y \in R(x)$ such that for each $i \leq n$ there exists a $z_i \in R^*(y)$ such that $\neg\psi_i \in z_i$, hence $\psi_i \notin z_i$. Now let P be the path component of y. If $P = P_i$ for some $i < n$, then as $y \in P_i$ and yR^*z_i, we get $z_i \in P_i$, and so $\psi_i \in z_i$ – which is false. Hence it must be that P is disjoint from P_i for all $i < n$, and so is a subset of P_n. But then as yR^*z_n we get $z_n \in P \subseteq P_n$, and so $\psi_n \in z_n$. That is also false, and shows that the assumption that $P_n \neq \emptyset$ is false. □

5.13 Completeness and finite model property for K4G$_n$

For the language $\mathcal{L}_\Box$ without $\langle t \rangle$, Theorem 5.17 provides a completeness theorem for any system extending K4G$_n$ by showing that any consistent formula φ is satisfiable in a locally n-connected weak canonical model (take a finite Var that includes all variables of φ and enough variables to have G_n as a formula in the weak language). But the "satisfiable" part of this depends on the Truth Lemma, which is unavailable in the presence of $\langle t \rangle$. We will need to apply filtration/reduction to establish completeness itself, as well as the finite model property.

Let L be a weak tangle logic that includes G_n; $\mathcal{F} = (W, R)$ an inner subframe of $\mathcal{F}_L$; and Φ a finite set of formulas that is closed under subformulas.

Recall that M is the set of all maximal clusters of $\mathcal{F}$, shown to be finite in Corollary 5.12. For each $x \in W$, define

$$M(x) = \{C \in M : C \subseteq R(x)\}.$$

Then $M(x)$ is finite, being a subset of M.

Define an equivalence relation $\approx$ on W by putting

[4] In that case P_n is the union of finitely many path components, by Lemma 5.14, but we do not need that fact.

$$x \approx y \text{ iff } x \cap \Phi = y \cap \Phi \text{ and } M(x) = M(y).$$

We then repeat the earlier standard transitive filtration construction, but using the finer relation $\approx$ in place of $\sim$. Thus we put $|x| = \{y \in W : x \approx y\}$ and $W_\Phi = \{|x| : x \in W\}$. The set W_Φ is finite, because the map $|x| \mapsto (x \cap \Phi, M(x))$ is a well-defined injection of W_Φ into the finite set $\wp\Phi \times \wp M$. The surjective function $f : W \to W_\Phi$ is given by $f(x) = |x|$.

Let $\mathcal{M}_\Phi = (W_\Phi, R_\Phi, h_\Phi)$, where $R_\Phi \subseteq W_\Phi \times W_\Phi$ is the transitive closure of $R_\lambda = \{(|x|, |y|) : xRy\}$, $h_\Phi(p) = \{|x| : p \in x\}$ for $p \in \Phi$, and $h_\Phi(p) = \emptyset$ otherwise.

We now verify that the pair $(\mathcal{M}_\Phi, f)$ as just defined satisfies the axioms (r1)–(r5) of a definable reduction of $\mathcal{F}$ via Φ.

(r1): *$p \in x$ iff $|x| \in h_\Phi(p)$, for all $p \in \mathsf{Var} \cap \Phi$.*

By definition of h_Φ.

(r2): *$|x| = |y|$ implies $x \cap \Phi = y \cap \Phi$.*

If $|x| = |y|$ then $x \approx y$, so $x \cap \Phi = y \cap \Phi$ by definition of $\approx$.

(r3): *xRy implies $|x|R_\Phi|y|$.*

xRy implies $|x|R_\lambda|y|$ and $R_\lambda \subseteq R_\Phi$.

(r4): *$|x|R_\Phi|y|$ implies $y \cap \Phi^t \subseteq x \cap \Phi^t$ and $\{\Diamond\varphi \in \Phi : \Diamond^*\varphi \in y\} \subseteq x$.*

The proof is the same as the proof given earlier of (r4) for the standard transitive filtration, but using $\approx$ in place of $\sim$ and the fact that $x \approx y$ implies $x \cap \Phi = y \cap \Phi$.

(r5): *For each subset C of W_Φ there is a formula φ that defines $f^{-1}(C)$ in W, i.e. $\varphi \in y$ iff $|y| \in C$.*

To see this, for each $x \in W$ let γ_x be the conjunction of $(x \cap \Phi) \cup \{\neg\psi : \psi \in \Phi \setminus x\}$. Then for any $y \in W$,

$$\gamma_x \in y \quad \text{iff} \quad x \cap \Phi = y \cap \Phi.$$

Next, let μ_x be the conjunction of the finite set of formulas

$$\{\Diamond\Box^*\alpha(C) : C \in M(x)\} \cup \{\neg\Diamond\Box^*\alpha(C) : C \in M \setminus M(x)\}.$$

Lemma 5.15 showed that each $C \in M$ has $C \in M(x)$ iff $\Diamond\Box^*\alpha(C) \in x$. From this it follows readily that for any $y \in W$,

$$\mu_x \in y \quad \text{iff} \quad M(x) = M(y).$$

So putting $\varphi_x = \gamma_x \wedge \mu_x$, we get that in general

$$\varphi_x \in y \quad \text{iff} \quad x \approx y \quad \text{iff} \quad |y| \in \{|x|\}.$$

Now if $C = \emptyset$, then $\bot$ defines $f^{-1}(C)$ in W. Otherwise if $C = \{|x_1|, \ldots, |x_n|\}$, then the disjunction $\varphi_{x_1} \vee \cdots \vee \varphi_{x_n}$ defines $f^{-1}(C)$ in W.

Consequently, the reduction $\mathcal{M}_t$ of $\mathcal{M}_\Phi$ satisfies the Reduction Lemma. We will show that G_n is valid in the frame of $\mathcal{M}_t$. But first we show that it is valid in the frame of $\mathcal{M}_\Phi$. Both cases involve some preliminary analysis, involving linking points of $R_\Phi(|y|)$ and $R_t(|y|)$ back to points of $R(y)$. This requires further work with maximal elements and clusters.

LEMMA 5.18. *For all $x, y \in W$, $|x| R^*_\Phi |y|$ implies $M(y) \subseteq M(x)$.*

Proof. If $|x| R^*_\Phi |y|$ there is a finite sequence $x = z_0, \dots, z_k = y$ for some $k \geq 1$ such that for all $i < k$, either $z_i \approx z_{i+1}$ or $z_i R z_{i+1}$. But $z_i \approx z_{i+1}$ implies $M(z_i) = M(z_{i+1})$, and $z_i R z_{i+1}$ implies $M(z_{i+1}) \subseteq M(z_i)$ by transitivity of R. This yields $M(z_k) \subseteq M(z_0)$ by induction on k. $\square$

LEMMA 5.19. *Suppose* $\mathsf{At} \subseteq \Phi$ *and $a \in W$ is R-maximal. Then for all $x \in W$, xRa iff $|x| R_\Phi |a|$.*

Proof. xRa implies $|x| R_\Phi |a|$ by (r3). For the converse, suppose $|x| R_\Phi |a|$ and let K be the maximal R-cluster of a.

If K is non-degenerate then $K \subseteq R(a)$, so $K \in M(a)$. Then from $|x| R_\Phi |a|$ we get $K \in M(x)$ by Lemma 5.18, implying xRa as required.

But if K is degenerate, then $K = \{a\}$ and $R(a) = M(a) = \emptyset$. Also $\Diamond\top \notin a$. Since $|x| R_\Phi |a|$, by definition of R_Φ there are $z, w \in W$ with $|x| R^*_\Phi |z|$ and $zRw \approx a$. As $\mathsf{At} \subseteq \Phi$, from $w \approx a$ we get $w \cap \mathsf{At} = a \cap \mathsf{At}$, i.e. $\tau(w) = \tau(a)$. In particular $\Diamond\top \notin w$, hence w is also R-maximal. Therefore a and w are maximal elements with the same atomic type, so $w = a$ by Lemma 5.10. Thus zRa and so $K \in M(z)$. Since $|x| R^*_\Phi |z|$ this implies $K \in M(x)$ by Lemma 5.18, giving the required xRa again. $\square$

LEMMA 5.20. *For any $y \in W$, let A be the set of all R-maximal points in $R(y)$. Then each point $v \in R_\Phi(|y|)$ has $v R^*_\Phi |a|$ for some $a \in A$.*

Proof. Let $v = |z| \in R_\Phi(|y|)$. By the Zorn property there exists an a with zR^*a and a is R-maximal. If $z = a$, then z is R-maximal, so as $|y| R_\Phi |z|$ we have $z \in R(y)$ by Lemma 5.19. Hence $z \in A$, so in this case we get $|z| R^*_\Phi |a|$ with $a \in A$ by taking $a = z$.

If however $z \neq a$, then zRa, hence $|z| R_\Phi |a|$ by (r3). Also, if C is the R-cluster of a, then $C \subseteq R(z)$ and C is maximal, hence $C \in M(z)$. But $|y| R_\Phi |z|$, so Lemma 5.18 then implies $C \in M(y)$, therefore $a \in R(y)$. So in this case we have $|z| R_\Phi |a|$ with $a \in A$. $\square$

THEOREM 5.21. *If* $\mathsf{At} \subseteq \Phi$*, the frame $\mathcal{F}_\Phi = (W_\Phi, R_\Phi)$ is locally n-connected.*

Proof. For any point $|y| \in W_\Phi$, we have to show that $R_\Phi(|y|)$ has at most n path components. But if it had more than n, then by picking points from different components we would get a sequence of more than n points no two of which were path connected. We show that this is impossible, by taking an arbitrary sequence $v_0, \dots, v_n$ of $n+1$ points in $R_\Phi(|y|)$, and proving that there must exist distinct i and j such that v_i and v_j are path connected in $R_\Phi(|y|)$.

For each $i \leq n$, by Lemma 5.20 there is an R-maximal $a_i \in R(y)$ with $v_i R^*_\Phi |a_i|$. This gives us a sequence $a_0, \dots, a_n$ of members of $R(y)$. But $R(y)$ has at most n path components, by Theorem 5.17. Hence there exist $i \neq j \leq n$ such that there is a connecting R-path $a_i = w_0, \dots, w_n = a_j$ between a_i and a_j that lies in $R(y)$. So for all $i < n$ we have yRw_i and either $w_i R w_{i+1}$ or $w_{i+1} R w_i$, hence $|y| R_\Phi |w_i|$ and either $|w_i| R_\Phi |w_{i+1}|$ or $|w_{i+1}| R_\Phi |w_i|$.

This shows that $|a_i|$ and $|a_j|$ are path connected in $R_\Phi(|y|)$ by the sequence $|w_0|, \dots, |w_n|$. Since $v_i R^*_\Phi |a_i|$ and $v_j R^*_\Phi |a_j|$, it follows that v_i and v_j are path connected in $R_\Phi(|y|)$, as required. $\square$

From this result we can infer that in the language $\mathcal{L}_\Box$, for all $n \geq 1$ the finite model property holds for K4G$_n$ and KD4G$_n$ over locally n-connected K4 and KD4 frames, respectively. For the proof, we take a consistent $\mathcal{L}_\Box$-formula φ and let Φ be the closure under $\mathcal{L}_\Box$-subformulas of $\mathsf{At} \cup \{\varphi\}$. Then Φ is finite and φ is satisfiable in the model $\mathcal{M}_\Phi$ (see the remarks about $\mathcal{M}_\Phi$ at the end of section 5.10). But the frame $\mathcal{F}_\Phi$ of $\mathcal{M}_\Phi$ is locally n-connected by the theorem just proved, so validates G$_n$. Together with the preservation of seriality by $\mathcal{F}_\Phi$, this implies the finite model property results for K4G$_n$ and KD4G$_n$.

Extending to the language $\mathcal{L}_{\Box\forall}$, and using that $\mathcal{F}_\Phi$ is path connected in the presence of axiom C, these finite model property results hold correspondingly for the four systems K4G$_n$.U, K4G$_n$.UC, KD4G$_n$.U, and KD4G$_n$.UC.

We turn now to the corresponding results for the versions of these systems that include the tangle connective.

LEMMA 5.22. *If $y \in W$ is the critical point for some R_Φ-cluster, then $z \in R(y)$ implies $|z| \in R_t(|y|)$.*

Proof. Let y be critical for cluster C. If $z \in R(y)$, then $|y|R_\Phi|z|$ (r3), so if $|z| \notin C$ then immediately $|y|R_t|z|$. But if $|z| \in C$, then $|z| \in C^\circ$ and again $|y|R_t|z|$. □

LEMMA 5.23. *Suppose $\Diamond\top \in \Phi$. Let $y \in W$ be a critical point, and $z, z' \in R(y)$. If z and z' are path connected in $R(y)$, then $|z|$ and $|z'|$ are path connected in $R_t(|y|)$.*

Proof. Let $z = z_0, \ldots, z_n = z'$ be a connecting path between z and z' within $R(y)$. The criticality of y ensures, by Lemma 5.22, that $|z_0|, \ldots, |z_n|$ are all in $R_t(|y|)$. We apply Lemma 5.7 to convert this sequence into a connecting R_t-path within $R_t(|y|)$.

For each $i < n$ we have $z_i R z_{i+1}$ or $z_{i+1} R z_i$, hence $|z_i|R_\Phi|z_{i+1}|$ or $|z_{i+1}|R_\Phi|z_i|$ by (r3). So if there is such an i that is "defective" in the sense that neither $|z_i|R_t|z_{i+1}|$ nor $|z_{i+1}|R_t|z_i|$, then by Lemma 5.7, which applies since $\Diamond\top \in \Phi$, there exists a v_i with $|z_i|R_t v_i$ and $|z_{i+1}|R_t v_i$. Then $v_i \in R_t(|y|)$ by transitivity of R_t, as $|z_i| \in R_t(|y|)$. We insert v_i between $|z_i|$ and $|z_{i+1}|$ in the sequence. Doing this for all defective $i < n$ turns the sequence into a connecting R_t-path in $R_t(|y|)$ with unchanged endpoints $|z|$ and $|z'|$. □

LEMMA 5.24. *Suppose $\Diamond\top \in \Phi$ and $a \in W$ is R-maximal. Then for all $x \in W$, $|x|R_t|a|$ iff $|x|R_\Phi|a|$.*

Proof. $|x|R_t|a|$ implies $|x|R_\Phi|a|$ by definition of R_t. For the converse, suppose $|x|R_\Phi|a|$, let C be the R_Φ-cluster of $|x|$, and let K be the maximal R-cluster of a.

If $|a| \notin C$, then since $|x|R_\Phi|a|$ it is immediate that $|x|R_t|a|$ as required. We are left with the case $|a| \in C$. Since $\Diamond\top \in \Phi$ and $|x|R_\Phi|a|$ we get $\Diamond\top \in x$ by (r4). As $|x|$ and $|a|$ both belong to C, Lemma 5.2 then gives $\Diamond\top \in a$. So $R(a) \neq \emptyset$, implying that $R(a) = K$ and $M(a) = \{K\}$. Moreover, since $|x|R_\Phi|a|$ we see that C is non-degenerate, so if y is the critical point for C then $|y|R_\Phi|a|$, hence $M(a) \subseteq M(y)$ by Lemma 5.18. Thus $K \in M(y)$, making yRa, hence $|a| \in C^\circ$ and so again $|x|R_t|a|$ as required. □

THEOREM 5.25. *If $\mathsf{At} \subseteq \Phi$, the frame $\mathcal{F}_t = (W_\Phi, R_t)$ is locally n-connected.*

Proof. This refines the proof of Theorem 5.21. If $u \in W_\Phi$, we have to show that $R_t(u)$ has at most n path components. Now if C is the R_Φ-cluster of u, then $R_t(u)$ is the union of the nucleus C° and all the R_Φ-clusters coming strictly R_Φ-after C. Hence $R_t(u) = R_t(w)$ for all $w \in C$. In particular, $R_t(u) = R_t(|y|)$ where y is the critical point of C. So we show that $R_t(|y|)$ has at most n path components. We take an arbitrary sequence $v_0, \ldots, v_n$ of $n+1$ points in $R_t(|y|)$, and prove that there must exist distinct i and j such that v_i and v_j are path connected in $R_t(|y|)$.

Let A be the set of all R-maximal points in $R(y)$. For each $i \leq n$ we have $v_i \in R_\Phi(|y|)$ and so by Lemma 5.20 there is an $a_i \in A \subseteq R(y)$ such that $v_i R_\Phi^* |a_i|$. Hence $v_i R_t^* |a_i|$ by Lemma 5.24. This gives us a sequence $a_0, \ldots, a_n$ of members of $R(y)$. But $R(y)$ has at most n path components, by Theorem 5.17. Hence there exist $i \neq j \leq n$ such that a_i and a_j are path connected in $R(y)$. Therefore by Lemma 5.23, $|a_i|$ and $|a_j|$ are path connected in $R_t(|y|)$. Since $v_i R_t^* |a_i|$ and $v_j R_t^* |a_j|$, and $v_i, v_j \in R_t(|y|)$, it follows that v_i and v_j are path connected in $R_t(|y|)$. That shows that $R_t(|y|)$ does not have more than n path components. □

This result combines with the analysis as in other cases to give the finite model property for the tangle systems $\mathrm{K4G}_n t$, $\mathrm{K4G}_n t.\mathrm{U}$, $\mathrm{K4G}_n t.\mathrm{UC}$, $\mathrm{KD4G}_n t$, $\mathrm{KD4G}_n t.\mathrm{U}$, and $\mathrm{KD4G}_n t.\mathrm{UC}$ for all $n \geq 1$.

6 More topology

The foregoing finite model property theorems will be instrumental in our completeness theorems for (some) topological spaces. Not surprisingly, we will also need some simple and standard topological definitions and results, together with some more substantial ones. The first one is very simple.

LEMMA 6.1. *Let X be a dense-in-itself T1 topological space. Then every non-empty open subset of X is infinite.*

Proof. Left to the reader. □

6.1 The $\langle d \rangle$ operator on sets

Let X be a topological space. For a set $S \subseteq X$, recall that $\langle d \rangle S = \{x \in X : S \cap O \setminus \{x\} \neq \emptyset$ for every open neighbourhood O of $x\}$, the set of strict limit points of S. The $\langle d \rangle$ operator has the following basic properties.

LEMMA 6.2. *Let $S, T \subseteq X$.*

1. $\mathrm{cl}\, S = S \cup \langle d \rangle S$.
2. *$\langle d \rangle$ is additive: $\langle d \rangle (S \cup T) = \langle d \rangle S \cup \langle d \rangle T$.*
3. *If X is dense in itself, then (i) $\mathrm{int}\, S \subseteq \langle d \rangle S$, and (ii) if S is open then $\langle d \rangle S = \mathrm{cl}\, S$.*

Proof. Easy. □

6.2 Regular open sets

Let X be a topological space. A regular open subset of X is one equal to the interior of its closure. We will mainly be interested in regular open subsets of open subspaces of X, so we give definitions directly for such situations.

DEFINITION 6.3. Let U be an open subset of X. A subset S of X is said to be a *regular open subset of* U if $S = \operatorname{int}(U \cap \operatorname{cl} S)$.

As 'int' is multiplicative and U is open, it is equivalent to say that $S = U \cap \operatorname{int} \operatorname{cl} S$, and we sometimes prefer this formulation. In such a case, $S \subseteq U$ and S is open. So $S = \operatorname{int}_U \operatorname{cl}_U S$: S is a regular open subset of the subspace U of X. It is worth noting that if $S \subseteq U$ is arbitrary then $\operatorname{int}_U \operatorname{cl}_U S$ is a regular open subset of U.

It is known (see, e.g., [13, chapter 10]) that for every open subset U of X, the set $RO(U)$ of regular open subsets of U is closed under the operations $+, \cdot, -, 0, 1$ defined by

- $S + S' = U \cap \operatorname{int} \operatorname{cl}(S \cup S')$
- $S \cdot S' = S \cap S'$
- $-S = U \setminus \operatorname{cl} S$
- $0 = \emptyset$ and $1 = U$,

and $(RO(U), +, \cdot, -, 0, 1)$ is a (complete) boolean algebra. We will also use the notation $RO(U)$ to denote this boolean algebra. The standard boolean ordering $\leq$ on $RO(U)$ coincides with set inclusion, because for $S, T \in RO(U)$ we have $S \leq T$ iff $S \cdot T = S$, iff $S \cap T = S$, iff $S \subseteq T$. We will need the following general lemma.

LEMMA 6.4. *Let $V \subseteq U$ be open subsets of X, and S, S' be regular open subsets of U.*

1. *If $T = U \setminus \operatorname{cl} S$, then T is also a regular open subset of U, with $S = U \setminus \operatorname{cl} T$ and $U \setminus S \subseteq \operatorname{cl} T$.*
2. *If $U \cap \operatorname{cl} S \cap \operatorname{cl} S' = \emptyset$, then $S + S' = S \cup S'$.*
3. *If $S \subseteq V$, then S is a regular open subset of V.*
4. *Every regular open subset of S is a regular open subset of U.*

Proof. 1. The first two points follow from boolean algebra considerations, and can easily be shown directly. The third point, $U \setminus S \subseteq \operatorname{cl} T$, follows from $U \setminus \operatorname{cl} T = S$.

2. Since $S, S' \leq S + S'$ and $\leq$ coincides with $\subseteq$, we obtain $S, S' \subseteq S + S'$ and so $S \cup S' \subseteq S + S'$. Conversely, it is easy to check[5] that

$$\operatorname{int} \operatorname{cl}(S \cup S') \subseteq \operatorname{int} \operatorname{cl} S \cup \operatorname{int} \operatorname{cl} S' \cup (\operatorname{cl} S \cap \operatorname{cl} S').$$

Since $U \cap \operatorname{cl} S \cap \operatorname{cl} S' = \emptyset$,

$$S + S' = U \cap \operatorname{int} \operatorname{cl}(S \cup S') \subseteq (U \cap \operatorname{int} \operatorname{cl} S) \cup (U \cap \operatorname{int} \operatorname{cl} S') = S \cup S',$$

as required.

[5]Indeed, $\Box\Diamond(p \vee q) \to \Box\Diamond p \vee \Box\Diamond q \vee (\Diamond p \wedge \Diamond q)$ is valid in S4 frames, so provable in S4. Since S4 is sound over X, the formula is valid in X.

3. $V \cap \operatorname{int} \operatorname{cl} S = (V \cap U) \cap \operatorname{int} \operatorname{cl} S = V \cap (U \cap \operatorname{int} \operatorname{cl} S) = V \cap S = S$.

4. Let T be a regular open subset of S. Clearly, $\operatorname{int} \operatorname{cl} T \subseteq \operatorname{int} \operatorname{cl} S$. So $U \cap \operatorname{int} \operatorname{cl} T = U \cap (\operatorname{int} \operatorname{cl} S \cap \operatorname{int} \operatorname{cl} T) = (U \cap \operatorname{int} \operatorname{cl} S) \cap \operatorname{int} \operatorname{cl} T = S \cap \operatorname{int} \operatorname{cl} T = T$.

□

6.3 Normal spaces

DEFINITION 6.5. A topological space X is said to be *Hausdorff* (or T2) if for every two distinct points $x_0, x_1 \in X$, there are disjoint open sets O_0, O_1 with $x_0 \in O_0$ and $x_1 \in O_1$, and *normal* (or T4) if it is Hausdorff and for every two disjoint closed subsets C_0, C_1 of X, there are disjoint open sets O_0, O_1 with $C_0 \subseteq O_0$ and $C_1 \subseteq O_1$.

Equivalently, X is normal iff it is Hausdorff and if $C \subseteq O \subseteq X$, C closed, and O open, then there is open P with $C \subseteq P \subseteq \operatorname{cl} P \subseteq O$.

LEMMA 6.6. *Let C_0, C_1 be disjoint closed subsets of a normal topological space X. Then there are regular open subsets O_0, O_1 of X with disjoint closures, such that $C_0 \subseteq O_0$ and $C_1 \subseteq O_1$.*

Proof. By normality, there are disjoint open sets $O_0^- \supseteq C_0$ and $U \supseteq C_1$. Then $O_0^- \subseteq X \setminus U$, a closed set. So $O_0 = \operatorname{int} \operatorname{cl} O_0^-$ is a regular open subset of X disjoint from U. We have $C_0 \subseteq O_0^- \subseteq O_0 \subseteq \operatorname{cl} O_0 \subseteq X \setminus U$, so $\operatorname{cl} O_0$ and C_1 are disjoint closed sets. By normality again, there are disjoint open sets $V \supseteq \operatorname{cl} O_0$ and $O_1^- \supseteq C_1$. Let $O_1 = \operatorname{int} \operatorname{cl} O_1^-$, a regular open subset of X disjoint from V. Then $C_1 \subseteq O_1^- \subseteq O_1 \subseteq \operatorname{cl} O_1 \subseteq X \setminus V$, so $\operatorname{cl} O_0 \cap \operatorname{cl} O_1 = \emptyset$. Now O_0, O_1 are as required. □

The following is well known (see, e.g., [28, III, 6.1]), but is so important for us that we include a quick proof.

LEMMA 6.7. *Every metric space is normal.*

Proof. Let X be a metric space. It is easy to check that X is Hausdorff, and we leave this to the reader. Let C, D be disjoint closed subsets of X. By symmetry, it is enough to show that there is open $O \supseteq C$ with $\operatorname{cl}(O) \cap D = \emptyset$. If $C = \emptyset$, take $O = \emptyset$. If $D = \emptyset$ take $O = X$. So we can suppose $C, D \neq \emptyset$, and thus define

$$O = \{x \in X : d(x, C) < d(x, D)/2\}$$

(recall from section 2.4 that $d(x, S) = \inf\{d(x, s) : s \in S\}$ for non-empty $S \subseteq X$). Then $C \subseteq O$, because if $x \in C$ then $d(x, C) = 0$, while $x \notin D$, so $d(x, D) > 0$ as D is closed. It is easily seen that O is open and $\operatorname{cl}(O) \subseteq \{x \in X : d(x, C) \leq d(x, D)/2\}$, so it is enough to show that this latter set is disjoint from D. If x is in both, then $d(x, C) \leq d(x, D)/2 = 0$ so $x \in C$ as C is closed. This contradicts the assumption that $C \cap D = \emptyset$. □

It follows that every metric space is T1, as we said earlier.

6.4 Tarski's theorem and relatives

The primary topological results needed later (for representing finite Kripke frames in proposition 7.10) are provided by the next theorem. A recent related result is [20, proposition 6.7].

THEOREM 6.8. *Let X be a dense-in-itself metric space.*

1. *Let $\mathbb{T}, \mathbb{U}$ be open subsets of X, with $\emptyset \neq \mathbb{T} \subseteq \mathbb{U}$. Let $k < \omega$. Then there are pairwise disjoint non-empty subsets $\mathbb{I}_0, \ldots, \mathbb{I}_k \subseteq \mathbb{T}$ satisfying*

$$\langle d \rangle \mathbb{I}_i = \mathrm{cl}(\mathbb{T}) \setminus \mathbb{U} \quad \textit{for each } i \leq k.$$

2. *Let $\mathbb{G}$ be a non-empty open subset of X, and let $r, s < \omega$. Then $\mathbb{G}$ can be partitioned into non-empty open subsets $\mathbb{G}_1, \ldots, \mathbb{G}_r$ and other non-empty sets $\mathbb{B}_0, \ldots, \mathbb{B}_s$ such that, letting*

$$D = \mathrm{cl}(\mathbb{G}) \setminus \bigcup_{1 \leq l \leq r} \mathbb{G}_l,$$

we have $\mathrm{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = D$ for each $i = 1, \ldots, r$, and $\langle d \rangle \mathbb{B}_j = D$ for each $j = 0, \ldots, s$.

Part 2 above is essentially known. Paraphrasing slightly, Tarski [35, satz 3.10] proved the following. Let X be a dense-in-itself normal topological space with a countable basis of open sets (see below). Then for every $r < \omega$, every non-empty open subset $\mathbb{G}$ of X can be partitioned into non-empty open sets $\mathbb{G}_1, \ldots, \mathbb{G}_r$ and a non-empty set $\mathbb{B}_0$ such that $\mathrm{cl}(\mathbb{G}) \setminus \mathbb{G} \subseteq \mathrm{cl}\, \mathbb{B}_0 \subseteq \mathrm{cl}\, \mathbb{G}_1 \cap \ldots \cap \mathrm{cl}\, \mathbb{G}_r$. Here and below, the empty intersection (when $r = 0$) is taken to be X. This statement is equivalent to the statement in part 2 of theorem 6.8 above in the case $s = 0$ and with $\langle d \rangle \mathbb{B}_j$ replaced by $\mathrm{cl}\, \mathbb{B}_j$.

A topological space (X, τ) has a countable basis of open sets iff there is countable $\tau_0 \subseteq \tau$ such that τ is the smallest topology on X containing τ_0. Given this and normality, Urysohn's theorem [37] yields that $\tau = \tau_d$ for some metric d on X. Any metric space is normal, and has a countable basis of open sets iff it is separable (see section 2.3). So Tarski's stipulation on X boils down to stipulating that X is a separable dense-in-itself metric space.

Removing the restriction to $s = 0$ but with the same hypotheses on X, McKinsey and Tarski [24, theorem 3.5] proved that for every $r, s < \omega$, every non-empty open set $\mathbb{G}$ can be partitioned into non-empty open sets $\mathbb{G}_1, \ldots, \mathbb{G}_r$ and non-empty sets $\mathbb{B}_0, \ldots, \mathbb{B}_s$ with $\mathrm{cl}(\mathbb{G}) \setminus \mathbb{G} \subseteq \mathrm{cl}\, \mathbb{B}_0 = \cdots = \mathrm{cl}\, \mathbb{B}_s \subseteq \mathrm{cl}\, \mathbb{G}_1 \cap \ldots \cap \mathrm{cl}\, \mathbb{G}_r$. This statement is equivalent to the statement of theorem 6.8(2) above, with $\langle d \rangle \mathbb{B}_j$ replaced by $\mathrm{cl}\, \mathbb{B}_j$. It was used in [24] to prove (in our terminology) that the $\mathcal{L}_\Box$-logic of X is S4.

Removing the assumption of separability, Rasiowa and Sikorski [28, III, 7.1] proved theorem 6.8(2) as formulated above, but with $\langle d \rangle \mathbb{B}_j$ replaced by $\mathrm{cl}\, \mathbb{B}_j$. Our use of $\langle d \rangle \mathbb{B}_j$ is only a formal strengthening of [28, III, 7.1], since the same effect can be achieved by first obtaining disjoint sets $\mathbb{B}^i_j$ with $\mathrm{cl}\, \mathbb{B}^i_j = D$ for $j = 0, \ldots, s$ and $i = 0, 1$, and then defining $\mathbb{B}_j = \mathbb{B}^0_j \cup \mathbb{B}^1_j$ for each j. As $\mathbb{B}^0_j \cap \mathbb{B}^1_j = \emptyset$, using lemma 6.2 we have

$$D \subseteq (D \setminus \mathbb{B}^0_j) \cup (D \setminus \mathbb{B}^1_j) = (\mathrm{cl}\, \mathbb{B}^0_j \setminus \mathbb{B}^0_j) \cup (\mathrm{cl}\, \mathbb{B}^1_j \setminus \mathbb{B}^1_j) \subseteq \underbrace{\langle d \rangle \mathbb{B}^0_j \cup \langle d \rangle \mathbb{B}^1_j}_{\langle d \rangle \mathbb{B}_j} \subseteq \mathrm{cl}\, \mathbb{B}^0_j \cup \mathrm{cl}\, \mathbb{B}^1_j = D,$$

so $\langle d \rangle \mathbb{B}_j = D$ as required. Given this, the reader may ask why we give a proof of part 2 at all. The answer is that we wish to make clear the affinity between the two parts of the theorem, as well as make our paper more self contained and explicit as to the topological arguments needed in our completeness proof.

Proof. We will get to the theorem shortly, but first, fix $k < \omega$. We define a game, $\mathcal{G}_k$, to build pairwise disjoint subsets $\mathbb{I}_0, \ldots, \mathbb{I}_k$ of X. The game has two players, $\forall$ (male) and $\exists$ (female), and ω rounds, numbered $0, 1, 2, \ldots$. At the start of round n (for each $n < \omega$), pairwise disjoint sets $I_0^n, \ldots, I_k^n \subseteq X$ are in play, satisfying

$$\langle d \rangle I_i^n = \emptyset \quad \text{for each } i \leq k. \tag{6.1}$$

Observe that each I_i^n is closed, because by lemma 6.2, $\operatorname{cl} I_i^n = I_i^n \cup \langle d \rangle I_i^n = I_i^n$. Also,

$$\operatorname{int}\Big(\bigcup_{j \leq k} I_j^n\Big) = \emptyset. \tag{6.2}$$

For if $U \subseteq \bigcup_{j \leq k} I_j^n$ is open, then by lemma 6.2 and (6.1),

$$U \subseteq \operatorname{cl} U = \langle d \rangle U \subseteq \langle d \rangle \bigcup_{j \leq k} I_j^n = \bigcup_{j \leq k} \langle d \rangle I_j^n = \emptyset.$$

The game starts off with all of the sets empty: $I_0^0 = \cdots = I_k^0 = \emptyset$. Round n is played as follows. Player $\forall$ moves first, by playing a triple $(\varepsilon_n, i_n, O_n)$, of his choice, where $\varepsilon_n > 0$ is a real number, $i_n \leq k$, and O_n is a non-empty open subset of X. Let

$$P_n = O_n \setminus \bigcup_{j \leq k} I_j^n. \tag{6.3}$$

Then $P_n \neq \emptyset$: for otherwise, $\emptyset \neq O_n \subseteq \bigcup_{j \leq k} I_j^n$, contradicting (6.2). Player $\exists$ responds to $\forall$'s move by using Zorn's lemma to choose a maximal subset $Z_n \subseteq P_n$ such that $d(x, y) \geq \varepsilon_n$ for each distinct $x, y \in Z_n$. Observe that

Z1. $\langle d \rangle Z_n = \emptyset$ (because for all $x \in X$, the set $N_{\varepsilon_n/2}(x) \cap Z_n$ has at most one element). Just as with I_i^n above, it follows that Z_n is closed.

Z2. Z_n is non-empty (because P_n is non-empty and any singleton subset of P_n satisfies the ε_n-condition).

Z3. $d(x, Z_n) < \varepsilon_n$ for every $x \in P_n$ (else x can be added to Z_n, contradicting its maximality). Recall again that $d(x, Z_n) = \inf\{d(x, z) : z \in Z_n\}$, which is defined because Z_n is non-empty.

Player $\exists$ then extends $I_{i_n}^n$ by Z_n, leaving the other sets I_i^n unchanged. Formally, she defines

$$\begin{aligned} I_{i_n}^{n+1} &= I_{i_n}^n \cup Z_n, \\ I_i^{n+1} &= I_i^n \qquad \text{for each } i \leq k \text{ with } i \neq i_n. \end{aligned}$$

This completes the round, and the sets $I_0^{n+1}, \ldots, I_k^{n+1}$ are passed to the start of round $n+1$. Note that (6.1) holds for these sets, since $\langle d \rangle I_{i_n}^{n+1} = \langle d \rangle I_{i_n}^n \cup \langle d \rangle Z_n = \emptyset$ by lemma 6.2, (6.1) for $I_{i_n}^n$, and Z1 above. Also, by (6.3), Z_n is disjoint from each I_i^n, so the I_i^{n+1} $(i \leq k)$ are pairwise disjoint.

At the end of the game, we define $\mathbb{I}_i = \bigcup_{n<\omega} I_i^n$ for each $i \leq k$. Plainly, $\mathbb{I}_0, \ldots, \mathbb{I}_k$ are pairwise disjoint.

We say that $\forall$ *plays well* in $\mathcal{G}_k$ if his choices of ε_n tend to zero, the set $\{n < \omega : i_n = i\}$ is infinite for each $i \leq k$, and his choices of O_n form a descending chain: $O_0 \supseteq O_1 \supseteq \cdots$.

It is clear by condition Z2 above that if $\forall$ plays well then $\mathbb{I}_0, \ldots, \mathbb{I}_k$ are all non-empty.

Claim. In any play (match?) of the game in which $\forall$ plays well, for each $i \leq k$ we have

$$\langle d\rangle \mathbb{I}_i = \bigcap_{n<\omega} \operatorname{cl} O_n.$$

Proof of claim. Let $n < \omega$. Define $I_i^{>n} = \mathbb{I}_i \setminus I_i^n$. This is the set of points that $\exists$ added to $\mathbb{I}_i$ in or after round n. By the game rules and because $\forall$ played well, $I_i^{>n} \subseteq \bigcup_{n\leq m<\omega} Z_m \subseteq \bigcup_{n\leq m<\omega} O_m = O_n$. Obviously, $\mathbb{I}_i = I_i^n \cup I_i^{>n}$. So by lemma 6.2 and (6.1),

$$\langle d\rangle \mathbb{I}_i = \langle d\rangle(I_i^n \cup I_i^{>n}) = \langle d\rangle I_i^n \cup \langle d\rangle I_i^{>n} = \langle d\rangle I_i^{>n} \subseteq \langle d\rangle O_n \subseteq \operatorname{cl} O_n.$$

This holds for all n, so $\langle d\rangle \mathbb{I}_i \subseteq \bigcap_{n<\omega} \operatorname{cl} O_n$.

Conversely, let $x \in \bigcap_{n<\omega} \operatorname{cl} O_n$. Let a real number $\varepsilon > 0$ be given. Since $\forall$ plays well, we can pick a round, say n, such that $\forall$ chose $\varepsilon_n \leq \varepsilon$ and $i_n = i$, and such that if $x \in \mathbb{I}_i$ then already $x \in I_i^n$. Since $x \in \operatorname{cl} O_n$, the set $N_\varepsilon(x) \cap O_n$ is non-empty, and plainly it is open. As before, (6.2) implies that $N_\varepsilon(x) \cap O_n \setminus \bigcup_{j\leq k} I_j^n$ is non-empty as well. Fix a point y in this set. Then $y \in P_n$ and $d(x, y) < \varepsilon$.

In round n, player $\exists$ picks $Z_n \subseteq P_n$ satisfying conditions Z1–Z3 above. Observe that $x \notin Z_n$, because otherwise, $x \in Z_n \subseteq \mathbb{I}_i$ (since $i_n = i$), so by assumption on n we have $x \in I_i^n$, so by (6.3), $x \notin P_n \supseteq Z_n$, a contradiction. Since $y \in P_n$, by Z3 we have $d(y, Z_n) < \varepsilon_n$. Since $d(x, y) < \varepsilon$, we have $d(x, Z_n) < \varepsilon + \varepsilon_n \leq 2\varepsilon$. So there is $z \in Z_n \subseteq \mathbb{I}_i$ with $z \neq x$ (since $x \notin Z_n$) and $d(x, z) < 2\varepsilon$. This holds for all $\varepsilon > 0$, and it follows that $x \in \langle d\rangle \mathbb{I}_i$, proving the claim.

Now we prove part 1 of the theorem. Suppose first that $\operatorname{cl}(\mathbb{T}) \setminus \mathbb{U} = \emptyset$. Noting that $\mathbb{T}$ is infinite (by lemma 6.1), we can take $\mathbb{I}_0, \ldots, \mathbb{I}_k$ to be disjoint singleton subsets of $\mathbb{T}$. Plainly, all requirements are met.

So suppose that $\operatorname{cl}(\mathbb{T}) \setminus \mathbb{U} \neq \emptyset$. Let $\forall$ and $\exists$ play the game $\mathcal{G}_k$. We suppose that $\forall$ plays well, and also so that for each $n < \omega$,

$$O_n = \mathbb{T} \cap \bigcup_{x\in\operatorname{cl}(\mathbb{T})\setminus\mathbb{U}} N_{\varepsilon_n}(x).$$

Note that O_n is open, and non-empty because $\operatorname{cl}(\mathbb{T}) \setminus \mathbb{U} \neq \emptyset$, so $\forall$ can legally play it. Then $\mathbb{I}_0, \ldots, \mathbb{I}_k$ are pairwise disjoint, and non-empty since $\forall$ plays well. We have $Z_n \subseteq O_n \subseteq \mathbb{T}$ for each n, so $\mathbb{I}_0, \ldots, \mathbb{I}_k$ are subsets of $\mathbb{T}$. By the claim, $\langle d\rangle \mathbb{I}_i = \bigcap_{n<\omega} \operatorname{cl} O_n$ for each $i \leq k$, so it suffices to show that $\bigcap_{n<\omega} \operatorname{cl} O_n = \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$.

Certainly, each $x \in \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$ lies in $\operatorname{cl} O_n$ for each n, because for every $\varepsilon > 0$,

$$O_n \cap N_\varepsilon(x) \supseteq \Big(\mathbb{T} \cap \bigcup_{y\in\operatorname{cl}(\mathbb{T})\setminus\mathbb{U}} N_{\varepsilon_n}(y)\Big) \cap N_{\min(\varepsilon,\varepsilon_n)}(x) = \mathbb{T} \cap N_{\min(\varepsilon,\varepsilon_n)}(x) \neq \emptyset.$$

So $\operatorname{cl}(\mathbb{T}) \setminus \mathbb{U} \subseteq \bigcap_{n<\omega} \operatorname{cl} O_n$. Conversely, first note that $O_0 \subseteq \mathbb{T}$, so $\bigcap_{n<\omega} \operatorname{cl} O_n \subseteq \operatorname{cl} O_0 \subseteq \operatorname{cl} \mathbb{T}$. It remains to show that $\mathbb{U} \cap \bigcap_{n<\omega} \operatorname{cl} O_n = \emptyset$. Suppose for contradiction that there is some $x \in \mathbb{U} \cap \bigcap_{n<\omega} \operatorname{cl} O_n$. As $\mathbb{U}$ is open, we can choose $\delta > 0$ with $N_\delta(x) \subseteq \mathbb{U}$. As $\forall$ played well, we can pick $n < \omega$ such that $\varepsilon_n \leq \delta$. Then $x \in \operatorname{cl} O_n$, so $d(x, O_n) = 0$. By definition of O_n, for each $y \in O_n$ we have $d(y, \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}) < \varepsilon_n$. So $d(x, \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}) < \varepsilon_n$ as well. As $\varepsilon_n \leq \delta$ and $N_\delta(x) \subseteq \mathbb{U}$, this is a contradiction. We conclude that indeed $\mathbb{U} \cap \bigcap_{n<\omega} \operatorname{cl} O_n = \emptyset$, so $\bigcap_{n<\omega} \operatorname{cl} O_n \subseteq \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$, as required. We have proved part 1 of the theorem.

To prove part 2, let $\forall$ and $\exists$ play $\mathcal{G}_{s+r}$. As we will see, $\forall$ will play so that $\mathbb{I}_0, \ldots, \mathbb{I}_{s+r} \subseteq \mathbb{G}$. In the end, $\mathbb{B}_1, \ldots, \mathbb{B}_s$ will be $\mathbb{I}_1, \ldots, \mathbb{I}_s$, $\mathbb{G}_1, \ldots, \mathbb{G}_r$ will be 'fattened' versions of $\mathbb{I}_{s+1}, \ldots, \mathbb{I}_{s+r}$, and $\mathbb{B}_0$ will be the rest of $\mathbb{G}$ (we will have $\mathbb{B}_0 \supseteq \mathbb{I}_0$). For the fattening, at the start of round n (for each $n < \omega$), for each $j = s+1, \ldots, s+r$, $\forall$ defines an auxiliary open set G_j^n such that

$$I_j^n \subseteq G_j^n \tag{6.4}$$

$$G_j^0 \subseteq G_j^1 \subseteq \cdots \tag{6.5}$$

$$I_0^n, \ldots, I_s^n, \operatorname{cl} G_{s+1}^n, \ldots, \operatorname{cl} G_{s+r}^n \text{ are pairwise disjoint subsets of } \mathbb{G}. \tag{6.6}$$

The sets G_j^n are for $\forall$'s own private use and are not formally part of the game. (If $r = 0$, there are no j in range and he does nothing.) At the start of round 0, he simply puts $G_{s+1}^0 = \cdots = G_{s+r}^0 = \emptyset$. Suppose we are at the start of round n, for arbitrary $n < \omega$, and that $\forall$ has defined open $G_j^n \supseteq I_j^n$ $(s+1 \leq j \leq s+r)$ satisfying (6.4)–(6.6). In round n he plays $(\varepsilon_n, i_n, O_n)$, where $i_0 = 0$,

$$O_n = \mathbb{G} \setminus \bigcup_{s+1 \leq j \leq s+r} \operatorname{cl} G_j^n, \tag{6.7}$$

and the ε_n, i_n are chosen so that overall, he plays well. By (6.5), $O_0 \supseteq O_1 \supseteq \cdots$, as required for him to play well. (We remark that if $r = 0$ then $O_n = \mathbb{G}$ for all n.)

We check that this is always a legal move for $\forall$. Certainly, O_n is open. We show that it is always non-empty. For $n = 0$ we plainly have $O_0 = \mathbb{G} \neq \emptyset$. In round 0, $\forall$ plays $i_0 = 0$, and $\exists$ defines $I_0^1 = Z_0 \neq \emptyset$ by condition Z2 above. Since the I_0^n form a chain, $I_0^n \supseteq I_0^1 \neq \emptyset$ for all $n > 0$, and by (6.6) and (6.7), $I_0^n \subseteq O_n$. So $O_n \neq \emptyset$ for all n.

Player $\exists$ continues round n by selecting $Z_n \subseteq P_n$ and defining $I_{i_n}^{n+1} = I_{i_n}^n \cup Z_n$ according to the rules.

It is now time for $\forall$ to define G_j^{n+1} for $j = s+1, \ldots, s+r$. If $i_n \leq s$, he leaves the sets unchanged, defining $G_j^{n+1} = G_j^n$ for all j. Trivially, conditions (6.4)–(6.5) continue to hold. We check (6.6). First, $Z_n \subseteq P_n$, so $I_{i_n}^{n+1}$ is disjoint from I_j^{n+1} for $i_n \neq j \leq s$. Second, if $s+1 \leq j \leq s+r$ then $I_{i_n}^{n+1} = I_{i_n}^n \cup Z_n \subseteq I_{i_n}^n \cup O_n$; by (6.6), $I_{i_n}^n$ is disjoint from $\operatorname{cl} G_j^n = \operatorname{cl} G_j^{n+1}$, and by (6.7), O_n is disjoint from $\operatorname{cl} G_j^{n+1}$ as well.

If instead, $i_n > s$, then $\forall$ defines $G_j^{n+1} = G_j^n$ for $j \neq i_n$, and uses normality of X to choose an open set $G_{i_n}^{n+1}$ satisfying

$$\overbrace{\operatorname{cl}(G_{i_n}^n) \cup Z_n}^{\text{closed}} \subseteq G_{i_n}^{n+1} \subseteq \operatorname{cl}(G_{i_n}^{n+1}) \subseteq \overbrace{\mathbb{G} \setminus \Big(\bigcup_{j \leq s} I_j^n \ \cup \bigcup_{\substack{s+1 \leq j \leq s+r \\ j \neq i_n}} \operatorname{cl}(G_j^n) \Big)}^{\text{open}}. \tag{6.8}$$

We need to check some things here. First, by condition Z1 above, Z_n is closed and so the left-hand side of (6.8) is closed. Similarly, we saw just after (6.1) that each I_j^n is closed, so the right-hand side of (6.8) is open. Second, it follows from (6.6) that $\operatorname{cl}(G_{i_n}^n)$ is contained in the right-hand side of (6.8). Also $Z_n \subseteq P_n \subseteq O_n$, and it follows from (6.3) and (6.7) that Z_n is contained in the right-hand side of (6.8) as well. So $G_{i_n}^{n+1}$ can be found as stated.

We also need to check (6.4)–(6.6) for the G_j^{n+1}. Condition (6.4) holds because $I_{i_n}^{n+1} = I_{i_n}^n \cup Z_n \subseteq G_{i_n}^n \cup Z_n \subseteq G_{i_n}^{n+1}$, and for $j \neq i_n$ we have $G_j^{n+1} = G_j^n \supseteq I_j^n = I_j^{n+1}$. Conditions (6.5) and (6.6) are clear from the definitions and (6.8).

As promised, at the end of play we define

$$\begin{aligned}
\mathbb{G}_i &= \bigcup_{n<\omega} G_{s+i}^n && \text{for } 1 \leq i \leq r, \\
\mathbb{B}_j &= \mathbb{I}_j && \text{for } 1 \leq j \leq s, \\
\mathbb{B}_0 &= \mathbb{G} \setminus \Big(\bigcup_{1\leq i\leq r} \mathbb{G}_i \cup \bigcup_{1\leq j\leq s} \mathbb{B}_j \Big) \\
D &= \mathrm{cl}(\mathbb{G}) \setminus \bigcup_{1\leq l\leq r} \mathbb{G}_l.
\end{aligned}$$

Note that $\mathbb{I}_{s+i} \subseteq \mathbb{G}_i$ for $1 \leq i \leq r$ by (6.4), and $\mathbb{I}_j \subseteq \mathbb{B}_j$ for $j \leq s$ by the definitions. Because $\forall$ played well, the $\mathbb{G}_j$ are non-empty (and plainly open) and the $\mathbb{B}_j$ are non-empty. It follows from (6.6) that together they partition $\mathbb{G}$.

For the final piece of the theorem, there are two preliminaries. First, we observe that each set $\mathbb{G}_i$ ($1 \leq i \leq r$) has a nice property. Each time $\forall$ plays $i_n = s+i$ in some round n, by (6.5), (6.8), and the definition of $\mathbb{G}_i$, for every $m \leq n$ we have $\mathrm{cl}\, G_{s+i}^m \subseteq \mathrm{cl}\, G_{s+i}^n \subseteq G_{s+i}^{n+1} \subseteq \mathbb{G}_i$. Since $\forall$ played $i_n = s+i$ infinitely often, it follows that

$$\mathrm{cl}\, G_{s+i}^m \subseteq \mathbb{G}_i \quad \text{for each } m < \omega \text{ and } 1 \leq i \leq r. \tag{6.9}$$

Second, we use this to show that

$$D = \bigcap_{n<\omega} \mathrm{cl}\, O_n. \tag{6.10}$$

Note that if $C \subseteq S \subseteq X$ and C is closed, then $S = C \cup (S \setminus C) \subseteq C \cup \mathrm{cl}(S \setminus C)$; the right-hand side is closed, so $\mathrm{cl}\, S \subseteq C \cup \mathrm{cl}(S \setminus C)$, whence $\mathrm{cl}(S) \setminus C \subseteq \mathrm{cl}(S \setminus C)$. Now, for each $n < \omega$ we have

$$\begin{aligned}
D &= \mathrm{cl}(\mathbb{G}) \setminus \textstyle\bigcup_{1\leq i\leq r} \mathbb{G}_i && \text{by definition} \\
&\subseteq \mathrm{cl}(\mathbb{G}) \setminus \textstyle\bigcup_{1\leq i\leq r} \mathrm{cl}\, G_{s+i}^n && \text{by (6.9)} \\
&\subseteq \mathrm{cl}\, \big(\mathbb{G} \setminus \textstyle\bigcup_{1\leq i\leq r} \mathrm{cl}\, G_{s+i}^n\big) && \text{by the observation above} \\
&= \mathrm{cl}\, O_n && \text{by (6.7).}
\end{aligned}$$

So $D \subseteq \bigcap_{n<\omega} \mathrm{cl}\, O_n$. Conversely, we certainly have $\bigcap_{n<\omega} \mathrm{cl}\, O_n \subseteq \mathrm{cl}\, O_0 = \mathrm{cl}\, \mathbb{G}$ since $O_0 = \mathbb{G}$. Now fix i with $1 \leq i \leq r$. By (6.7), for each $n < \omega$ we have $G_{s+i}^n \cap O_n = \emptyset$, so as G_{s+i}^n is open, $G_{s+i}^n \cap \mathrm{cl}\, O_n = \emptyset$. It follows that

$$\mathbb{G}_i \cap \bigcap_{n<\omega} \mathrm{cl}\, O_n = (\bigcup_{n<\omega} G_{s+i}^n) \cap \bigcap_{n<\omega} \mathrm{cl}\, O_n = \emptyset.$$

This holds for each i, so $\bigcap_{n<\omega} \mathrm{cl}\, O_n \subseteq \mathrm{cl}(\mathbb{G}) \setminus \bigcup_{1\leq i\leq r} \mathbb{G}_i = D$, proving (6.10).

Now we can finish easily. For each $0 \leq j \leq s$, we plainly have $\mathbb{B}_j \subseteq \mathbb{G} \setminus \bigcup_{1\leq l\leq r} \mathbb{G}_l \subseteq D$. Since D is closed, $\langle d\rangle \mathbb{B}_j \subseteq \mathrm{cl}\, \mathbb{B}_j \subseteq D$. Conversely, by (6.10) and the claim, $D = \bigcap_{n<\omega} \mathrm{cl}\, O_n = \langle d\rangle \mathbb{I}_j \subseteq \langle d\rangle \mathbb{B}_j$.

Similarly, take i with $1 \leq i \leq r$. Since the $\mathbb{G}_l$ $(1 \leq l \leq r)$ are pairwise disjoint open subsets of $\mathbb{G}$, we have $\operatorname{cl}\mathbb{G}_i \subseteq \operatorname{cl}(\mathbb{G}) \setminus \bigcup_{l\neq i}\mathbb{G}_l$ and hence $\operatorname{cl}(\mathbb{G}_i)\setminus\mathbb{G}_i \subseteq \operatorname{cl}(\mathbb{G})\setminus\bigcup_{1\leq l\leq r}\mathbb{G}_l = D$. Conversely, by (6.10), the claim, and lemma 6.2 we have $D = \bigcap_{n<\omega}\operatorname{cl}O_n = \langle d\rangle\mathbb{I}_{s+i} \subseteq \langle d\rangle\mathbb{G}_i \subseteq \operatorname{cl}\mathbb{G}_i$. By definition, $D \cap \mathbb{G}_i = \emptyset$. So $D \subseteq \operatorname{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i$, as required. □

COROLLARY 6.9. *Let $\mathbb{U}$ be an open subspace of a dense-in-itself metric space X, and suppose that $\mathbb{S}_0, \mathbb{S}_1$ are open subsets of $\mathbb{U}$ such that $\mathbb{U}\cap\operatorname{cl}\mathbb{S}_0\cap\operatorname{cl}\mathbb{S}_1 = \emptyset$ and $\mathbb{T} = \mathbb{U}\setminus\operatorname{cl}(\mathbb{S}_0\cup\mathbb{S}_1) \neq \emptyset$. Then there are regular open subsets $\mathbb{U}_0, \mathbb{U}_1$ of $\mathbb{U}$ such that $\mathbb{U}\cap\operatorname{cl}\mathbb{U}_0\cap\operatorname{cl}\mathbb{U}_1 = \emptyset$, and for each $i = 0, 1$:*

1. *$\mathbb{U}\cap\operatorname{cl}\mathbb{S}_i \subseteq \mathbb{U}_i$,*
2. *writing $\mathbb{T}_i = \mathbb{U}_i \setminus \operatorname{cl}\mathbb{S}_i$, we have $\mathbb{T}_i \neq \emptyset$ and $\operatorname{cl}(\mathbb{T}) \setminus \mathbb{U} \subseteq \operatorname{cl}\mathbb{T}_i$.*

Proof. Since $\mathbb{T}$ is a non-empty open subset of $\mathbb{U}$, we can use theorem 6.8 to choose disjoint non-empty subsets $\mathbb{I}_0, \mathbb{I}_1 \subseteq \mathbb{T}$ such that $\langle d\rangle\mathbb{I}_0 = \langle d\rangle\mathbb{I}_1 = \operatorname{cl}(\mathbb{T}) \setminus \mathbb{U}$.

We now work in the subspace $\mathbb{U}$. Recall that $\operatorname{cl}_{\mathbb{U}}$ denotes the closure operator in the subspace topology on $\mathbb{U}$, so $\operatorname{cl}_{\mathbb{U}} K = \mathbb{U}\cap\operatorname{cl}K$ for subsets $K \subseteq \mathbb{U}$. The sets

$$\operatorname{cl}_{\mathbb{U}}\mathbb{S}_0,\ \operatorname{cl}_{\mathbb{U}}\mathbb{S}_1,\ \mathbb{I}_0,\ \mathbb{I}_1$$

are pairwise disjoint (by assumptions) and closed in $\mathbb{U}$. (Each $\mathbb{I}_i$ is closed in $\mathbb{U}$ because by lemma 6.2, $\operatorname{cl}_{\mathbb{U}}\mathbb{I}_i = \mathbb{U}\cap\operatorname{cl}\mathbb{I}_i = \mathbb{U}\cap(\mathbb{I}_i\cup\langle d\rangle\mathbb{I}_i) = \mathbb{U}\cap(\mathbb{I}_i\cup(\operatorname{cl}(\mathbb{T})\setminus\mathbb{U})) = \mathbb{U}\cap\mathbb{I}_i = \mathbb{I}_i$.) Hence, $\mathbb{I}_0\cup\operatorname{cl}_{\mathbb{U}}\mathbb{S}_0$ and $\mathbb{I}_1\cup\operatorname{cl}_{\mathbb{U}}\mathbb{S}_1$ are disjoint closed subsets of $\mathbb{U}$. The subspace $\mathbb{U}$ is a metric space in its own right, and so, by lemma 6.7, normal. Using lemma 6.6 in $\mathbb{U}$, we can find regular open subsets $\mathbb{U}_0, \mathbb{U}_1$ of $\mathbb{U}$ with

$$\mathbb{I}_i\cup\operatorname{cl}_{\mathbb{U}}\mathbb{S}_i \subseteq \mathbb{U}_i \subseteq \mathbb{U} \quad \text{for } i = 0, 1, \tag{6.11}$$

and $\operatorname{cl}_{\mathbb{U}}\mathbb{U}_0\cap\operatorname{cl}_{\mathbb{U}}\mathbb{U}_1 = \emptyset$. Working back in X again, this says that

$$\mathbb{U}\cap\operatorname{cl}\mathbb{U}_0\cap\operatorname{cl}\mathbb{U}_1 = \emptyset. \tag{6.12}$$

Now for each $i = 0, 1$, write $\mathbb{T}_i = \mathbb{U}_i \setminus \operatorname{cl}\mathbb{S}_i$. By definition, $\mathbb{I}_i \subseteq \mathbb{U}_i$. Also, $\mathbb{I}_i\cap(\mathbb{U}\cap\operatorname{cl}\mathbb{S}_i) = \emptyset$, and since $\mathbb{I}_i \subseteq \mathbb{U}$, this gives $\mathbb{I}_i\cap\operatorname{cl}\mathbb{S}_i = \emptyset$. Hence, $\mathbb{I}_i \subseteq \mathbb{T}_i$, so $\mathbb{T}_i \neq \emptyset$. We now obtain

$$\operatorname{cl}(\mathbb{T})\setminus\mathbb{U} = \langle d\rangle\mathbb{I}_i \subseteq \operatorname{cl}\mathbb{I}_i \subseteq \operatorname{cl}\mathbb{T}_i. \tag{6.13}$$

Lines (6.11), (6.12), and (6.13), together with $\mathbb{T}_i \neq \emptyset$, establish the corollary. □

7 Representations of frames over topological spaces

Our next aim is to use the results of the preceding section to construct a 'representation' from an arbitrary dense-in-itself metric space to any given finite connected locally connected KD4 Kripke frame. The notion of representation is chosen so as to preserve $\mathcal{L}^{\mu}_{[d]\forall}$-formulas, and this will allow us to prove completeness theorems in the next two sections.

Until the end of section 7.6, we fix a topological space X and a finite Kripke frame $\mathcal{F} = (W, R)$. We will frequently regard the elements of W as propositional atoms.

7.1 Representations

The following definition seems to originate with Shehtman: see equation (71) in [30, §5, p.25].

DEFINITION 7.1. A map $\rho : X \to W$ is said to be a *representation of $\mathcal{F}$ over X* if for every $x \in X$ and $w \in W$ we have

$$(X, \rho^{-1}), x \models \langle d \rangle w \iff R(\rho(x), w).$$

Here, ρ^{-1} assigns an atom $w \in W$ to the possibly empty subset $\{x \in X : \rho(x) = w\}$ of X. The condition says that for every $x \in X$, the set of points of W with preimages under ρ in every open neighbourhood of x but distinct from x itself is precisely $R(\rho(x))$. Equivalently, $\langle d \rangle \rho^{-1}(w) = \rho^{-1}(R^{-1}(w))$ for every $w \in W$, where R^{-1} is the converse relation of R.

Note that ρ need not be surjective. Indeed, the empty map is vacuously a representation of $\mathcal{F}$ over the empty space — and we definitely do allow empty representations.

It can be checked that if $\rho : X \to W$ is a representation then $R \restriction \operatorname{rng} \rho$ is transitive. Endow W with the topology generated by $\{R(w) : w \in W\}$ (so the open sets are those $A \subseteq W$ such that $a \in A$ implies $R(a) \subseteq A$). Then every representation of $\mathcal{F}$ over X is an interior map from X to W: that is, a map that is both continuous and open. (Many other topological completeness proofs use interior maps.) The converse, however, does not hold in general. See [2, 22] for more information.

Although Shehtman uses the term 'd-p-morphism' (when ρ is surjective), here we will call ρ a 'representation' because it is closely related to the representations of algebras of relations seen in algebraic logic. Indeed, if ρ is a surjective representation of (W, R) over X then ρ^{-1} induces an embedding from $\wp(W)$ into $\wp(X)$ that preserves the algebraic structure with which these power sets can be naturally endowed.

7.2 Representations over subspaces

Our main interest is in representations over X itself, but representations over subspaces are also useful in proofs. Given a subspace U of X, a map $\rho : U \to W$ induces a well defined assignment $\rho^{-1} : W \to \wp(X)$ by $\rho^{-1}(w) = \{x \in X : x \in U \text{ and } \rho(x) = w\}$, for $w \in W$. Put simply, preimages under ρ of elements of W are obviously subsets of U, but they are also subsets of X, and so ρ^{-1} can be regarded equally as an assignment into U or X, as appropriate. The following easy lemma gives some connections between the two views.

LEMMA 7.2. *Let U be a subspace of X and let $\rho : U \to W$ be a map. Let $x \in U$ and $w \in W$ be arbitrary.*

1. *If $(U, \rho^{-1}), x \models \langle d \rangle w$ then $(X, \rho^{-1}), x \models \langle d \rangle w$.*
2. *If U is open in X, then $(U, \rho^{-1}), x \models \langle d \rangle w$ iff $(X, \rho^{-1}), x \models \langle d \rangle w$.*

Proof. For the first part, assume that $(U, \rho^{-1}), x \models \langle d \rangle w$ and let O be any open neighbourhood of x in X. Then $O \cap U$ is an open neighbourhood of x in U, so by assumption, there is $y \in O \cap U \setminus \{x\}$ with $(U, \rho^{-1}), y \models w$. Then $y \in O \setminus \{x\}$ and $(X, \rho^{-1}), y \models w$. Hence, $(X, \rho^{-1}), x \models \langle d \rangle w$.

For the second part, assume that $(X, \rho^{-1}), x \models \langle d \rangle w$. Let N be an arbitrary open neighbourhood of x in U, so that $N = O \cap U$ for some open neighbourhood O of x in X. As U is assumed open in X, we see that N is also open in X, so by assumption, there is $y \in N \setminus \{x\}$

with $(X, \rho^{-1}), y \models w$. Plainly, $(U, \rho^{-1}), y \models w$. This shows that $(U, \rho^{-1}), x \models \langle d \rangle w$, and the converse follows from the first part. $\square$

By part 2 of the lemma, if ρ is a representation of $\mathcal{F}$ over an open subspace U of X, then $(X, \rho^{-1}), x \models \langle d \rangle w$ iff $R(\rho(x), w)$ for every $x \in U$ and $w \in W$. So we can work in (X, ρ^{-1}) instead of (U, ρ^{-1}). To avoid too much jumping around between subspaces, we will do this below, often without mention. Part 3 of the next lemma makes it a little more explicit. The lemma gives some general information on how representations of different generated subframes of $\mathcal{F}$ over different subspaces of X are related.

LEMMA 7.3. *Let $\mathcal{G} = (W', R')$ be a generated subframe of $\mathcal{F}$. Let T, U, and U_i ($i \in I$) be open subspaces of X, with $T \subseteq U = \bigcup_{i \in I} U_i$. Finally, let $\rho : U \to W'$ be a map. Then:*

1. *ρ is a representation of $\mathcal{F}$ over U iff it is a representation of $\mathcal{G}$ over U.*
2. *ρ is a representation of $\mathcal{F}$ over U iff for each $i \in I$, the restriction $\rho \restriction U_i$ is a representation of $\mathcal{F}$ over U_i.*
3. *If $\rho \restriction T$ is a representation of $\mathcal{F}$ over T, then $(X, \rho^{-1}), x \models \langle d \rangle w$ iff $R(\rho(x), w)$, for each $x \in T$ and $w \in W$.*

Proof. Simple. $\square$

7.3 Representations preserve formulas

Here, we will show that surjective representations preserve all formulas of $\mathcal{L}^{\mu}_{[d]\forall}$. Since representations are like p-morphisms, albeit between different kinds of structure, this is entirely expected and the proof is essentially quite standard — see [30, lemma 20] and [2, corollary 2.9], for example. We do need, however, that $\mathcal{F}$ is finite. We will be able to handle larger sublanguages of $\mathcal{L}^{\mu\langle t \rangle\langle dt \rangle}_{\square[d]\forall}$ by using the translations of section 4.

Let us explain the setting. Suppose we are given a representation $\rho : X \to W$ of $\mathcal{F}$ over X. Recall that Var is our fixed base set of propositional variables, or atoms. For each assignment $h : \mathsf{Var} \to \wp(W)$ of atoms in Var into W, the map $\rho^{-1} \circ h : \mathsf{Var} \to \wp(X)$ is an assignment of atoms into X, given of course by

$$(\rho^{-1} \circ h)(p) = \{x \in X : \rho(x) \in h(p)\}, \qquad \text{for each } p \in \mathsf{Var}.$$

So ρ, or rather ρ^{-1}, gives us a way to transform an assignment into $\mathcal{F}$ to one into X, and then to evaluate a formula in the resulting model on X. The following definition encapsulates when we get the same result as in the original model on $\mathcal{F}$:

DEFINITION 7.4. Let $\rho : X \to W$ be a map, and let φ be a formula of $\mathcal{L}^{\mu\langle t \rangle\langle dt \rangle}_{\square[d]}$. We say that ρ *preserves* φ if for every assignment $h : \mathsf{Var} \to \wp(W)$ and every $x \in X$,

$$(X, \rho^{-1} \circ h), x \models \varphi \quad \text{iff} \quad (W, R, h), \rho(x) \models \varphi. \tag{7.1}$$

We are now ready for our main preservation result.

PROPOSITION 7.5. *Let $\rho : X \to W$ be a surjective representation of $\mathcal{F}$ over X. Then ρ preserves every formula of $\mathcal{L}^{\mu}_{[d]\forall}$.*

Proof. The proof is by induction on φ. The atomic and boolean cases are easy and left to the reader. Let φ be a formula, and inductively assume (7.1) for every assignment $h : \mathsf{Var} \to \wp(W)$ and every $x \in X$. It is sufficient to consider the cases $\langle d \rangle\varphi$, $\forall\varphi$, and $\mu q\varphi$.

First, consider $\langle d \rangle\varphi$. Fix h, x. Suppose that $(W, R, h), \rho(x) \models \langle d \rangle\varphi$. Choose $w \in R(\rho(x))$ with $(W, R, h), w \models \varphi$. As ρ is a representation, $(X, \rho^{-1}), x \models \langle d \rangle w$. So for every open neighbourhood O of x, there is $y \in O \setminus \{x\}$ with $\rho(y) = w$. Since $(W, R, h), w \models \varphi$, for any such y we inductively have $(X, \rho^{-1} \circ h), y \models \varphi$. It follows that $(X, \rho^{-1} \circ h), x \models \langle d \rangle\varphi$.

Conversely, suppose that $(X, \rho^{-1} \circ h), x \models \langle d \rangle\varphi$. Let $[\![\varphi]\!] = \{y \in X : (X, \rho^{-1} \circ h), y \models \varphi\}$. As $\mathcal{F}$ is finite and $\langle d \rangle$ is additive (lemma 6.2(2)), we have

$$\begin{aligned} x \in \langle d \rangle [\![\varphi]\!] = \langle d \rangle ([\![\varphi]\!] \cap X) &= \langle d \rangle \Big([\![\varphi]\!] \cap \bigcup_{w \in W} \rho^{-1}(w) \Big) \\ &= \langle d \rangle \Big(\bigcup_{w \in W} ([\![\varphi]\!] \cap \rho^{-1}(w)) \Big) = \bigcup_{w \in W} \langle d \rangle ([\![\varphi]\!] \cap \rho^{-1}(w)). \end{aligned}$$

So we can take $w \in W$ with $x \in \langle d \rangle([\![\varphi]\!] \cap \rho^{-1}(w))$. Then $(X, \rho^{-1}), x \models \langle d \rangle w$, so as ρ is a representation, $R(\rho(x), w)$. Moreover, $[\![\varphi]\!] \cap \rho^{-1}(w) \neq \emptyset$. Take any $y \in [\![\varphi]\!] \cap \rho^{-1}(w)$. Then $(X, \rho^{-1} \circ h), y \models \varphi$ and $\rho(y) = w$. Inductively, $(W, R, h), w \models \varphi$. By Kripke semantics, $(W, R, h), \rho(x) \models \langle d \rangle\varphi$, as required.

Next, consider $\forall\varphi$. Then $(X, \rho^{-1} \circ h), x \models \forall\varphi$ iff $(X, \rho^{-1} \circ h), y \models \varphi$ for all $y \in X$, iff $(W, R, h), \rho(y) \models \varphi$ for all $y \in X$ (by the inductive hypothesis (7.1)), iff $(W, R, h), w \models \varphi$ for all $w \in W$ (since ρ is surjective), iff $(W, R, h), \rho(x) \models \forall\varphi$.

Finally consider the case $\mu q\varphi$, assumed well formed. Fix arbitrary $h : \mathsf{Var} \to \wp(W)$. We define an assignment $h^\alpha : \mathsf{Var} \to \wp(W)$ for each ordinal α. For each atom $p \neq q$, we set $h^\alpha(p) = h(p)$. We define $h^\alpha(q)$ by induction on α as follows:

- $h^0(q) = \emptyset$,
- $h^{\alpha+1}(q) = \{w \in W : (W, R, h^\alpha), w \models \varphi\}$,
- $h^\delta(q) = \bigcup_{\alpha<\delta} h^\alpha(q)$ for limit ordinals δ.

Of course, W is finite, but we need all ordinals for the argument below. Let $\eta = \rho^{-1} \circ h : \mathsf{Var} \to \wp(X)$. Define an assignment $\eta^\alpha : \mathsf{Var} \to \wp(X)$ in the same way as for h^α: let $\eta^\alpha(p) = \eta(p)$ for all atoms $p \neq q$ and all α, and

- $\eta^0(q) = \emptyset$,
- $\eta^{\alpha+1}(q) = \{x \in X : (X, \eta^\alpha), x \models \varphi\}$,
- $\eta^\delta(q) = \bigcup_{\alpha<\delta} \eta^\alpha(q)$ for limit ordinals δ.

Claim. $\eta^\alpha(q) = \rho^{-1}(h^\alpha(q))$ for each ordinal α.
Proof of claim. By induction on α. For $\alpha = 0$ this is saying that $\rho^{-1}(\emptyset) = \emptyset$, which is true. Assume the result for α inductively. So $\eta^\alpha = \rho^{-1} \circ h^\alpha$. We now obtain

$$\begin{array}{rcll} \eta^{\alpha+1}(q) & = & \{x \in X : (X, \eta^\alpha), x \models \varphi\} & \text{by definition of } \eta^{\alpha+1} \\ & = & \{x \in X : (X, \rho^{-1} \circ h^\alpha), x \models \varphi\} & \text{since } \eta^\alpha = \rho^{-1} \circ h^\alpha \\ & = & \{x \in X : (W, R, h^\alpha), \rho(x) \models \varphi\} & \text{by inductive hypothesis (7.1)} \\ & = & \{x \in X : \rho(x) \in h^{\alpha+1}(q)\} & \text{by definition of } h^{\alpha+1} \\ & = & \rho^{-1}(h^{\alpha+1}(q)). & \end{array}$$

For limit δ we have

$$\rho^{-1}(h^\delta(q)) = \rho^{-1}(\bigcup_{\alpha<\delta} h^\alpha(q)) = \bigcup_{\alpha<\delta} \rho^{-1}(h^\alpha(q)) =_{IH} \bigcup_{\alpha<\delta} \eta^\alpha(q) = \eta^\delta(q).$$

This completes the induction on α, and proves the claim.

By semantics of μ, we have $(X,\eta), x \models \mu q\varphi$ iff $x \in \bigcup_{\alpha\in\mathrm{On}} \eta^\alpha(q)$, iff $x \in \bigcup_\alpha \rho^{-1}(h^\alpha(q))$ by the claim, iff $\rho(x) \in \bigcup_\alpha h^\alpha(q)$, iff $(W,R,h), \rho(x) \models \mu q\varphi$. This completes the induction and proves the proposition. □

7.4 Basic representations

Certain very primitive representations called *basic representations* will play an important role later, because they can easily be extended to more interesting representations.

DEFINITION 7.6. Let S, U be open subspaces of X, with $S \subseteq U$, and let $\sigma : S \to W$ be a representation of $\mathcal{F}$ over S. We say that σ is *U-basic* if for every $x \in U$ and $w, v \in W$, if $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$ then Rwv.

Note that we use $\Diamond$ and not $\langle d\rangle$ here.

REMARK 7.7. In the setting of this definition:

1. Vacuously, if σ is empty then it is U-basic.
2. More generally, but equally trivially, if $\operatorname{rng}\sigma$ is contained in a nondegenerate cluster C in $\mathcal{F}$, then σ is U-basic. For, $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$ implies that $w, v \in \operatorname{rng}\sigma \subseteq C$, and so Rwv as C is a nondegenerate cluster.

We remark (but will not formally use) that σ is U-basic iff $\operatorname{rng}\sigma$ is a (possibly empty) union of R-maximal clusters in $\mathcal{F}$ whose preimages under σ have pairwise disjoint closures within U. Moreover, each such preimage is a regular open subset of S.

7.5 Full representations

In induction proofs, we often need a stronger inductive hypothesis than formally required for the final result. This will be the case in proposition 7.10 below, and the notion of *T-full* representation will be used to formulate it.

DEFINITION 7.8. Let $T \subseteq U$ be open subspaces of X. A representation $\rho : U \to W$ of $\mathcal{F}$ over U is said to be *T-full* if:

1. for every $x \in \mathrm{cl}(T) \setminus U$ and $w \in W$, we have $(X, \rho^{-1}), x \models \langle d\rangle w$,
2. if T is non-empty then $\rho : U \to W$ is surjective.

Every representation is vacuously $\emptyset$-full.

7.6 Full representability

DEFINITION 7.9. We say that $\mathcal{F}$ is *fully representable (over X)* if whenever

1. $U \subseteq X$ is open,
2. S is a regular open subset of U,
3. $\sigma : S \to W$ is a U-basic representation of $\mathcal{F}$ over S,
4. $T = U \setminus \operatorname{cl} S$,

then σ extends to a T-full representation $\rho : U \to W$ of $\mathcal{F}$ over U.

Notice that in the boolean algebra $RO(U)$ of regular open subsets of U, we have $T = -S$, so $\{S, T\}$ is a partition of 1. That is, $S, T \in RO(U)$, $S \cdot T = 0$, and $S + T = 1$.

In proposition 7.10 below, we will fulfil our main aim, to prove (surjective) representability of every finite connected locally connected KD4-frame; we are going to do it by induction on the size of the frame; we appear to need a stronger inductive hypothesis, namely full representability, than is needed for the conclusion; T-fullness and extending σ are mainly to do with this, but the σ part is also helpful in the proof of strong completeness in theorem 9.1 later. Note that if $\mathcal{F}$ is fully representable over X, and $X \neq \emptyset$, then by taking $U = X$ and $S = \sigma = \emptyset$, we see that there exists a surjective representation of $\mathcal{F}$ over X. So we do obtain our desired conclusion from the stronger hypothesis of full representability.

7.7 Main proposition

The following proposition has relatives in the literature: see, e.g., [24, theorem 3.7], [30, proposition 22], [22, lemma 4.4], and [20, lemma 6.9]. It actually holds for any dense-in-itself topological space X for which theorem 6.8 and corollary 6.9 can be proved.

PROPOSITION 7.10. *Suppose that X is a dense-in-itself metric space. Then every finite connected locally connected* KD4 *frame $\mathcal{F} = (W, R)$ is fully representable over X.*

Proof. The proof is by induction on the number of worlds in $\mathcal{F}$. Let $\mathcal{F} = (W, R)$ be a finite connected locally connected KD4 frame, and assume the result inductively for all smaller frames. Note that R is transitive. Recall that we write

- $R^\circ = \{(w, v) \in W^2 : Rwv \wedge Rvw\}$,
- $R^\bullet = \{(w, v) \in W^2 : Rwv \wedge \neg Rvw\}$,

and for $w \in W$,

- $\mathcal{F}(w)$ for the subframe $(R(w), R \restriction R(w))$ of $\mathcal{F}$ with domain $R(w)$,
- $\mathcal{F}^*(w)$ for the subframe $(R^*(w), R \restriction R^*(w)) = (R(w) \cup \{w\}, R \restriction R(w) \cup \{w\})$ of $\mathcal{F}$ generated by w.

Let $U \subseteq X$ be open, let S be a regular open subset of U, and let $\sigma : S \to W$ be a U-basic representation of $\mathcal{F}$ over S. Write

$$T = U \setminus \operatorname{cl} S.$$

We need to extend σ to a T-full representation $\rho : U \to W$ of $\mathcal{F}$ over U.

If $T = \emptyset$, then $U \subseteq \operatorname{cl} S$, so $S = \operatorname{int}(U \cap \operatorname{cl} S) = \operatorname{int} U = U$. Thus, $\sigma : S \to W$ is already a representation of $\mathcal{F}$ over U, and it is vacuously T-full. So we can take $\rho = \sigma$. We are done.

So assume from now on that $T \neq \emptyset$. There are three cases.

Case 1: $\mathcal{F} = \mathcal{F}^*(w_0)$ for some reflexive $w_0 \in W$ Choose such a w_0 (it may not be unique). Then $R(w_0) = W$ and $w_0 \in R^\circ(w_0)$ since w_0 is reflexive. So $R^\circ(w_0) \neq \emptyset$. Since T is clearly a non-empty open set, we can use theorem 6.8(2) to partition T into non-empty open sets $G_{v^\bullet}$ ($v^\bullet \in R^\bullet(w_0)$) and other non-empty sets B_{v° ($v^\circ \in R^\circ(w_0)$) such that for each $v^\bullet \in R^\bullet(w_0)$ and $v^\circ \in R^\circ(w_0)$ we have

$$\mathrm{cl}(G_{v^\bullet}) \setminus G_{v^\bullet} = \langle d\rangle B_{v^\circ} = \mathrm{cl}(T) \setminus \bigcup_{v \in R^\bullet(w_0)} G_v = D, \text{ say.} \tag{7.2}$$

For each $v^\bullet \in R^\bullet(w_0)$, the frame $\mathcal{F}^*(v^\bullet)$ is connected (as it is rooted) and locally connected KD4 (as it is a generated subframe of $\mathcal{F}$). Since w_0 is a world of $\mathcal{F}$ but not of $\mathcal{F}^*(v^\bullet)$, the frame $\mathcal{F}^*(v^\bullet)$ is smaller than $\mathcal{F}$. By the inductive hypothesis, $\mathcal{F}^*(v^\bullet)$ is fully representable over X. So, taking the regular open subset 'S' of $G_{v^\bullet}$ to be $\emptyset$ and 'T' to be $G_{v^\bullet} \setminus \mathrm{cl}\,\emptyset = G_{v^\bullet}$, we can find a $G_{v^\bullet}$-full representation $\rho_{v^\bullet}$ of $\mathcal{F}^*(v^\bullet)$ over $G_{v^\bullet}$.

Define $\rho : U \to W$ by:

$$\rho(x) = \begin{cases} \rho_{v^\bullet}(x), & \text{if } x \in G_{v^\bullet} \text{ for some (unique) } v^\bullet \in R^\bullet(w_0), \\ v^\circ, & \text{if } x \in B_{v^\circ} \text{ for some (unique) } v^\circ \in R^\circ(w_0), \\ \sigma(x), & \text{if } x \in S, \\ w_0, & \text{otherwise,} \end{cases}$$

for each $x \in U$. The map ρ is well defined because the $G_{v^\bullet}$, the B_{v°, and S are pairwise disjoint, and plainly it is total and extends σ.

We aim to show that ρ is a T-full representation of $\mathcal{F}$ over U. The following claim will help.

Claim. Let $x \in D$ (see (7.2)). Then $(X, \rho^{-1}), x \models \langle d\rangle w$ for every $w \in W$.
Proof of claim. Let $x \in D$ and $w \in W$ be given. There are two cases. The first is when $w \in R^\bullet(w_0)$. Now (7.2) gives $x \in \mathrm{cl}\, G_w \setminus G_w$. As ρ_w is a G_w-full representation of $\mathcal{F}^*(w)$, a frame of which w is a world, we have $(X, \rho_w^{-1}), x \models \langle d\rangle w$, and hence $(X, \rho^{-1}), x \models \langle d\rangle w$ (since $\rho_w \subseteq \rho$).

The second case is when $w \notin R^\bullet(w_0)$. Since $w \in W = R(w_0) = R^\bullet(w_0) \cup R^\circ(w_0)$, we have $w \in R^\circ(w_0)$. By (7.2), $x \in \langle d\rangle B_w$ (since $x \in D$). Since $\rho \restriction B_w$ has constant value w, we obtain again that $(X, \rho^{-1}), x \models \langle d\rangle w$. This proves the claim.

We now check that ρ is a representation of $\mathcal{F}$ over U. Let $x \in U$ and $w \in W$. We require $(X, \rho^{-1}), x \models \langle d\rangle w$ iff $R(\rho(x), w)$. There are four cases.

1. Suppose that $x \in G_{v^\bullet}$ for some $v^\bullet \in R^\bullet(w_0)$. Since $G_{v^\bullet}$ is open and $\rho \restriction G_{v^\bullet} = \rho_{v^\bullet}$, a representation over $G_{v^\bullet}$ of the generated subframe $\mathcal{F}^*(v^\bullet)$ of $\mathcal{F}$, lemma 7.3 yields $(X, \rho^{-1}), x \models \langle d\rangle w$ iff $R(\rho(x), w)$.

2. Suppose that $x \in B_{v^\circ}$ for some $v^\circ \in R^\circ(w_0)$. Then $\rho(x) = v^\circ$. As $v^\circ \in R^\circ(w_0)$, we have $Rv^\circ w_0$. By transitivity of R, we have $R(\rho(x), w)$ for every $w \in W$. So we need to prove that $(X, \rho^{-1}), x \models \langle d\rangle w$ for every $w \in W$. But $x \in B_{v^\circ} \subseteq D$ by definition of D (7.2), so this follows from the claim.

3. If $x \in S$, then since S is open and $\rho \restriction S = \sigma$, a representation of $\mathcal{F}$ over S, the result follows from lemma 7.3 again.

4. Suppose finally that $x \in U \setminus (S \cup T)$. Then $\rho(x) = w_0$. Since $R(w_0, w)$ for all $w \in W$, we require that $(X, \rho^{-1}), x \models \langle d \rangle w$ for all $w \in W$ as well.

   Now as S is a regular open subset of U, by lemma 6.4 we obtain $U \setminus S = \operatorname{cl} T$. Hence, $x \in \operatorname{cl} T \setminus T \subseteq D$ by (7.2). As in case 2, the claim now gives $(X, \rho^{-1}), x \models \langle d \rangle w$ for all $w \in W$.

So ρ is indeed a representation of $\mathcal{F}$ over U. We check that it is T-full. First let $x \in \operatorname{cl} T \setminus U$. Then $x \in \operatorname{cl} T \setminus T \subseteq D$ by (7.2). By the claim, $(X, \rho^{-1}), x \models \langle d \rangle w$ for every $w \in W$, as required.

We also need that ρ is surjective. Take any $x \in B_{w_0}$. Then $x \in D$ by definition of D in (7.2). By the claim, $(X, \rho^{-1}), x \models \langle d \rangle w$, and so $\rho^{-1}(w) \neq \emptyset$, for every $w \in W$. Hence, ρ is surjective.

Case 2: $\boldsymbol{\mathcal{F} = \mathcal{F}^*(w_0)}$ for some irreflexive $\boldsymbol{w_0 \in W}$ Choose such a w_0 (it is unique this time). Then W is the disjoint union of $\{w_0\}$ and $R(w_0)$. Using theorem 6.8(1), select non-empty $I \subseteq T$ with

$$\langle d \rangle I = \operatorname{cl} T \setminus U. \tag{7.3}$$

Write

$$\begin{array}{rcl} U' & = & U \setminus I, \\ T' & = & T \setminus I. \end{array}$$

We aim to use the inductive hypothesis on these sets and $\sigma : S \to \mathcal{F}(w_0)$, so we check the necessary conditions.

Claim 1. U' is open, S is a regular open subset of U', and $T' = U' \setminus \operatorname{cl} S$.
Proof of claim. First, U' is open. For, by lemma 6.2 and (7.3),

$$U \setminus \operatorname{cl} I = U \setminus (I \cup \langle d \rangle I) = U \setminus (I \cup (\operatorname{cl}(T) \setminus U)) = U \setminus I = U',$$

and the left-hand side is open.

We are given that S is a regular open subset of U. Since $S \subseteq U$ and $I \subseteq T = U \setminus \operatorname{cl} S$, we have $S \subseteq U \setminus I = U'$. By lemma 6.4(3), S is a regular open subset of U'.

Finally, $U' \setminus \operatorname{cl} S = (U \setminus I) \setminus \operatorname{cl} S = (U \setminus \operatorname{cl} S) \setminus I = T \setminus I = T'$. This proves the claim.

Claim 2. σ is a U'-basic representation of $\mathcal{F}(w_0)$ over S.
Proof of claim. First we show that $\sigma : S \to R(w_0)$. We know that $\sigma : S \to W = \{w_0\} \cup R(w_0)$. Assume for contradiction that there is some $x \in S$ with $\sigma(x) = w_0$. Then plainly, $x \in U$ and $(X, \sigma^{-1}), x \models \Diamond w_0$. As σ is a U-basic representation of $\mathcal{F}$ over S, we obtain $R w_0 w_0$, contradicting the choice of w_0 as irreflexive. So indeed, $\operatorname{rng} \sigma \subseteq W \setminus \{w_0\} = R(w_0)$. Since σ is a representation of $\mathcal{F}$ over S, by lemma 7.3 it is also a representation (over S) of the generated subframe $\mathcal{F}(w_0)$ of $\mathcal{F}$. It is trivially U'-basic, since if $x \in U'$, $w, v \in R(w_0)$, and $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$, then $x \in U$ and $w, v \in W$ as well, so Rwv since σ is U-basic. This proves the claim.

In summary, U' is open, S is a regular open subset of U', σ is a U'-basic representation of $\mathcal{F}(w_0)$ over S, and $T' = U' \setminus \operatorname{cl} S$.

Now $\mathcal{F}(w_0)$ is smaller than $\mathcal{F}$ (since $w_0 \notin R(w_0)$), connected (since $\mathcal{F}$ is locally connected), and locally connected KD4 (since it is a generated subframe of $\mathcal{F}$). By the inductive hypothesis, $\mathcal{F}(w_0)$ is fully representable over X.

So σ extends to a T'-full representation $\rho' : U' \to R(w_0)$ of $\mathcal{F}(w_0)$ over U'. By T'-fullness,

$$(X, \rho'^{-1}), x \models \langle d \rangle v \text{ for every } v \in R(w_0) \text{ and } x \in \operatorname{cl} T' \setminus U'. \tag{7.4}$$

We extend ρ' to a map $\rho : U \to W$ by defining

$$\rho(x) = \begin{cases} \rho'(x), & \text{if } x \in U', \\ w_0, & \text{if } x \in I, \end{cases}$$

for $x \in U$. This is plainly well defined and total. Since ρ extends ρ', it also extends σ. We will show that ρ is a T-full representation of $\mathcal{F}$ over U. To do it, we need another claim.

Claim 3. $\operatorname{cl} T \setminus U \subseteq \operatorname{cl} I \subseteq \operatorname{cl} T' \setminus U'$.
Proof of claim. By (7.3) and lemma 6.2, we have $\operatorname{cl} T \setminus U = \langle d \rangle I \subseteq \operatorname{cl} I$.

Using openness of $T = T' \cup I$, the assumption that X is dense in itself, and lemma 6.2(3,2), we have $I \subseteq T \subseteq \operatorname{cl} T = \langle d \rangle T = \langle d \rangle T' \cup \langle d \rangle I$. But by (7.3), $I \cap \langle d \rangle I \subseteq U \cap \operatorname{cl} T \setminus U = \emptyset$. So in fact, $I \subseteq \langle d \rangle T' \subseteq \operatorname{cl} T'$. Hence, $\operatorname{cl} I \subseteq \operatorname{cl} T'$. Since $I \cap U' = \emptyset$ and U' is open (claim 1), we have $\operatorname{cl} I \cap U' = \emptyset$. So $\operatorname{cl} I \subseteq \operatorname{cl} T' \setminus U'$, proving the claim.

Claim 4. ρ is a representation of $\mathcal{F}$ over U.
Proof of claim. Let $x \in U$. We require $(X, \rho^{-1}), x \models \langle d \rangle w$ iff $R(\rho(x), w)$, for each $w \in W$.

There are two cases here. The first is when $x \in I$. Then $\rho(x) = w_0$, so we require first that $(X, \rho^{-1}), x \models \langle d \rangle w$ for each $w \in R(w_0)$. So pick any $w \in R(w_0)$. By claim 3, $x \in I \subseteq \operatorname{cl} I \subseteq \operatorname{cl} T' \setminus U'$, so by (7.4), $(X, \rho'^{-1}), x \models \langle d \rangle w$. As $\rho' \subseteq \rho$, the result follows.

We also require that $(X, \rho^{-1}), x \not\models \langle d \rangle w$ for each $w \in W \setminus R(w_0)$ — that is, $(X, \rho^{-1}), x \not\models \langle d \rangle w_0$. But as $x \in U$, we have $x \notin \operatorname{cl} T \setminus U = \langle d \rangle I$ by (7.3). Since $\rho^{-1}(w_0) = I$, we do indeed have $(X, \rho^{-1}), x \not\models \langle d \rangle w_0$.

The second case is when $x \notin I$. In this case, $x \in U'$, an open set, and $\rho \restriction U' = \rho'$, a representation over U' of the generated subframe $\mathcal{F}(w_0)$ of $\mathcal{F}$. By lemma 7.3, $(X, \rho^{-1}), x \models \langle d \rangle w$ iff $R(\rho(x), w)$ for every $w \in W$, as required. The claim is proved.

Claim 5. ρ is T-full.
Proof of claim. Let $x \in \operatorname{cl} T \setminus U$ and $w \in W$. We require $(X, \rho^{-1}), x \models \langle d \rangle w$.

Suppose first that $w = w_0$. By (7.3), $x \in \langle d \rangle I$. Since $I = \rho^{-1}(w_0)$, we obtain $(X, \rho^{-1}), x \models \langle d \rangle w_0$. Suppose instead that $w \in R(w_0)$. By claim 3, $x \in \operatorname{cl} T' \setminus U'$. So by (7.4), $(X, \rho'^{-1}), x \models \langle d \rangle w$. As $\rho' \subseteq \rho$, we obtain $(X, \rho^{-1}), x \models \langle d \rangle w$ as required.

We must also show that $\rho(U) = W$. Well, $I \neq \emptyset$, and it follows from claim 3 that $T' \neq \emptyset$ as well. As ρ' is T'-full, $\rho'(U') = R(w_0)$. So

$$\rho(U) = \rho'(U') \cup \rho(I) = R(w_0) \cup \{w_0\} = W,$$

as required. This proves the claim and completes case 2 of proposition 7.10. Only case 3 remains, but this is the hardest case.

Case 3: otherwise As $\mathcal{F}$ is finite and connected, we can choose worlds $a_0, b_0, a_1, b_1, \ldots, b_{n-1}, a_n \in W$, for some least possible $n < \omega$, such that Ra_ib_i and $Ra_{i+1}b_i$ for each $i < n$, each b_i is R-maximal (so that $R^\bullet(b_i) = \emptyset$), and $W = \bigcup_{i \leq n} R^*(a_i)$. By the case assumption, $n \geq 1$.

Write $\mathcal{F}^*(a_0)$ as $\mathcal{F}_0 = (W_0, R_0)$, say. Let $\mathcal{F}_1 = (W_1, R_1)$ be the smallest generated subframe of $\mathcal{F}$ containing $a_1, \ldots, a_n$. We have $W_0 \cup W_1 = W$ and $b_0 \in W_0 \cap W_1$. Plainly, $\mathcal{F}_0$ and $\mathcal{F}_1$ are connected generated subframes of $\mathcal{F}$. Therefore, they are locally connected KD4 frames. By minimality of n, they are proper subframes of $\mathcal{F}$. By the inductive hypothesis, $\mathcal{F}_0$ and $\mathcal{F}_1$ are fully representable over X. Our plan is to combine suitable representations of them to give a representation of $\mathcal{F}$ over U.

Recall that S is a regular open subset of U and $\sigma : S \to W$ is a U-basic representation of $\mathcal{F}$. We use W_0, W_1 to split S (and, later, σ) in two. Let

$$\begin{array}{rcl} S_0 & = & \sigma^{-1}(W_0) = \{x \in S : \sigma(x) \in W_0\}, \\ S_1 & = & S \setminus S_0. \end{array}$$

So $\sigma(S_0) \subseteq W_0$ and $\sigma(S_1) \subseteq W \setminus W_0 \subseteq W_1$. Also, $S_0 = S \setminus S_1$.

Claim 1. S_0 and S_1 are regular open subsets of U, and $U \cap \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1) = \emptyset$.
Proof of claim. We prove the last point first. Suppose for contradiction that there is some $x \in U \cap \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1)$. As $x \in \mathrm{cl}\, S_0$, we have $(X, \sigma^{-1}), x \models \Diamond \bigvee_{w \in W_0} w$. As $\Diamond$ is additive, it follows that there is some $w_0 \in W_0$ such that $(X, \sigma^{-1}), x \models \Diamond w_0$. Similarly, as $x \in \mathrm{cl}\, S_1$ and $\sigma(S_1) \subseteq W \setminus W_0$, there is some $w_1 \in W \setminus W_0$ with $(X, \sigma^{-1}), x \models \Diamond w_1$. As σ is a U-basic representation, we obtain Rw_0w_1. Since $\mathcal{F}_0$ is a generated subframe of $\mathcal{F}$, this implies that $w_1 \in W_0$, a contradiction. So $U \cap \mathrm{cl}(S_0) \cap \mathrm{cl}(S_1) = \emptyset$ as required.

Now let $i < 2$. We show that S_i is regular open in U. First note that S_i is open. To see this, observe that

$$\begin{array}{rcll} S_i & \subseteq & S \cap U \cap \mathrm{cl}\, S_i & \text{as } S_i \subseteq S \subseteq U \text{ by definition and assumption} \\ & \subseteq & S \cap U \setminus \mathrm{cl}\, S_{1-i} & \text{by the first part} \\ & = & S \setminus \mathrm{cl}\, S_{1-i} & \text{as } S \subseteq U \text{ by assumption} \\ & \subseteq & S \setminus S_{1-i} & \text{as } S_{1-i} \subseteq \mathrm{cl}\, S_{1-i} \\ & = & S_i & \text{by definition of } S_i. \end{array}$$

Hence, $S_i = S \setminus \mathrm{cl}\, S_{1-i}$, an open set.

It follows that $\mathrm{cl}(S_i) \cap S_{1-i} = \emptyset$, so $S_i \subseteq S \cap \mathrm{cl}\, S_i \subseteq S \setminus S_{1-i} = S_i$. Thus, $S \cap \mathrm{cl}\, S_i = S_i$, and so $\mathrm{int}(S \cap \mathrm{cl}\, S_i) = \mathrm{int}\, S_i = S_i$ as S_i is open. So S_i is regular open in S, and as S is regular open in U, lemma 6.4(4) yields that S_i is regular open in U. The claim is proved.

The claim and the assumption at the outset that $T \neq \emptyset$ are more than enough to apply corollary 6.9, to obtain open subsets U_i, T_i of U, for $i = 0, 1$, satisfying the following conditions:

C1. $U \cap \mathrm{cl}\, U_0 \cap \mathrm{cl}\, U_1 = \emptyset$,

C2. $U \cap \mathrm{cl}\, S_i \subseteq U_i$,

C3. $T_i = U_i \setminus \mathrm{cl}\, S_i \neq \emptyset$,

C4. $\mathrm{cl}(T) \setminus U \subseteq \mathrm{cl}(T_i)$,

C5. U_i is a regular open subset of U.

We now work in the boolean algebra $RO(U)$ of regular open subsets of U. By C5, we have $U_0, U_1 \in RO(U)$. We define further elements of $RO(U)$:

C6. $M = -(U_0 + U_1)$,

C7. $V_i = M + U_i$ for $i = 0, 1$.

The main property of these sets is as follows.

Claim 2. $\{M, S_0, S_1, T_0, T_1\}$ is a partition of 1 in the boolean algebra $RO(U)$. That is, the five elements are pairwise disjoint regular open subsets of U, with

$$U = \underbrace{\overbrace{S_0 + T_0}^{U_0} + M}_{V_0} + \overbrace{M + \underbrace{S_1 + T_1}_{U_1}}^{V_1}. \tag{7.5}$$

Proof of claim. Let $i < 2$. By claim 1 and condition C5 above, $S_i, U_i \in RO(U)$. By this and condition C3,

$$T_i = U_i \setminus \mathrm{cl}\, S_i = U_i \cap U \setminus \mathrm{cl}\, S_i = U_i \cdot -S_i \in RO(U). \tag{7.6}$$

So $S_i \cdot T_i = \emptyset$ and, since $S_i \subseteq U_i$ by condition C2, also $U_i = U_i \cdot S_i + U_i \cdot -S_i = S_i + T_i$. Condition C1 above gives $U_0 \cdot U_1 = \emptyset$. By definition, $M = -(U_0 + U_1)$, so $M \in RO(U)$ and M is disjoint from T_i, S_i. Also, $U = U_0 + U_1 + M = S_0 + T_0 + S_1 + T_1 + M$. It is now plain that $M + S_i + T_i = M + U_i = V_i$. This proves the claim.

We aim to apply the inductive hypothesis to $V_i, M + S_i, T_i, \mathcal{F}_i$, for each $i = 0, 1$. We will need a V_i-basic representation of $\mathcal{F}_i$ over $M + S_i$, and the next claim helps us get one.

Claim 3. For each $i < 2$ we have $U \cap \mathrm{cl}\, M \cap \mathrm{cl}\, S_i = \emptyset$, and $M + S_i = M \cup S_i$ in $RO(U)$.
Proof of claim. By definition, $M = -(U_0 + U_1) = U \setminus \mathrm{cl}(U_0 + U_1) \subseteq U \setminus U_i$. Since U_i is open, $\mathrm{cl}\, M \cap U_i = \emptyset$. But $U \cap \mathrm{cl}\, S_i \subseteq U_i$ by condition C2 above, so $U \cap \mathrm{cl}\, M \cap \mathrm{cl}\, S_i = \emptyset$. By lemma 6.4, $M + S_i = M \cup S_i$. This proves the claim.

So all we need is to find suitable representations over M and S_i and take their union.

Clearly, $\mathcal{F}^*(b_0)$ is a subframe of $\mathcal{F}_0$, and so a *proper* subframe of $\mathcal{F}$. It is obviously connected (since rooted), and a generated subframe of $\mathcal{F}$, so a locally connected KD4 frame. By the inductive hypothesis, it is fully representable over X. So we can find an (M-full) representation $\mu : M \to R(b_0)$ of $\mathcal{F}^*(b_0)$ over M.

For each $i < 2$ let

$$\sigma_i = (\sigma \restriction S_i) \; : \; S_i \to W_i.$$

Claim 4. For each $i < 2$, $\mu \cup \sigma_i : M \cup S_i \to W_i$ is a well defined V_i-basic representation of $\mathcal{F}_i$ over $M \cup S_i$.
Proof of claim. Since $\mathcal{F}^*(b_0)$ is a generated subframe of $\mathcal{F}_i$, it follows from lemma 7.3(1) that μ is a representation of $\mathcal{F}_i$ over M. Similarly, σ_i is a representation of $\mathcal{F}_i$ over S_i. Since M and S_i are disjoint open sets, $\mu \cup \sigma_i : M \cup S_i \to W_i$ is well defined and, by lemma 7.3(2), a representation of $\mathcal{F}_i$ over $M \cup S_i$.

To prove that it is V_i-basic, let $x \in V_i$ and $v, w \in W_i$ be given, and suppose that $(X, (\mu \cup \sigma_i)^{-1}), x \models \Diamond w \wedge \Diamond v$. We require Rwv.

Plainly, $x \in \mathrm{cl}(M \cup S_i) = \mathrm{cl}\, M \cup \mathrm{cl}\, S_i$, and $x \in V_i \subseteq U$. But $U \cap \mathrm{cl}\, M \cap \mathrm{cl}\, S_i = \emptyset$ by claim 3. So there are two possibilities.

The first one is that $x \notin \mathrm{cl}\, M$. In this case, we must have $(X, \sigma_i^{-1}), x \models \Diamond w \wedge \Diamond v$. As $\sigma_i \subseteq \sigma$, we also have $(X, \sigma^{-1}), x \models \Diamond w \wedge \Diamond v$. As σ is U-basic, we obtain Rwv.

The other possibility is that $x \notin \mathrm{cl}\, S_i$. So $(X, \mu^{-1}), x \models \Diamond w \wedge \Diamond v$. Since μ is a representation of $\mathcal{F}^*(b_0)$, we have $w, v \in R(b_0)$. But b_0 is R-maximal, so $R^\bullet(b_0) = \emptyset$. Hence, $w \in R^\circ(b_0)$, so Rwb_0, and since Rb_0v, we deduce Rwv by transitivity. (Essentially we are using that $\mathcal{F}^*(b_0)$ is a non-degenerate cluster.) This proves the claim.

In summary, for each $i < 2$ we have:

- V_i is open (by claim 2)
- $M+S_i, V_i \in RO(U)$ and $M+S_i \subseteq V_i$, so by lemma 6.4, $M+S_i$ is a regular open subset of V_i
- working in $RO(U)$, we have $V_i = (M+S_i)+T_i$ and $(M_i+S_i)\cdot T_i = \emptyset$ by claim 2. So $T_i = V_i \cdot -(M+S_i) = V_i \cap U \setminus \mathrm{cl}(M+S_i) = V_i \setminus \mathrm{cl}(M+S_i)$.
- $M+S_i = M\cup S_i$ (by claim 3), and $\mu\cup\sigma_i : M\cup S_i \to W_i$ is a V_i-basic representation of $\mathcal{F}_i$ over $M+S_i$ (by claim 4)

So for each $i<2$, recalling that $\mathcal{F}_i$ is fully representable, we see that $\mu\cup\sigma_i : M\cup S_i \to W_i$ extends to a T_i-full representation $\rho_i : V_i \to W_i$ of $\mathcal{F}_i$ over V_i. We have

$$(X,\rho_i^{-1}), x \models \langle d\rangle w \quad \text{for every } w\in W_i \text{ and } x\in \mathrm{cl}\,T_i\setminus V_i. \tag{7.7}$$

Finally define

$$\rho = \rho_0\cup\rho_1 : U \to W. \tag{7.8}$$

We check first that ρ is well defined and total. Working in $RO(U)$ again, we have $\mathrm{dom}\,\rho_0 \cap \mathrm{dom}\,\rho_1 = V_0\cap V_1 = V_0\cdot V_1 = M$ by (7.5). But $\rho_0 \restriction M = \mu = \rho_1\restriction M$. So ρ is well defined. Also, $V_i = -U_{1-i} = U\setminus \mathrm{cl}\,U_{1-i}$ (for $i=0,1$) by (7.5), and $U\cap\mathrm{cl}\,U_0\cap\mathrm{cl}\,U_1=\emptyset$ by condition C1 above, so

$$\mathrm{dom}\,\rho = V_0\cup V_1 = (U\setminus\mathrm{cl}\,U_1)\cup(U\setminus\mathrm{cl}\,U_0) = U\setminus(\mathrm{cl}\,U_1\cap\mathrm{cl}\,U_0) = U. \tag{7.9}$$

Hence, ρ is total. Plainly, ρ extends σ, since $\rho=\rho_0\cup\rho_1 \supseteq (\mu\cup\sigma_0)\cup(\mu\cup\sigma_1)=\mu\cup\sigma$.

Claim 5. ρ is a representation of $\mathcal{F}$ over U.
Proof of claim. Let $i<2$. Then $\rho\restriction V_i = \rho_i$, a representation of $\mathcal{F}_i$ over V_i. By lemma 7.3(1), this is also a representation of $\mathcal{F}$ over V_i, which is an open set by claim 2. By (7.9), $U = V_0\cup V_1$, so by lemma 7.3(2), ρ is a representation of $\mathcal{F}$ over U, proving the claim.

Claim 6. ρ is T-full.
Proof of claim. Let $x\in\mathrm{cl}\,T\setminus U$. We require $(X,\rho^{-1}),x\models\langle d\rangle w$ for every $w\in W$.

For each $i<2$, as $\mathrm{cl}\,T\setminus U\subseteq \mathrm{cl}\,T_i$ by condition C4 above, and $x\notin U\supseteq V_i$, we have $x\in\mathrm{cl}\,T_i\setminus V_i$. Since $\rho_i\subseteq\rho$, it follows from (7.7) that $(X,\rho^{-1}),x\models\langle d\rangle w$ for every $w\in W_i$. This holds for each $i=0,1$. Since $W_0\cup W_1 = W$, we have $(X,\rho^{-1}),x\models\langle d\rangle w$ for every $w\in W$.

Finally, we show that $\rho(U)=W$. Since each ρ_i is a T_i-full representation of $\mathcal{F}_i$ over V_i, and $T_i\neq\emptyset$ by condition C3, by (7.9) we obtain $\rho(U)=\rho(V_0)\cup\rho(V_1)=\rho_0(V_0)\cup\rho_1(V_1) = W_0\cup W_1 = W$. This proves the claim, and with it, proposition 7.10. □

REMARK 7.11. We end with some technical remarks on the definition of 'fully representable' (definition 7.9) and its relation to the proof just completed. They are not needed later, and the reader can of course skip them if desired.

It is very helpful throughout the proof that U is open — see, e.g., lemma 7.3. However, we cannot assume in definition 7.9 that U is regular open in X. For if we did, then in case 2 of the proof, we have $\operatorname{cl} I \subseteq \operatorname{cl} T' \subseteq \operatorname{cl} U'$ by claim 3 and $T' \subseteq U'$, so $U' \neq U = \operatorname{int}\operatorname{cl} U = \operatorname{int}(\operatorname{cl} U' \cup \operatorname{cl} I) = \operatorname{int}\operatorname{cl} U'$. Therefore, U' is not regular open in X, and we can not apply the inductive hypothesis to it. We use that X is dense in itself to show that $I \subseteq \operatorname{cl} T'$.

At least according to the construction we gave, S should be open. In case 1, if S is not open then there is $x \in S \setminus \operatorname{int} S \subseteq \operatorname{cl}(U \setminus S)$, and a little thought shows that $(X, \rho^{-1}), x \models \langle d \rangle w_0$ for any such x. For ρ to be a representation, we would need $R(\rho(x), w_0)$. Since $\rho \supseteq \sigma$ and $x \in S$, this says that $R(\sigma(x), w_0)$, which we have no reason to suppose is true.

The problem if S is not regular open in U is that, again in case 1, we used that $U \setminus S = \operatorname{cl} T$. If this were to fail, there may be points $x \in U \setminus (S \cup \operatorname{cl} T)$ (so $x \in U \cap \operatorname{int}\operatorname{cl} S$). We have to define ρ on these x, and defining $\rho(x) = w_0$ as in the proof may not give a representation. However, as σ is U-basic, it is possible to define $\rho(x)$ using σ instead. This effectively extends σ to $U \cap \operatorname{int}\operatorname{cl} S$. So we can assume without loss of generality that S is regular open in U. It is therefore easier to do so and avoid the problem completely.

We could just suppose in definition 7.9 that S is regular open in X, but we cannot suppose this of U, and we have to work in $RO(U)$, so there is little gain in doing so.

We need that σ is U-basic in order that in case 3, the subsets S_0, S_1 have disjoint closures in U. This in turn is needed to apply normality in the proof of corollary 6.9.

We cannot assume instead in definition 7.9 that σ is X-basic, because in case 3, we cannot guarantee that $\mu \cup \sigma_i$ is X-basic. This is because we do not know that $M \cap \operatorname{cl} S_i = \emptyset$, but only that $U \cap M \cap \operatorname{cl} S_i = \emptyset$. We could solve this problem by assuming further that $\operatorname{cl} S \subseteq U$ (which implies that S is regular open in X), but this weakens the proposition sufficiently to cause trouble in theorem 9.1 later, where we would need to ensure that $\operatorname{cl} S_n \cup \operatorname{cl} S_{n+1} \subseteq U_n$ for each n.

Finally, we mention that actually $\rho(T) = W$ when $T \neq \emptyset$ — not only ρ but also $\rho \restriction T$ is surjective. We might try to drop the second, surjectivity part of definition 7.8 and simply prove it from the first part, as in cases 1 and 2 of the proof, but it is not clear how to do this in case 3.

8 Weak completeness

We are now ready to prove our first tranche of main results, showing that Hilbert systems for various sublanguages of $\mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ are sometimes sound and always complete over any non-empty dense-in-itself metric space. Several of the proofs use the translations $-^d$ and $-^\mu$ of section 4. We establish only weak completeness. We will discuss strong completeness later, in section 9.4.

Here and later, we include 't' in the name of a Hilbert system to indicate that it includes the tangle axioms **Fix** and **Ind** of section 5.3. Recall that by lemma 6.7, metric spaces, regarded as topological spaces, are T1.

8.1 Weak completeness for $\mathcal{L}^\mu_\Box$ and $\mathcal{L}^{\langle t\rangle}_\Box$

The pioneering result in this field was the theorem of [24] that the $\mathcal{L}_\Box$-logic of every separable dense-in-itself metric space is S4. The assumption of separability was removed in [28]. We begin by generalising this theorem, establishing (weak) completeness results for $\mathcal{L}^\mu_\Box$ and $\mathcal{L}^{\langle t\rangle}_\Box$ over any dense-in-itself metric space. We will go on to prove strong completeness in theorem 9.3.

THEOREM 8.1. *Let X be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* S4μ *is sound and complete over X for $\mathcal{L}_\Box^\mu$-formulas.*
2. *The Hilbert system* S4t *is sound and complete over X for $\mathcal{L}_\Box^{\langle t\rangle}$-formulas.*

Proof. For part 1, soundness is easy to check and indeed we have already mentioned it in corollary 4.7. For completeness, let φ be an $\mathcal{L}_\Box^\mu$-formula that is not a theorem of S4μ. By theorem 3.7, we can find a finite S4 frame $\mathcal{F} = (W, R)$, an assignment h into $\mathcal{F}$, and a world $w \in W$ with $(W, R, h), w \models \neg\varphi$. By replacing $\mathcal{F}$ by $\mathcal{F}(w)$, we can suppose that w is a root of $\mathcal{F}$ — this can be justified in a standard way using lemma 2.1. Since $\mathcal{F}$ is rooted, it is clearly connected. Since it is reflexive and transitive, it is a locally connected KD4 frame. So by proposition 7.10, $\mathcal{F}$ is fully representable over X. So, taking $U = X$ and $S = \sigma = \emptyset$ in the definition of 'fully representable' (definition 7.9), we may choose an X-full, hence surjective, representation ρ of $\mathcal{F}$ over X. Choose $x \in X$ with $\rho(x) = w$. Then

$$\begin{array}{llll}(W,R,h),w \models \varphi & \text{iff} & (W,R,h),w \models \varphi^d & \text{by lemma 4.4, since } \mathcal{F} \text{ is reflexive,}\\ & \text{iff} & (X,\rho^{-1}\circ h),x \models \varphi^d & \text{by proposition 7.5, since } \varphi^d \in \mathcal{L}^\mu_{[d]\forall},\\ & \text{iff} & (X,\rho^{-1}\circ h),x \models \varphi & \text{by lemma 4.5, since } X \text{ is T1.}\end{array}$$

We obtain $(X, \rho^{-1} \circ h), x \models \neg\varphi$. Thus, φ is not valid over X, proving completeness.

The proof of part 2 is similar. The differences are: φ is assumed to be an $\mathcal{L}_\Box^{\langle t\rangle}$-formula that is not a theorem of S4t; we use the results of section 5.8 in place of theorem 3.7 to obtain a finite S4 Kripke model satisfying $\neg\varphi$ at a root; and having obtained a surjective representation ρ of $\mathcal{F}$ over X and $x \in X$ with $\rho(x) = w$, we use the additional translation $-^\mu$ from section 4, as follows. Note that $\varphi \in \mathcal{L}_\Box^{\langle t\rangle}$, $\varphi^d \in \mathcal{L}_{[d]}^{\langle dt\rangle}$, and $(\varphi^d)^\mu \in \mathcal{L}^\mu_{[d]} \subseteq \mathcal{L}^\mu_{[d]\forall}$.

$$\begin{array}{llll}(W,R,h),w \models \varphi & \text{iff} & (W,R,h),w \models \varphi^d & \text{by lemma 4.4, since } \mathcal{F} \text{ is reflexive,}\\ & \text{iff} & (W,R,h),w \models (\varphi^d)^\mu & \text{by lemma 4.2, since } \mathcal{F} \text{ is transitive,}\\ & \text{iff} & (X,\rho^{-1}\circ h),x \models (\varphi^d)^\mu & \text{by proposition 7.5, since } (\varphi^d)^\mu \in \mathcal{L}^\mu_{[d]\forall},\\ & \text{iff} & (X,\rho^{-1}\circ h),x \models \varphi^d & \text{by lemma 4.2 again,}\\ & \text{iff} & (X,\rho^{-1}\circ h),x \models \varphi & \text{by lemma 4.5, since } X \text{ is T1.}\end{array}$$

□

8.2 Weak completeness for $\mathcal{L}_{\Box\forall}$ and $\mathcal{L}_{\Box\forall}^{\langle t\rangle}$

Completeness for languages with $\forall$ follows the same lines, although soundness requires that the space be connected.

THEOREM 8.2. *Let X be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* S4.UC *is complete over X for $\mathcal{L}_{\Box\forall}$-formulas, and sound if X is connected.*[6]

[6] In [31, theorem 18], Shehtman states this result when X is additionally assumed separable. However, [20, footnote 7] states that [31] "contains a stronger claim: [the $\mathcal{L}_{\Box\forall}$-logic of X is S4.UC] for any connected dense-in-itself separable metric X. However, recently we found a gap in the proof of Lemma 17 from that paper. Now we state the main result only for the case $X = \mathbb{R}^n$; a proof can be obtained by applying the methods of the present Chapter, but we are planning to publish it separately."

2. *The Hilbert system* S4t.UC *is complete over* X *for* $\mathcal{L}^{\langle t\rangle}_{\Box\forall}$*-formulas, and sound if* X *is connected.*

Proof. For part 1, soundness when X is connected is again clear: connectedness is needed so that the C axiom is valid in X. For completeness, even when X is not connected, suppose that $\varphi \in \mathcal{L}_{\Box\forall}$ is not a theorem of S4.UC. By the results of section 5.10, or by [31, theorem 10], S4.UC has the finite model property, so we can find a finite connected S4 frame $\mathcal{F} = (W, R)$, an assignment h into $\mathcal{F}$, and a world $w \in W$ such that $(W, R, h), w \models \neg\varphi$. The proof that φ is not valid in X is now exactly as in theorem 8.1.

Part 2 is proved similarly, using the results of section 5.10 to obtain a finite model. □

We have no results for $\mathcal{L}^{\mu}_{\Box\forall}$ because we are not aware of any completeness theorem for this language with respect to finite connected S4 frames. If one is proved in future, we could take advantage of it.

8.3 Weak completeness for $\mathcal{L}_{[d]}$ and $\mathcal{L}^{\langle dt\rangle}_{[d]}$

In one way this is even easier, as we do not need the translation φ^d. But again, soundness requires a condition on the space.

THEOREM 8.3. *Let* X *be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* KD4G$_1$ *is complete over* X *for* $\mathcal{L}_{[d]}$*-formulas, and sound if* G$_1$ *is valid in* X.
2. *The Hilbert system* KD4G$_1t$ *is complete over* X *for* $\mathcal{L}^{\langle dt\rangle}_{[d]}$*-formulas, and sound if* G$_1$ *is valid in* X.

Proof. For part 1, soundness is clear. For completeness, even when X does not validate G$_1$, suppose that $\varphi \in \mathcal{L}_{[d]}$ is not a theorem of KD4G$_1$. As we mentioned in section 5.12, KD4G$_1$ has the finite model property [30, theorem 15], so we can find a finite KD4G$_1$ frame $\mathcal{F} = (W, R)$, an assignment h into $\mathcal{F}$, and a world $w \in W$ such that $(W, R, h), w \models \neg\varphi$. As usual, by replacing $\mathcal{F}$ by $\mathcal{F}(w)$, we can suppose that $\mathcal{F}$ is connected. It is also locally connected because it validates G$_1$ (see fact 5.9). Using proposition 7.10, let ρ be a surjective representation of $\mathcal{F}$ over X. Let $x \in X$ satisfy $\rho(x) = w$. Then $(X, \rho^{-1} \circ h), x \models \neg\varphi$ by proposition 7.5. So φ is not valid in X.

The proof of part 2 is similar, except that we use the results of section 5.13 to obtain a finite model, and in order to apply proposition 7.5, we first use the translation $-^{\mu}$ to turn $\varphi \in \mathcal{L}^{\langle dt\rangle}_{[d]}$ into an $\mathcal{L}^{\mu}_{[d]}$-formula φ^{μ} equivalent to φ in transitive frames and in X. □

REMARK 8.4. Theorem 8.3(1) is related to earlier work of Shehtman [30]. In [30, theorem 23, p.39], the following is proved for the language $\mathcal{L}_{[d]}$:

(i) Let X be a topological space having an open set homeomorphic to some $\mathbb{R}^n$, $n > 0$. Then $L(D(X)) \subseteq D4\mathrm{G}_1$ [the $\mathcal{L}_{[d]}$-logic of X is contained in KD4G$_1$].

(ii) If additionally X satisfies conditions of lemma 2 then $L(D(X)) = D4\mathrm{G}_1$.

Lemma 2 [30, p.3] states the following.

Let X be a topological space satisfying the following condition: for any open U and any $x \in U$ there is open $V \subseteq U$ such that $x \in V$ and $(V \setminus \{x\})$ is connected [as a subspace of X]. Then $X \models \mathrm{G}_1$.

Shehtman's results (i), (ii) above follow from theorem 8.3(1). We remark that the converse of his lemma 2 fails in general — a counterexample is given by the subspace $X = \mathbb{R}^2 \setminus \{(1/n, y) : n$ a positive integer, $y \in \mathbb{R}\}$ of $\mathbb{R}^2$. [22, theorems 3.12, 3.14] give a characterisation of when a topological space validates G_n, for $n \geq 1$.

Shehtman [30, p.43] also states two open problems:

1. To describe all [$\mathcal{L}_{[d]}$-]logics [of] dense-in-itself metric spaces X. In particular, is [K]$D4\mathrm{G}_1$ the greatest of them?
2. Is theorem 23(ii) extended to the infinite dimensional case? In particular, does it hold for Hilbert space ℓ_2 (with the weak or with the strong topology)?

Theorem 8.3(1) appears to resolve problem 2 and the second part of problem 1, both positively.

Shehtman also proved in [30, theorem 29] that the $\mathcal{L}_{[d]}$-logic of any separable zero-dimensional dense-in-itself metric space is KD4. This does not follow from theorem 8.3.

8.4 Weak completeness for $\mathcal{L}_{[d]\forall}$ and $\mathcal{L}^{\langle dt\rangle}_{[d]\forall}$

The following is now purely routine.

THEOREM 8.5. *Let X be a non-empty dense-in-itself metric space.*

1. *The Hilbert system* $\mathrm{KD4G_1.UC}$ *is complete over X for $\mathcal{L}_{[d]\forall}$-formulas, and sound if X is connected and validates* G_1.
2. *The Hilbert system* $\mathrm{KD4G}_1t.\mathrm{UC}$ *is complete over X for $\mathcal{L}^{\langle dt\rangle}_{[d]\forall}$-formulas, and sound if X is connected and validates* G_1.

Proof. The finite model property for $\mathrm{KD4G_1.UC}$ and $\mathrm{KD4G}_1t.\mathrm{UC}$ follows from the results of section 5.13. There are no other new elements in the proof, so we leave it to the reader. $\square$

9 Strong completeness

Here, we will prove that $\mathrm{KD4G}_1t$ is strongly complete over any non-empty dense-in-itself metric space X: any countable $\mathrm{KD4G}_1t$-consistent set of $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas is satisfiable over X. The analogous results for $\mathcal{L}^{\mu}_{\Box}$ and the weaker languages $\mathcal{L}_{[d]}$ and $\mathcal{L}^{\langle t\rangle}_{\Box}$ will follow. The analogous result for $\mathcal{L}_{\Box}$ also follows, but this is a known result, proved recently by Kremer [17]. We will then show that strong completeness frequently fails for languages with $\forall$.

9.1 The problem

Let us outline a naïve approach to the problem. It does not work, but it will illustrate the difficulty we face and motivate the formal proof later.

Let Γ be a countable $\mathrm{KD4G}_1t$-consistent set of $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas. For simplicity, assume that Γ is maximal consistent. Write Γ as the union of an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ of finite sets. Fix $x \in X$. By weak completeness (theorem 8.3), each Γ_n ($n < \omega$) is satisfiable at x, so we can find an assignment g_n on X with $(X, g_n), x \models \Gamma_n$. Suppose we could build a new assignment g that *behaves like* g_n *for larger and larger* n, as we approach x. Then we might hope that $(X, g), x \models \Gamma_n$ for all n, and so $(X, g), x \models \Gamma$.

To define such a g, we choose a countable sequence $X = S_0 \supseteq S_1 \supseteq \cdots$ of open neighbourhoods of x, such that

S1. every open neighbourhood of x contains some S_n (that is, the S_n form a 'base of open neighbourhoods' of x).

X is a metric space, so we can do this. Since we can make the S_n as small as we like, and the Γ_n are finite sets, we can suppose that for each $n < \omega$:

S2. for each $[d]\varphi \in \Gamma_n$, we have $(X, g_n), y \models \varphi$ for every $y \in S_n \setminus \{x\}$,

S3. for each $\langle d\rangle\varphi \in \Gamma_n$, there is $y \in S_n \setminus \operatorname{cl} S_{n+1}$ with $(X, g_n), y \models \varphi$.

We can now define a new assignment g by 'using g_n within S_n', for each $n < \omega$. More precisely, we let

$$g \restriction (S_n \setminus S_{n+1}) = g_n \restriction (S_n \setminus S_{n+1})$$

for each $n < \omega$. We also need to define g at x itself, but we can use Γ to determine truth values of atoms there.

Now we try to prove that $\varphi \in \Gamma$ iff $(X, g), x \models \varphi$ for all formulas φ, by induction on φ. The atomic and boolean cases are easy. Consider the case $\langle d\rangle\varphi$.

If $\langle d\rangle\varphi \in \Gamma$, then $\langle d\rangle\varphi \in \Gamma_n$ for all large enough n, so by S3, there is $y \in S_n \setminus \operatorname{cl} S_{n+1}$ with $(X, g_n), y \models \varphi$. As $S_n \setminus \operatorname{cl} S_{n+1}$ is open and g_n agrees with g on it, it follows that $(X, g), y \models \varphi$. This holds for cofinitely many n, so $(X, g), x \models \langle d\rangle\varphi$.

Conversely, if $(X, g), x \models \langle d\rangle\varphi$, then for infinitely many n, there is $y \in S_n \setminus S_{n+1}$ with $(X, g), y \models \varphi$. *If we could find such a* $y \in S_n \setminus \operatorname{cl} S_{n+1}$, then as above, $(X, g_n), y \models \varphi$, and it would follow by S2 and maximality of Γ that $\langle d\rangle\varphi \in \Gamma$.

But it may be that we can only find such $y \in \operatorname{cl} S_{n+1}$. The truth of φ at such y may not be preserved when we change from g to g_n, because it may depend on points in S_{n+1}, and at such points, g agrees with g_{n+1}, not g_n. (We cannot just make S_{n+1} smaller to take the witnesses y out of $\operatorname{cl} S_{n+1}$, because g will then change, and we may no longer have $(X, g), y \models \varphi$.)

So we would like to arrange a *smooth transition* between g_n and g_{n+1}, avoiding unpleasant discontinuities. It would be sufficient if there is some closed $T_{n+1} \subseteq S_{n+1}$ such that g_n and g_{n+1} agree on the 'buffer zone' $S_{n+1} \setminus T_{n+1}$. Much of the formal proof below is aimed at achieving something like this for atoms occurring in Γ_n — see claim 3 especially.

9.2 Strong completeness for $\mathcal{L}^{\langle dt\rangle}_{[d]}$

THEOREM 9.1 (strong completeness)**.** *Let X be a non-empty dense-in-itself metric space. Then the Hilbert system* $\mathrm{KD4G_1}t$ *is strongly complete over X for $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas, and sound if* $\mathrm{G_1}$ *is valid in X.*

Proof. For soundness, see theorem 8.3. For strong completeness, let Γ be a countable $\mathrm{KD4G_1}t$-consistent set of $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas. We show that Γ is satisfiable over X. We can suppose without loss of generality that Γ is maximal consistent. Since Γ is countable, we can write it as $\Gamma = \bigcup_{n<\omega} \Gamma_n$, where $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ is a chain of finite sets. Let L_n be the finite set of atoms occurring in formulas in Γ_n, for each $n < \omega$. So $L_0 \subseteq L_1 \subseteq \cdots$. For each $n < \omega$, as Γ_n is $\mathrm{KD4G}_1t$-consistent, by the results of section 5.13 there is a finite Kripke model $\mathcal{M}_n = (W_n, R_n, h_n)$ whose frame (W_n, R_n) validates $\mathrm{KD4G}_1$, and a world $w_n \in W_n$ with

$$\mathcal{M}_n, w_n \models \Gamma_n.$$

We can assume without loss of generality that the W_n $(n < \omega)$ are pairwise disjoint. For each n, fix an arbitrary $e_n \in W_n$ with $R_n w_n e_n$ and such that e_n is R_n-maximal — that is, $R_n^\bullet(e_n) = \emptyset$.

For $i \leq j < \omega$ and $w \in W_j$ write

$$\begin{array}{rcll} \mathrm{tp}_i(w) & = & \{p \in L_i : \mathcal{M}_j, w \models p\} & \in \wp L_i \\ \tau_i^j & = & \{\mathrm{tp}_i(w) : w \in R_j(e_j)\} & \in \wp\wp L_i \end{array}$$

So $\mathrm{tp}_i(w)$ is the 'atomic type' of w in $\mathcal{M}_j$ with respect to the finite set L_i of atoms. We do not need to write $\mathrm{tp}_i^j(w)$ since the W_n are pairwise disjoint so j is determined by w. And τ_i^j is the set of such types that occur as types of points in the cluster $R_j(e_j)$.

Claim 1. We can suppose without loss of generality that $\tau_i^j = \tau_i^i$ whenever $i \leq j < \omega$.
Proof of claim. Essentially König's tree lemma. We will define by induction infinite sets $\omega = I_{-1} \supseteq I_0 \supseteq I_1 \supseteq \cdots$. We let $i_n = \min I_n$, and we will arrange that $0 = i_{-1} < i_0 < i_1 < \cdots$ and $i_n \geq n$ for all n. Let $n < \omega$ and suppose that we are given I_{n-1} and $i_{n-1} = \min I_{n-1} \geq n-1$ inductively. Using that $\wp\wp L_n$ is finite, choose infinite $I_n \subseteq I_{n-1} \setminus \{i_{n-1}\}$ such that $\tau_n^i \in \wp\wp L_n$ is constant for all $i \in I_n$. The term τ_n^i is defined for all $i \in I_n$, because $i \geq \min I_n > i_{n-1} \geq n-1$ and so $i \geq n$. Of course define $i_n = \min I_n$. Then $i_n > i_{n-1}$ and $i_n \geq n$ as required. This completes the definition. Now replace $\mathcal{M}_n, w_n, e_n$ by $\mathcal{M}_{i_n}, w_{i_n}, e_{i_n}$ for each $n < \omega$. Do not change Γ_n or L_n. Since $n \leq i_n$, we have $\Gamma_n \subseteq \Gamma_{i_n}$, and consequently we still have $\mathcal{M}_n, w_n \models \Gamma_n$ for each n. And if $r \leq s < \omega$ we have $i_r, i_s \in I_r$, so $\tau_r^{i_r} = \tau_r^{i_s}$, and consequently after replacement, $\tau_r^r = \tau_r^s$. This proves the claim.

For each $n < \omega$, define the frames

$$\begin{array}{rcl} \mathcal{F}_n & = & (R_n(w_n), R_n \restriction R_n(w_n)), \\ \mathcal{C}_n & = & (R_n(e_n), R_n \restriction R_n(e_n)). \end{array}$$

$\mathcal{F}_n$ is a generated subframe of (W_n, R_n), so also a $\mathrm{KD4G}_1$-frame; it is connected since (W_n, R_n) validates G_1. As e_n is R_n-maximal, $\mathcal{C}_n$ is a nondegenerate cluster, so trivially a connected $\mathrm{KD4G}_1$-frame, and (as R_n is transitive) a generated subframe of $\mathcal{F}_n$. We conclude from proposition 7.10 that $\mathcal{F}_n$ and $\mathcal{C}_n$ are fully representable over X, for all $n < \omega$.

Now fix arbitrary $x_0 \in X$. Let O be an open neighbourhood of x_0. Since X is a metric space, all singletons are closed, and since it is dense in itself, lemma 6.1 tells us that O is infinite, so we can pick $y \in O \setminus \{x_0\}$. Then $O \setminus \{y\}$ is open, $\{x_0\} \subseteq O \setminus \{y\}$, and $\{x_0\}$ is closed. By lemma 6.7, X is normal, so there is open P with $x_0 \in P \subseteq \mathrm{int}\,\mathrm{cl}\,P \subseteq \mathrm{cl}\,P \subseteq O \setminus \{y\} \subset O$ (the last inclusion being strict). Note that $\mathrm{int}\,\mathrm{cl}\,P$ is regular open in X. So *every open neighbourhood of x_0 properly contains the closure of some regular open neighbourhood of x_0.* Using this repeatedly, we may choose regular open subsets O_n, P_n of X (for $n < \omega$) containing x_0, with $O_0 = X$, and with the following properties:

1. $\mathrm{cl}\,O_{n+1} \subset P_n$ and $\mathrm{cl}\,P_n \subset O_n$ (the inclusions are strict) for each $n < \omega$.

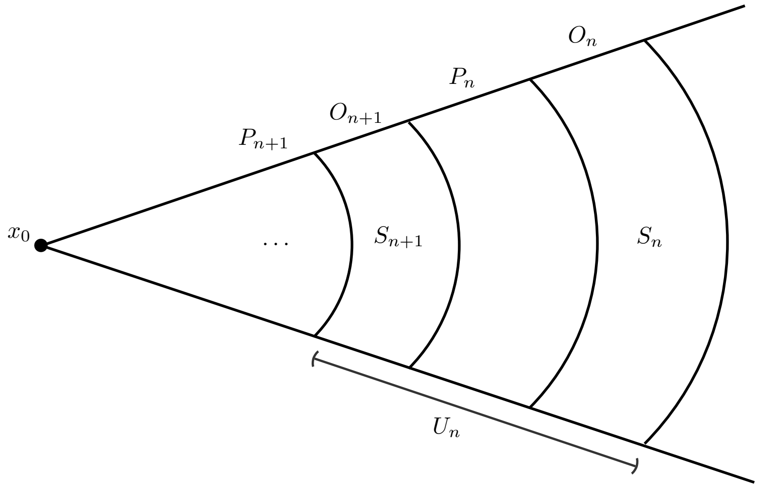


Figure 1: rough guide to the sets O_n, P_n, U_n, S_n

2. $O_n \subseteq N_{1/n}(x_0)$ for each $n > 0$.

It follows that for every open neighbourhood O of x_0, there is $n < \omega$ with $O_n \subseteq O$. That is, the O_n form a base of open neighbourhoods of x_0.

For each $n < \omega$ define open sets

$$\begin{aligned} U_n &= O_n \setminus \operatorname{cl} P_{n+1}, \\ S_n &= O_n \setminus \operatorname{cl} P_n. \end{aligned}$$

See figure 1. It is easily seen that

$$\bigcup_{n<\omega} (O_n \setminus O_{n+1}) = X \setminus \{x_0\}, \tag{9.1}$$

$$\bigcup_{n \le m < \omega} U_m = O_n \setminus \{x_0\} \quad \text{for each } n < \omega. \tag{9.2}$$

The following claim lists some more basic facts about our situation.

Claim 2. For each $n < \omega$:

1. $U_n \cap U_{n+1} = S_{n+1} \neq \emptyset$.

2. $S_n \cup S_{n+1} \subseteq U_n$,

3. $\operatorname{cl} S_n \cap \operatorname{cl} S_{n+1} = \emptyset$,

4. S_n, S_{n+1}, and $S_n \cup S_{n+1}$ are regular open subsets of U_n,

5. $U_n \setminus \operatorname{cl}(S_n \cup S_{n+1}) \neq \emptyset$.

Proof of claim.

1. Easy.

2. From the definitions we have $S_n = O_n \setminus \mathrm{cl}\, P_n \subseteq O_n \setminus \mathrm{cl}\, P_{n+1} = U_n$ and $S_{n+1} = O_{n+1} \setminus \mathrm{cl}\, P_{n+1} \subseteq O_n \setminus \mathrm{cl}\, P_{n+1} = U_n$.

3. It is clear that
$$\mathrm{cl}\, S_n \subseteq \mathrm{cl}\, O_n \setminus P_n. \tag{9.3}$$
Applying this for $n+1$ and n gives $\mathrm{cl}\, S_{n+1} \cap \mathrm{cl}\, S_n \subseteq \mathrm{cl}\, O_{n+1} \setminus P_n \subseteq P_n \setminus P_n = \emptyset$.

4. O_n and P_n are regular open subsets of X, so by lemma 6.4, $S_n = O_n \setminus \mathrm{cl}\, P_n$ is a regular open subset of X too. Since $\mathrm{cl}\, S_n \cap \mathrm{cl}\, S_{n+1} = \emptyset$ by part 2, lemma 6.4(2) yields that $S_n \cup S_{n+1}$ is also a regular open subset of X. Since each of these three sets is a subset of U_n by part 2, by lemma 6.4(3) it is also regular open in U_n.

5. By (9.3) (for n and $n+1$), $\mathrm{cl}\, S_n$ and $\mathrm{cl}\, S_{n+1}$ are disjoint from $P_n \setminus \mathrm{cl}\, O_{n+1}$, so by additivity of closure, $U_n \setminus \mathrm{cl}(S_n \cup S_{n+1}) = U_n \setminus (\mathrm{cl}\, S_n \cup \mathrm{cl}\, S_{n+1}) \supseteq P_n \setminus \mathrm{cl}\, O_{n+1} \neq \emptyset$.

Claim 3. There are surjective representations ρ_n of $\mathcal{F}_n$ over U_n $(n < \omega)$ such that

1. $\rho_n \restriction S_{n+1}$ is a representation of $\mathcal{C}_n$ over S_{n+1},

2. $\mathrm{tp}_n(\rho_n(x)) = \mathrm{tp}_n(\rho_{n+1}(x))$ for all $x \in S_{n+1}$.

Proof of claim. We define the ρ_n by induction on n. First let $n = 0$. Since $\mathcal{C}_0$ is fully representable over X, we can choose a representation $\sigma : S_1 \to \mathcal{C}_0$. Because $\mathcal{C}_0$ is a nondegenerate cluster, σ is actually a U_0-basic representation (see remark 7.7). By claim 2, S_1 is a regular open subset of U_0, and $U_0 \setminus \mathrm{cl}\, S_1 \neq \emptyset$. Now $\mathcal{F}_0$ is also fully representable over X, so σ extends to a surjective representation ρ_0 of $\mathcal{F}_0$ over U_0. Clearly, condition 1 above is met.

Let $n < \omega$ and assume inductively that for each $m \leq n$, a surjective representation ρ_m of $\mathcal{F}_m$ over U_m has been constructed, such that $\rho_m \restriction S_{m+1}$ is a representation of $\mathcal{C}_m$ over S_{m+1} and $tp_m(\rho_m(x)) = tp_m(\rho_{m+1}(x))$ for all $x \in S_{m+1}$ whenever $m < n$. We will define ρ_{n+1} to continue the sequence.

Note first that since $\mathcal{C}_n$ is a non-degenerate cluster, $\rho_n \restriction S_{n+1}$ is U_n-basic — see remark 7.7. It is also surjective. For, let $w \in R_n(e_n)$ be given. Take $x \in S_{n+1}$ (note that S_{n+1} is non-empty by claim 2). As $\mathcal{C}_n$ is a non-degenerate cluster, $R_n(\rho_n(x), w)$, so as $\rho_n \restriction S_{n+1}$ is a representation, $(S_{n+1}, (\rho_n \restriction S_{n+1})^{-1}), x \models \langle d \rangle w$. This certainly implies that $\rho_n(y) = w$ for some $y \in S_{n+1}$.

For each $w \in R_n(e_n)$, define

$$\begin{array}{rcllcl} D_w & = & \{x \in S_{n+1} : \rho_n(x) = w\} & \subseteq & S_{n+1}, \\ H_w & = & \{v \in R_{n+1}(e_{n+1}) : \mathrm{tp}_n(v) = \mathrm{tp}_n(w)\} & \subseteq & W_{n+1}, \\ \mathcal{H}_w & = & (H_w, R_{n+1} \restriction H_w). & & \end{array}$$

See figure 2. Because $\rho_n \restriction S_{n+1}$ is surjective onto $\mathcal{C}_n$, each set D_w is non-empty, and plainly, S_{n+1} is partitioned by the D_w $(w \in R_n(e_n))$. Because $\tau_n^{n+1} = \tau_n^n$, each H_w is non-empty and $\bigcup_{w \in R_n(e_n)} H_w = R_{n+1}(e_{n+1})$. (The sets H_w may not be pairwise disjoint, but any two of them are equal or disjoint.)

Let $w \in R_n(e_n)$ and consider D_w as a subspace of X. We show that it is dense in itself. Let $x \in D_w$ and suppose for contradiction that $\{x\}$ is open in D_w. So there is open $O \subseteq X$ with $O \cap D_w = \{x\}$, and as S_{n+1} is open, we can suppose that $O \subseteq S_{n+1}$. Now by the inductive hypothesis, $\rho_n \restriction S_{n+1}$ is a representation of $\mathcal{C}_n$ over S_{n+1}. Because $\mathcal{C}_n$ is a non-degenerate

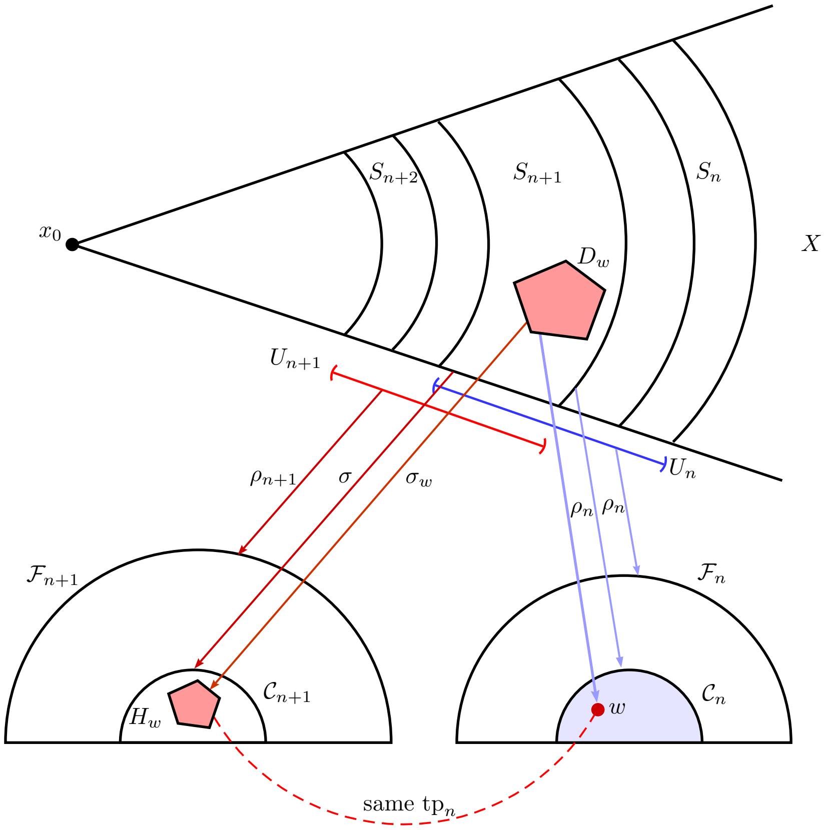


Figure 2: illustration for claim 3

cluster, $R_n ww$, so $(X, (\rho_n \restriction S_{n+1})^{-1}), x \models \langle d \rangle w$. So there is $y \in O \setminus \{x\}$ with $\rho_n(y) = w$. But then $y \in O \cap D_w = \{x\}$, a contradiction.

So D_w is a dense-in-itself metric space in its own right. Since $\mathcal{C}_{n+1}$ is a nondegenerate cluster, so is its subframe $\mathcal{H}_w$. Hence, $\mathcal{H}_w$ is trivially a finite connected KD4G$_1$ frame. So by proposition 7.10, there is a surjective representation

$$\sigma_w : D_w \to H_w$$

of $\mathcal{H}_w$ over D_w. We have $(D_w, \sigma_w^{-1}), x \models \langle d \rangle v$ for every $x \in D_w$ and $v \in H_w$. By lemma 7.2,

$$(X, \sigma_w^{-1}), y \models \langle d \rangle v \quad \text{for every } x \in D_w \text{ and } v \in H_w. \tag{9.4}$$

Now let

$$\sigma = \Big(\bigcup_{w \in R_n(e_n)} \sigma_w \Big) \ : \ S_{n+1} \to R_{n+1}(e_{n+1}).$$

The sets D_w partition S_{n+1}, so σ is a well defined and total map. It has the following property. Let $x \in S_{n+1}$. Writing $\rho_n(x) = w$, say, we have $x \in D_w$ and $\sigma(x) = \sigma_w(x) \in H_w$, so $\mathrm{tp}_n(\sigma(x)) = \mathrm{tp}_n(w)$ by definition of H_w. That is,

$$\mathrm{tp}_n(\sigma(x)) = \mathrm{tp}_n(\rho_n(x)) \quad \text{for each } x \in S_{n+1}. \tag{9.5}$$

We show that σ is a representation of $\mathcal{C}_{n+1}$ over S_{n+1}. Since $\mathcal{C}_{n+1}$ is a non-degenerate cluster, we need show only that $(X, \sigma^{-1}), x \models \langle d \rangle v$ for every $x \in S_{n+1}$ and $v \in R_{n+1}(e_{n+1})$.

So take such x, v. Suppose that $\rho_n(x) = w$, say, so $x \in D_w$. Choose $w' \in R_n(e_n)$ such that $v \in H_{w'}$ (it may not be unique). As $\mathcal{C}_n$ is a cluster, $R_n(w, w')$. As $\rho_n \restriction S_{n+1}$ is a representation of $\mathcal{C}_n$ over S_{n+1}, we have $(X, (\rho_n \restriction S_{n+1})^{-1}), x \models \langle d \rangle w'$. That is, $x \in \langle d \rangle D_{w'}$. But by (9.4), $(X, \sigma^{-1}), y \models \langle d \rangle v$ for every $y \in D_{w'}$. It follows that $(X, \sigma^{-1}), x \models \langle d \rangle \langle d \rangle v$, and hence $(X, \sigma^{-1}), x \models \langle d \rangle v$ as required.

So σ is indeed a representation of $\mathcal{C}_{n+1}$ over S_{n+1}. As $\mathcal{C}_{n+1}$ is fully representable over X, we may choose a representation σ' of $\mathcal{C}_{n+1}$ over S_{n+2}. By claim 2, $S_{n+1} \cap S_{n+2} = \emptyset$, so by lemma 7.3, $\sigma \cup \sigma'$ is a well defined representation of $\mathcal{C}_{n+1}$ over the regular open subset $S_{n+1} \cup S_{n+2}$ of U_{n+1}. Also, $U_{n+1} \setminus \mathrm{cl}(S_{n+1} \cup S_{n+2}) \neq \emptyset$. And since $\mathcal{C}_{n+1}$ is a nondegenerate cluster, $\sigma \cup \sigma'$ is U_{n+1}-basic (see remark 7.7 again). We can now use the fact that $\mathcal{F}_{n+1}$ is fully representable over X to extend $\sigma \cup \sigma'$ is to a surjective representation ρ_{n+1} of $\mathcal{F}_{n+1}$ over U_{n+1}. Then $\rho_{n+1} \restriction S_{n+2} = \sigma'$ is a representation of $\mathcal{C}_{n+1}$ over S_{n+2}, and by (9.5), $tp_n(\rho_n(x)) = \mathrm{tp}_n(\sigma(x)) = tp_n(\rho_{n+1}(x))$ for all $x \in S_{n+1}$. This proves claim 3.

Let $n < \omega$. Define an assignment g_n on U_n by

$$g_n(p) = \rho_n^{-1}(h_n(p)) \text{ for each atom } p. \tag{9.6}$$

By the claim, if $p \in L_n$, then for each $x \in S_{n+1}$ we have $x \in g_n(p)$ iff $\rho_n(x) \in h_n(p)$, iff $p \in \mathrm{tp}_n(\rho_n(x)) = \mathrm{tp}_n(\rho_{n+1}(x))$, iff $\rho_{n+1}(x) \in h_{n+1}(p)$, iff $x \in g_{n+1}(p)$. So

$$S_{n+1} \cap g_n(p) = S_{n+1} \cap g_{n+1}(p) \quad \text{for each } p \in L_n. \tag{9.7}$$

Finally, define an assignment g on X as follows. Let p be an atom.

- For $x \in X \setminus \{x_0\}$, define $x \in g(p)$ iff $x \in g_n(p)$, where $x \in O_n \setminus O_{n+1}$.

  Since the $O_n \setminus O_{n+1}$ are pairwise disjoint, and $\bigcup_{n<\omega}(O_n \setminus O_{n+1}) = X \setminus \{x_0\}$ by (9.1), this is well defined.

- Define $(X, g), x_0 \models p$ iff $p \in \Gamma$.

Claim 4. Let $n < \omega$, let $x \in U_n$, and let φ be a formula whose atoms lie in L_n. Then $(X, g), x \models \varphi$ iff $\mathcal{M}_n, \rho_n(x) \models \varphi$.
Proof of claim. Let $p \in L_n$ be arbitrary. Recall that $U_n = O_n \setminus \mathrm{cl}\, P_{n+1}$. By definition of g, if $x \in O_n \setminus O_{n+1}$ then $x \in g(p)$ iff $x \in g_n(p)$. If instead $x \in O_{n+1}$, then $x \in O_{n+1} \setminus \mathrm{cl}\, P_{n+1} = S_{n+1} \subseteq O_{n+1} \setminus O_{n+2}$, and since $p \in L_{n+1}$ too, the definition of g gives $x \in g(p)$ iff $x \in g_{n+1}(p)$. But by (9.7), this is iff $x \in g_n(p)$ again. So g and g_n agree on U_n as far as atoms in L_n are concerned, and as U_n is open, it follows easily that $(X, g), x \models \varphi$ iff $(U_n, g_n), x \models \varphi$. Since ρ_n is a representation over U_n of the generated subframe $\mathcal{F}_n$ of (W_n, R_n), by lemma 7.3 it is also a representation of (W_n, R_n) over U_n. So by (9.6) and proposition 7.5, $(U_n, g_n), x \models \varphi$ iff $\mathcal{M}_n, \rho_n(x) \models \varphi$. This proves the claim.

Claim 5. For all φ we have $(X, g), x_0 \models \varphi$ iff $\varphi \in \Gamma$.
Proof of claim. By induction on φ. For atoms, the result follows from the definition of g. The boolean operators are handled in the usual way by induction, using the maximal consistency of Γ; they are the only cases in which the inductive hypothesis is used.

We now tackle the case $[d]\varphi$. It is sufficient (and seems more intuitive) to deal with $\langle d\rangle\varphi$ instead. Suppose first that $\langle d\rangle\varphi \in \Gamma$. Choose $n < \omega$ such that $\langle d\rangle\varphi \in \Gamma_n$. Let $i \geq n$ be arbitrary. Then $\langle d\rangle\varphi \in \Gamma_i$, so $\mathcal{M}_i, w_i \models \langle d\rangle\varphi$, and hence there is $v \in R_i(w_i)$ with $\mathcal{M}_i, v \models \varphi$. As $\rho_i : U_i \to R_i(w_i)$ is surjective (see claim 3), there is $x \in U_i$ with $\rho_i(x) = v$. Since $\langle d\rangle\varphi \in \Gamma_i$, the atoms of φ lie in L_i, so claim 4 applies: $(X, g), x \models \varphi$. We conclude that for every $i \geq n$ there is $x \in U_i$ with $(X, g), x \models \varphi$. As $U_i \subseteq O_i \setminus \{x_0\}$ and the O_i form a base of neighbourhoods of x_0, it follows that $(X, g), x_0 \models \langle d\rangle\varphi$.

Conversely, suppose that $(X, g), x_0 \models \langle d\rangle\varphi$. For each $n < \omega$, O_n is an open neighbourhood of x_0, so there is $x \in O_n \setminus \{x_0\}$ with $(X, g), x \models \varphi$. Since $O_n \setminus \{x_0\} = \bigcup_{n \leq i < \omega} U_i$ by (9.2), we have $x \in U_i$ for some $i \geq n$. It follows that there are infinitely many $i < \omega$ such that $(X, g), x \models \varphi$ for some $x \in U_i$. Since the atoms of φ lie in L_i for cofinitely many i, there must be infinitely many i with $\mathcal{M}_i, v \models \varphi$ for some $v \in R_i(w_i)$ (by claim 4), and so $\mathcal{M}_i, w_i \models \langle d\rangle\varphi$ (by Kripke semantics), and so $\neg\langle d\rangle\varphi \notin \Gamma_i$ (since $\mathcal{M}_i, w_i \models \Gamma_i$). Since Γ is the union of the chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$, we have $\neg\langle d\rangle\varphi \notin \Gamma$. As Γ is maximal consistent, it follows that $\langle d\rangle\varphi \in \Gamma$.

Finally, consider the case $\langle dt\rangle\Delta$, where Δ is any non-empty finite set of formulas. Suppose first that $\langle d\rangle\Delta \in \Gamma$. We only sketch the proof here, referring the reader to the case of $\langle d\rangle\varphi$ for more details. Pick any $\delta \in \Delta$. Then as in the case of $\langle d\rangle\varphi$, each of the following holds for cofinitely many $i < \omega$:

- $\langle dt\rangle\Delta \in \Gamma_i$
- $\mathcal{M}_i, w_i \models \langle dt\rangle\Delta$
- there is $v \in R_i(w_i)$ with $\mathcal{M}_i, v \models \delta \wedge \langle dt\rangle\Delta$
- there is $x \in U_i$ with $(X, g), x \models \delta \wedge \langle dt\rangle\Delta$.

As the latter holds for every $\delta \in \Delta$, it follows that $(X, g), x_0 \models \langle dt\rangle\Delta$.

Conversely, suppose $(X, g), x_0 \models \langle dt\rangle\Delta$. Then as in the $\langle d\rangle\varphi$ case, there are infinitely many $i < \omega$ such that $(X, g), x \models \langle dt\rangle\Delta$ for some $x \in U_i$. Since the atoms of $\langle dt\rangle\Delta$ lie in L_i for cofinitely many $i < \omega$, it follows by claim 4 that there are infinitely many i such that there is $v \in R_i(w_i)$ with $\mathcal{M}_i, v \models \langle dt\rangle\Delta$, and hence — by the semantics of $\langle dt\rangle$ — $\mathcal{M}_i, w_i \models \langle dt\rangle\Delta$. As in the $\langle d\rangle\varphi$ case, we obtain $\neg\langle dt\rangle\Delta \notin \Gamma_i$ for infinitely many i, so $\neg\langle dt\rangle\Delta \notin \Gamma$, and so $\langle dt\rangle\Delta \in \Gamma$ by maximal consistency of Γ. The claim is proved, and the theorem with it. □

9.3 Strong completeness for $\mathcal{L}_{[d]}$

We can now easily derive the analogous result for 'modal' $\mathcal{L}_{[d]}$-formulas, essentially by showing that $\mathrm{KD4G_1}t$ is a conservative extension of $\mathrm{KD4G_1}$.

THEOREM 9.2. *Let X be a non-empty dense-in-itself metric space. Then the Hilbert system* $\mathrm{KD4G_1}$ *is strongly complete over X for $\mathcal{L}_{[d]}$-formulas, and sound if* $\mathrm{G_1}$ *is valid in X.*

Proof. For soundness, see theorem 8.3. For strong completeness, let Γ be a countable $\mathrm{KD4G_1}$-consistent set of $\mathcal{L}_{[d]}$-formulas. Let $\Gamma_0 \subseteq \Gamma$ be finite and put $\gamma = \bigwedge \Gamma_0$. Then γ is $\mathrm{KD4G_1}$-consistent, so by the results of section 5.13 it is satisfied in some finite $\mathrm{KD4G_1}$-frame $\mathcal{F}$. Plainly, $\mathcal{F}$ is also a $\mathrm{KD4G_1}t$-frame, and it follows that γ is $\mathrm{KD4G_1}t$-consistent. So Γ is $\mathrm{KD4G_1}t$-consistent. By theorem 9.1, Γ is satisfiable over X. □

9.4 Strong completeness for $\mathcal{L}_{\Box}^{\langle t\rangle}$ and $\mathcal{L}_{\Box}^{\mu}$

This also follows, using the translations $-^d$ and $-^t$ of section 4.

THEOREM 9.3. *Let X be any dense-in-itself metric space.*

1. *The Hilbert system* S4t *is sound and strongly complete over X for $\mathcal{L}_{\Box}^{\langle t\rangle}$-formulas.*
2. *The Hilbert system* S4μ *is sound and strongly complete over X for $\mathcal{L}_{\Box}^{\mu}$-formulas.*
3. *(Kremer, [17]) The Hilbert system* S4 *is sound and strongly complete over X for $\mathcal{L}_{\Box}$-formulas.*

Proof. Soundness is clear in all cases: cf. theorem 8.1. We prove strong completeness. For part 1, let φ be an S4t-consistent $\mathcal{L}_{\Box}^{\langle t\rangle}$-formula. By the results of section 5.8, φ is satisfiable in some finite S4 Kripke frame $\mathcal{F}$. Recall from section 4 the translation $-^d$: it takes $\mathcal{L}_{\Box}^{\langle t\rangle}$-formulas to $\mathcal{L}_{[d]}^{\langle dt\rangle}$-formulas. Since $\mathcal{F}$ is reflexive, it follows from lemma 4.4 that φ^d is equivalent to φ in $\mathcal{F}$. So φ^d is satisfiable in $\mathcal{F}$. Plainly, $\mathcal{F}$ is also a KD4G$_1t$ frame, so φ^d is KD4G$_1t$-consistent.

Since $-^d$ commutes with $\wedge$, it is now easily seen that if $\Gamma \subseteq \mathcal{L}_{\Box}^{\langle t\rangle}$ is a countable S4t-consistent set then $\Gamma^d = \{\gamma^d : \gamma \in \Gamma\} \subseteq \mathcal{L}_{[d]}^{\langle dt\rangle}$ is a countable KD4G$_1t$-consistent set. By theorem 9.1, Γ^d is satisfiable over X. Since X is T1, by lemma 4.5 each $\gamma \in \Gamma$ is equivalent to γ^d in X, so Γ is also satisfiable over X.

For part 2, for a set $\Gamma \subseteq \mathcal{L}_{\Box}^{\mu}$ we write $\Gamma^t = \{\gamma^t : \gamma \in \Gamma\} \subseteq \mathcal{L}_{\Box}^{\langle t\rangle}$, where the translation $-^t : \mathcal{L}_{\Box}^{\mu} \to \mathcal{L}_{\Box}^{\langle t\rangle}$ is as in section 4.3. Let $\Gamma \subseteq \mathcal{L}_{\Box}^{\mu}$ be a countable S4μ-consistent set. Let $\Gamma_0 \subseteq \Gamma$ be any finite subset. By assumption, the formula $\bigwedge \Gamma_0$ is S4μ-consistent. So by theorem 3.7, there is a finite S4 frame $\mathcal{F}$ in which $\bigwedge \Gamma_0$ is satisfied. By fact 4.6, φ^t is equivalent to φ in $\mathcal{F}$, for each $\varphi \in \mathcal{L}_{\Box}^{\mu}$. So $\bigwedge(\Gamma_0^t)$ is also satisfied in $\mathcal{F}$. Since $\mathcal{F}$ is plainly an S4t frame, it follows that $\bigwedge(\Gamma_0^t)$ is S4t-consistent. As Γ_0 was arbitrary, Γ^t is S4t-consistent.

By part 1, Γ^t is satisfied in X. But by corollary 4.7, each $\gamma \in \Gamma$ is equivalent to γ^t in X. So Γ is also satisfied in X.

Part 3 can be proved similarly, by showing in the same way that for $\mathcal{L}_{\Box}$-formulas, S4-consistency implies S4t-consistency, and then appealing to part 1. $\Box$

9.5 Universal modality

We do not include the universal modality in our strong completeness results, for good reason.

THEOREM 9.4. *There is a set Σ of $\mathcal{L}_{\Box\forall}$-formulas such that for every non-empty compact locally connected dense-in-itself metric space X, each finite subset of Σ is satisfiable in X, but Σ as a whole is not.*

Compact means that if $\mathcal{S}$ is a set of open sets with $\bigcup \mathcal{S} = X$, then $X = \bigcup \mathcal{S}_0$ for some finite $\mathcal{S}_0 \subseteq \mathcal{S}$. *Locally connected* means that every open neighbourhood of a point x contains a connected (in the subspace topology) open neighbourhood of x. An example of a compact locally connected dense-in-itself metric space is the subspace $[0, 1]$ of $\mathbb{R}$.

Proof. The proof is based on the following model $\mathcal{M} = (W, R, h)$, where we suppose that $\mathsf{Var} = \{\mathsf{r}, \mathsf{g}, \mathsf{b}\} \cup \{p_i : i < \omega\}$.

1. $W = \{a_n, b_n : n < \omega\}$, where the a_n and b_n are pairwise distinct
2. R is the reflexive closure of $\{(a_n, b_n), (a_n, b_{n+1}) : n < \omega\}$
3. $h(\mathsf{r}) = \{b_{3n} : n < \omega\}$, $h(\mathsf{g}) = \{b_{3n+1} : n < \omega\}$, $h(\mathsf{b}) = \{b_{3n+2} : n < \omega\}$, and $h(p_n) = \{b_{3n}, b_{3n+1}\}$ for each $n < \omega$.

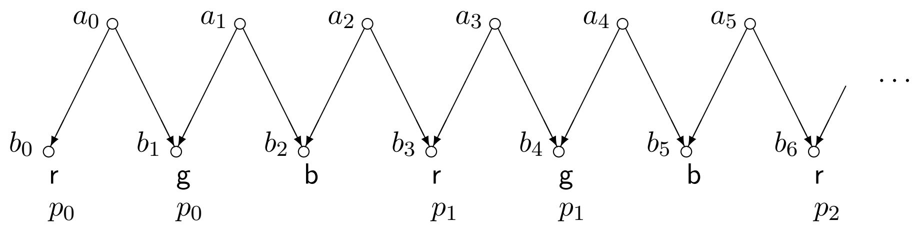


Figure 3: $\mathcal{M}$

The model is shown in figure 3 — it goes off to the right forever, roughly repeating after every three steps. Of course R is reflexive. Note that the underlying frame is connected.

We let Σ be the set comprising the following formulas:

Σ1. $\exists(\Diamond p_i \wedge \Diamond\mathsf{r} \wedge \Diamond\mathsf{g})$ for each $i < \omega$

Σ2. $\forall\neg(\Diamond p_i \wedge \Diamond p_j)$ for $i < j < \omega$

Σ3. $\forall\neg(\Diamond\mathsf{r} \wedge \Diamond\mathsf{g} \wedge \Diamond\mathsf{b})$

Σ4. $\forall(\Diamond p_i \wedge \Box\neg\mathsf{b} \to \Box\Diamond p_i)$ for $i < \omega$.

They are plainly valid in $\mathcal{M}$. Hence Σ is satisfied in $\mathcal{M}$, at every point. Moreover, any finite subset $\Sigma_0 \subseteq \Sigma$ is satisfied in a finite submodel of $\mathcal{M}$ obtained by taking a large enough 'initial segment' of $\mathcal{M}$ ending on the right at a b-world. Check especially formulas of the form $\forall\exists$. In particular, Σ4 is valid in such a submodel. Or one can use that it is a generated submodel. The submodel is finite and its frame validates S4.UC, so every formula satisfied in it — for example, $\bigwedge \Sigma_0$ — is S4.UC-consistent. Hence, by theorem 8.2, every finite subset of Σ is satisfiable in X.

Assume for contradiction that Σ is satisfied in some model (X, h) on X. Below, we will write $x \models \varphi$ instead of $(X, h), x \models \varphi$. By Σ1, for each $i < \omega$ there is $x_i \in X$ with $x_i \models \Diamond p_i \wedge \Diamond\mathsf{r} \wedge \Diamond\mathsf{g}$. As X is compact, it contains a point z such that for every open neighbourhood N of z, the set $\{i < \omega : x_i \in N\}$ is infinite. Then $z \models \Diamond\mathsf{r} \wedge \Diamond\mathsf{g}$ as well. By Σ3, $z \models \Box\neg\mathsf{b}$. As X is locally connected, there is a connected open neighbourhood N of z with $y \models \neg\mathsf{b}$ for all $y \in N$.

Take $i < j < \omega$ with $x_i, x_j \in N$. Let $U = \{x \in N : x \models \Diamond p_i\}$. Then U is an open subset of N, because for every $u \in U$ we have $u \models \Diamond p_i \wedge \Box\neg\mathsf{b}$, and Σ4 gives $u \models \Box\Diamond p_i$. And $N \setminus U$ is also open, because $U' = \{x \in X : x \models \Diamond p_i\}$ is closed and $N \setminus U = N \setminus U'$. We have $x_i \in U$, but by Σ2, $x_j \in N \setminus U$. So N is the union of two disjoint non-empty open sets (U and $N \setminus U$), contradicting its connectedness. $\Box$

COROLLARY 9.5. *Let X be a non-empty compact locally connected dense-in-itself metric space, and $\mathcal{L} \subseteq \mathcal{L}^{\mu\langle t\rangle\langle dt\rangle}_{\Box[d]\forall}$ a language containing $\mathcal{L}_{\Box\forall}$ or $\mathcal{L}_{[d]\forall}$. Then no Hilbert system for $\mathcal{L}$ is sound and strongly complete over X.*

Proof. Assume for contradiction that the Hilbert system H is sound and strongly complete over X. Let Σ be as in theorem 9.4 (use the translation $-^d$ if necessary to ensure it is a set of $\mathcal{L}$-formulas). Since every finite subset of Σ is satisfiable in X, and H is sound over X, it follows that Σ is H-consistent. But H is strongly complete over X, so Σ is satisfiable over X, contradicting the theorem. □

10 Conclusion

This paper has presented some completeness theorems for various spatial logics over dense-in-themselves metric spaces. Table 1 summarises them. The numbers in parentheses refer to our earlier results. The first line of the table is of course known, included here to give a more complete picture. For handy reference, table 2 summarises the ingredients of each logic.

Language	Logic	sound	complete	strongly complete
$\mathcal{L}_\Box$	S4	yes	yes [24]	yes [17]
$\mathcal{L}_\Box^\mu$	S4μ	yes	yes (8.1)	yes (9.3)
$\mathcal{L}_\Box^{\langle t\rangle}$	S4t	yes	yes (8.1)	yes (9.3)
$\mathcal{L}_{\Box\forall}$	S4.UC	if X connected	yes (8.2)	not in general (9.5)
$\mathcal{L}_{\Box\forall}^{\langle t\rangle}$	S4t.UC	if X connected	yes (8.2)	not in general (9.5)
$\mathcal{L}_{[d]}$	KD4G$_1$	if G$_1$ valid in X	yes (8.3)	yes (9.2)
$\mathcal{L}_{[d]}^{\langle dt\rangle}$	KD4G$_1t$	if G$_1$ valid in X	yes (8.3)	yes (9.1)
$\mathcal{L}_{[d]\forall}$	KD4G$_1$.UC	if X connected & validates G$_1$	yes (8.5)	not in general (9.5)
$\mathcal{L}_{[d]\forall}^{\langle dt\rangle}$	KD4G$_1t$.UC	if X connected & validates G$_1$	yes (8.5)	not in general (9.5)

Table 1: Soundness and completeness for a non-empty dense-in-itself metric space X

S4	$\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$
S4μ	fixed point axiom and rule: see definition 3.1
t	tangled closure axioms from section 5.3
U	$\forall\varphi \to \Box\varphi$, S5 axioms for $\forall$, $\forall$-generalisation rule
C	$\forall(\Box^*\varphi \vee \Box^*\neg\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$, where $\Box^*\varphi = \varphi \wedge \Box\varphi$
G$_1$	all uniform substitution instances of $\big([d]\bigvee_{i=0}^{1}\Box Q_i\big) \to \bigvee_{i=0}^{1}[d]\neg Q_i$, where $Q_i = p_i \wedge \neg p_{1-i}$ $(i = 0, 1)$

Table 2: Parts of the logics

There are of course many problems left open by our work, and we present some of them here.

10.1 Extensions

PROBLEM 10.1. *Can the results be extended to more general topological spaces?*

For example, consider the topological space T defined as follows. For ordinals α, β write ${}^{\alpha}\beta$ for the set of all maps $f : \alpha \to \beta$. The set of points of T is $\bigcup_{n \leq \omega} {}^{n}2$, and the open sets are unions of sets of the form $\{f \in T : f \supseteq g\}$ for some $g \in \bigcup_{n<\omega} {}^{n}2$. This space is not even T1, though it is T0 (that is, no two distinct points have the same open neighbourhoods) and dense in itself.

PROBLEM 10.2. *What is the logic of T in the various languages discussed above?*

PROBLEM 10.3. *Can the results be extended to stronger languages, for example, the mu-calculus with $[d]$ and/or $\forall$, languages with the difference modality or graded modalities, hybrid languages, and so on?* Results of Kudinov [18, 19] are relevant. Recently, Kudinov and Shehtman [20] proved numerous results about logics of topology with $\Box$, $[d]$, $\forall$, and the 'difference modality' $[\neq]$. In particular, they determine the logic of $\mathbb{R}^n$ for $n \geq 2$ in the language with $[d]$ and $[\neq]$. However, results for general dense-in-themselves metric spaces appear to be lacking.

10.2 Strong completeness

Our definition of strong completeness is limited to countable sets of formulas. We have not investigated the extent to which the strong completeness results in section 9 generalise to uncountable sets, but an argument based on the Erdős–Rado theorem [7] will show that for any given dense-in-itself topological space X and any Hilbert system H that is sound over X, there is an (uncountable) cardinal κ such that the set $\{\Diamond p_i : i < \kappa\} \cup \{\Box\neg(p_i \wedge p_j) : i < j < \kappa\}$ is H-consistent but not satisfiable in X. So strong completeness will fail over any given X, for large enough sets of formulas.

PROBLEM 10.4. *Let X be a dense-in-itself metric space. For which uncountable cardinals κ can our strong completeness results for X be extended to sets of at most κ formulas?*

Our strong completeness results for languages with $[d]$ are limited to logics with G_1. We could ask for more:

PROBLEM 10.5. *Let X be a dense-in-itself metric space and let $\mathcal{L}$ be $\mathcal{L}_{[d]}$ or $\mathcal{L}_{[d]}^{\langle dt \rangle}$. Is the $\mathcal{L}$-logic of X strongly complete over X?*

By theorems 9.1 and 9.2, the answer is 'yes' if X validates G_1.

We saw in corollary 9.5 that in the language $\mathcal{L}_{\Box\forall}$, there are many dense-in-themselves metric spaces over which S4.UC is not strongly complete. So we ask:

PROBLEM 10.6. *Can strong completeness for languages with $\forall$ be proved for each dense-in-itself metric space in some reasonably large class, and for $\mathbb{R}^n$ for $n \geq 1$?*

PROBLEM 10.7. *Is* S4.UC *strongly complete for Kripke semantics in the language $\mathcal{L}_{\Box\forall}$?*

Even without $\forall$, the example in section 5.4 can be used to show that strong completeness fails in Kripke semantics for all our systems for languages containing $\mathcal{L}_{\Box}^{\langle t \rangle}$. But we saw that strong completeness does hold for some of these systems over dense-in-themselves metric spaces. Taking the example of S4t for $\mathcal{L}_{\Box}^{\langle t \rangle}$, it is striking that this logic is sound and complete for two different semantics (the class of finite S4 frames, and any non-empty dense-in-itself metric space), but strongly complete for only the latter.

PROBLEM 10.8. *Is there any general connection between strong completeness for topological semantics and for Kripke semantics?*

10.3 Complexity

Decidability of the logics in table 1 follows from the finite model property results of section 5 and their finite (schema) axiomatisations. But we have not investigated their complexity.

PROBLEM 10.9. *What is the complexity of the logics discussed in this paper?*

Of course, the complexity of some are known (e.g., S4 is PSPACE-complete).

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