# Mosaics and step-by-step — Remarks on 'A modal logic of relations' by Venema & Marx

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Mosaics were introduced by Németi in [N1], and developed further in [N2]. They are very useful in decidability and completeness proofs. Recently, Andréka and Németi used them in proving the 'finite base' property for several logics related to cylindric algebras, such as $\mathsf{Crs}$ and (of interest here) the modal logic $\Theta_{MLR_n}(\mathsf{LC}_n)$. These results appear in [N3], and are proved in [AN].

Venema & Marx's paper [VM], in this volume, used mosaics to prove completeness and decidability theorems (theorems 4.2, 5.1) for the logic $\Theta_{MLR_n}(\mathsf{LC}_n)$, with modal syntax $\{\wedge, \neg, \iota\delta_{pq}, \bigcirc_\sigma, \Diamond_q : p, q < n, \sigma : n \to n\}$ (the Modal Logic of $n$-ary Relations), and with semantics given by the Locally Cubic $n$-dimensional frames. In this short note, we present alternative proofs of these results. Our treatment still uses mosaics, but has two, perhaps novel, features:

- the use of 'step-by-step' is largely (in the case of a single formula, completely) avoided;
- it provides a simple proof of the finite base property for this logic. This is the result of Andréka and Németi referred to above, and in [VM, remark 5.2].

Ideas and theorems of Hrushovski [Hr] and Herwig [He] are used in our proofs.

**Notation** We will refer freely to [VM] for definitions, notation, and properties of mosaics. Fix the dimension, $n > 2$. The letters $p, q$ denote numbers $< n$. The $q$th element of a sequence $\bar{s}$ is written as $s_q$. All mosaics are coherent unless otherwise stated. A map is (by definition) a certain set of ordered pairs (a binary relation), and we will use its form here. Many maps will act on the right. The identity map on a set $X$ is denoted by $Id_X$.

**DEFINITION 1** A *linked system of mosaics* (LSM) is an object of the form

$$(I, (X_i, \ell_i), \nu_{ij} : i, j \in I),$$

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where $I$ is a non-empty set, for each $i \in I$, $(X_i, \ell_i)$ is a mosaic as defined in [VM, definition 4.3], and for $i, j \in I$, $\nu_{ij}$ is a 1–1 partial map from $X_i$ into $X_j$, with the following properties.

1. $\nu_{ii} = Id_{X_i}$ for each $i \in I$
2. $\nu_{ji} = \nu_{ij}^{-1}$ for all $i, j \in I$
3. $\nu_{ij} \circ \nu_{jk} \subseteq \nu_{ik}$ for all $i, j, k \in I$.

   That is, if $x \in dom(\nu_{ij})$ and $x\nu_{ij} \in dom(\nu_{jk})$, then $x \in dom(\nu_{ik})$ and $x\nu_{ij}\nu_{jk} = x\nu_{ik}$. Note that we write maps on the right here.
4. If $i, j \in I$ and $\bar{s} \in {}^n(dom(\nu_{ij}))$ then $\ell_i(\bar{s}) = \ell_j(\bar{s}\nu_{ij})$.

   Here and below, $\bar{s}\nu_{ij}$ denotes the sequence $(s_0\nu_{ij}, \ldots, s_{n-1}\nu_{ij})$.
5. (no defects) If $i \in I, \bar{s} \in {}^nX_i$, $q < n$, and $\Diamond_q\varphi \in \ell_i(\bar{s})$, then there are $j \in I$ and $\bar{t} \in {}^nX_j$ such that $\varphi \in \ell_j(\bar{t})$, and $s_p \in dom(\nu_{ij})$ and $s_p\nu_{ij} = t_p$ for all $p \neq q$.

**LEMMA 2 (cf. [VM, lemma 4.5])** *A set $\Gamma$ of formulas is satisfiable in a locally cubic model iff there exists a LSM for $\Gamma$ (i.e., the LSM has a mosaic $(X_i, \ell_i)$ with $\ell_i(\bar{s}) \supseteq \Gamma$ for some $\bar{s} \in {}^nX_i$).*

**Proof.**

'$\Rightarrow$' is easy. If $\mathfrak{N} = (\mathfrak{C}_n^W(U), V)$ is a locally cubic model of $\Gamma$, then let $I$ be the set of all cubicles $X \subseteq U$ (so ${}^nX \subseteq W$) with $0 < |X| \leq n$. Define a mosaic $(X, \ell_X)$ for each $X \in I$, by

$$\ell_X(\bar{s}) = \{\varphi : \varphi \text{ a formula}, \mathfrak{N}, \bar{s} \Vdash \varphi\}, \text{ for each } \bar{s} \in {}^nX.$$

Define $\nu_{XY} : X \to Y$ to be simply $Id_{X\cap Y}$. It's easy to check that the system $(I, (X, \ell_X), \nu_{XY} : X, Y \in I)$ is an LSM for $\Gamma$. For example, $Id_{X\cap Y} \circ Id_{Y\cap Z} \subseteq Id_{X\cap Z}$.

'$\Leftarrow$': let $(I, (X_i, \ell_i), \nu_{ij} : i, j \in I)$ be an LSM for $\Gamma$. By renaming elements if need be, we can assume that the $X_i$ are pairwise disjoint. Define a binary relation $\sim$ on $\bigcup_{i\in I} X_i$ by:

$$x \sim y \iff x \in dom(\nu_{ij}) \text{ and } x\nu_{ij} = y.$$

Clearly, $\sim$ is an equivalence relation on $\bigcup_{i\in I} X_i$. Let

$$Z = (\bigcup_{i\in I} X_i)/\sim.$$

We write $x/{\sim}$ for the $\sim$-class of an element $x \in \bigcup X_i$. We use this notation for $n$-sequences, too: so if $\bar{s} \in {}^nZ$ and $\bar{x} \in {}^n(\bigcup_{i\in I} X_i)$, we write $\bar{s} = \bar{x}/{\sim}$ if $s_q = x_q/{\sim}$ for all $q < n$.

$Z$ will be the base of our model of $\Gamma$. Its domain will be the set

$$W = \{\bar{x}/{\sim} : \bar{x} \in {}^nX_i \text{ for some } i \in I\}.$$

So $W$ consists of the sequences in ${}^nZ$ that come from a single $X_i$, modulo $\sim$. Clearly, if $\bar{s} \in W$ then $\mathcal{C}(\bar{s}) \subseteq W$ (see [VM, equation (18)] for the notation), so $W$ is locally cubic.

Define accessibility relations for $Id_{pq}, \bigcirc_\sigma, \Diamond_q$ on $W$ in the standard way (see [VM, definition 3.2]). Now associate with each $\bar{s} \in W$ a MCS $\Gamma_{\bar{s}}$ as follows.

- Choose $\bar{x} \in {}^nX_i$ for some $i$ such that $\bar{s} = \bar{x}/{\sim}$.
- Let $\Gamma_{\bar{s}} = \ell_i(\bar{x})$.

This is well-defined, for if $\bar{x} \in {}^nX_i, \bar{y} \in {}^nX_j$ and $\bar{x}/{\sim} = \bar{y}/{\sim}$, then by definition of '$\sim$' we have $\bar{x}\nu_{ij} = \bar{y}$. So $\ell_i(\bar{x}) = \ell_j(\bar{y})$ by the fourth condition in the definition of LSM. Finally, define a valuation $V$ on atoms $r$ by $V(r) = \{\bar{s} \in W : r \in \Gamma_{\bar{s}}\}$. This yields a locally cubic model $\mathfrak{N} = (\mathfrak{C}_n^W(Z), V)$ with base $Z$.

**Truth lemma:** for all $\bar{s} \in W$ and all formulas $\varphi$, $\mathfrak{N}, \bar{s} \Vdash \varphi \iff \varphi \in \Gamma_{\bar{s}}$.

**Proof:** By induction on $\varphi$. For $\varphi$ a propositional atom or a boolean combination of simpler formulas, the argument is standard. We consider the other cases.

- $\varphi = \iota\delta_{pq}$. Choose $i$ and $\bar{x} \in {}^nX_i$ such that $\bar{x}/{\sim} = \bar{s}$. So $\Gamma_{\bar{s}} = \ell_i(\bar{x})$. Then $\mathfrak{N}, \bar{s} \Vdash \iota\delta_{pq}$ iff $s_p = s_q$ iff $x_p \sim x_q$. By definition of $\sim$, this is iff $x_p = x_q$, since $\nu_{ii} = Id_{X_i}$. This is iff $\iota\delta_{pq} \in \ell_i(\bar{x}) = \Gamma_{\bar{s}}$.
- $\varphi = \bigcirc_\sigma\psi$. Assume inductively that the result holds for $\psi$. Let $i \in I$, $\bar{x}, \bar{y} \in {}^nX_i$, $\bar{t} \in {}^nZ$ be such that $\bar{x}/{\sim} = \bar{s}$, $\bar{y} = \bar{x} \circ \sigma$, $\bar{t} = \bar{s} \circ \sigma$. Note that $\bar{t} = \bar{y}/{\sim}$. Then $\bigcirc_\sigma\psi \in \Gamma_{\bar{s}} = \ell_i(\bar{x})$ iff $\psi \in \ell_i(\bar{y}) = \Gamma_{\bar{t}}$ (because $(X_i, \ell_i)$ is a mosaic), iff $\mathfrak{N}, \bar{t} \Vdash \psi$ (by the inductive hypothesis), iff $\mathfrak{N}, \bar{s} \Vdash \bigcirc_\sigma\psi$ (as $\mathfrak{N}$ is a model).
- $\varphi = \Diamond_q\psi$. Assume inductively that the result holds for $\psi$, and choose $i$ and $\bar{x} \in {}^nX_i$ such that $\bar{x}/{\sim} = \bar{s}$.

  First assume $\mathfrak{N}, \bar{s} \Vdash \Diamond_q\psi$. So for some $\bar{t} \in W$ with $\bar{t} \equiv_q \bar{s}$, we have $\mathfrak{N}, \bar{t} \Vdash \psi$. Inductively, $\psi \in \Gamma_{\bar{t}}$, so by axiom CM1 of the system $QAX_n$ [VM, definition 4.1] we have $\Diamond_q\psi \in \Gamma_{\bar{t}}$.

  Choose $j \in I$ and $\bar{y} \in {}^nX_j$ such that $\bar{y}/{\sim} = \bar{t}$. So if $p < n, p \neq q$, we have $x_p \sim y_p$, so $x_p \in dom(\nu_{ij})$ and $x_p\nu_{ij} = y_p$. Choose $q' \neq q$, and write $\bar{y}'$ for the tuple $\bar{y}$ but with its $q$th entry $y_q$ replaced by $y_{q'}$. We know that $\Diamond_q\psi \in \ell_j(\bar{y})$ $(= \Gamma_{\bar{t}})$. As $(X_j, \ell_j)$ is a mosaic, it follows that $\Diamond_q\psi \in \ell_j(\bar{y}')$. But $\bar{y}'\nu_{ji}$ exists and is $\bar{x}'$, where $\bar{x}'$ is the tuple $\bar{x}$ altered in the same way that $\bar{y}$ was to get $\bar{y}'$. By definition of LSMs, $\ell_i(\bar{x}') = \ell_j(\bar{y}')$, and so $\Diamond_q\psi \in \ell_i(\bar{x}')$. As $(X_i, \ell_i)$ is a mosaic, $\Diamond_q\psi \in \ell_i(\bar{x}) = \Gamma_{\bar{s}}$.

Conversely, assume that $\Diamond_q\psi \in \Gamma_{\bar{s}} = \ell_i(\bar{x})$. By definition of LSM, there are $j \in I$ and $\bar{y} \in {}^nX_j$ with $x_p \in dom(\nu_{ij})$, $x_p\nu_{ij} = y_p$ for all $p < n, p \neq q$, and $\psi \in \ell_j(\bar{y})$. Let $\bar{t} = \bar{y}/\sim$. Then $\bar{t} \in W$, and $\psi \in \Gamma_{\bar{t}}$. Inductively, $\mathfrak{N}, \bar{t} \Vdash \psi$. As $\bar{t} \equiv_q \bar{s}$ by definition of $\sim$, we obtain $\mathfrak{N}, \bar{s} \Vdash \Diamond_q\psi$.

This proves the truth lemma. Lemma 2 now follows, as $\Gamma \subseteq \Gamma_{\bar{s}}$ for some $\bar{s} \in W$. ■

**LEMMA 3 (cf. [VM, Lemma 4.6])** *A set $\Sigma$ of formulas is consistent iff there exists an LSM for it.*

**Proof.**

'$\Leftarrow$' is trivial. For '$\Rightarrow$', assume that $\Sigma$ is consistent. So there is a mosaic $(X, \ell)$ with $\Sigma \subseteq \ell(\bar{s})$ for some $\bar{s} \in {}^nX$ (for a proof, see [VM, lemma 4.6]).

Say that two mosaics are *isomorphic* if there is a bijection between their base sets that preserves the labelling. Choose representatives $(X_i, \ell_i)$ $(i \in I)$ for the isomorphism types of mosaic — there is only a *set* of isomorphism types of mosaic, though it will usually have size $2^\omega$. We can assume that the base sets of the $X_i$ are pairwise disjoint.

Form a model-theoretic structure $K$ as follows. $K$ is analogous to the 'universal SSM' built in [VM, lemma 4.6]. The signature $L$ of $K$ consists of an $n$-ary relation symbol for each MCS $\Gamma$. We identify $\Gamma$ with the relation symbol for it. The domain of $K$ is the disjoint union $\bigcup_{i\in I} X_i$. (As is usual in model theory, we will use the same notation for the structure $K$ as for its domain or universe). And if $\bar{x} \in {}^nK$ and $\Gamma \in L$, we define

$$K \models \Gamma(\bar{x}) \iff \exists i \in I(\bar{x} \in {}^nX_i \wedge \ell_i(\bar{x}) = \Gamma).$$

By the proof of [VM, lemma 4.6], $K$ has the following properties:

($*$) $\Sigma \subseteq \ell_i(\bar{x})$ for some $i \in I$ and $\bar{x} \in {}^nX_i$

($\dagger$) for all $i \in I$, all $\bar{x} \in {}^nX_i$, and all $\Diamond_q\varphi \in \ell_i(\bar{x})$, there are $j \in I$ and $\bar{y} \in {}^nX_j$ with $\varphi \in \ell_j(\bar{y})$, and a partial isomorphism[1] $\alpha$ of $K$ such that $x_p\alpha$ is defined and is $y_p$, for each $p < n$ with $p \neq q$.

We can replace $I$ by a countable subset and keep these properties.[2] Thus, we can assume that $K$ is countable. Since only countably many relations in $L$ now have a non-empty interpretation in $K$, we can assume that $L$ is countable, too.

[1] Recall that a partial isomorphism of a relational structure is a partial 1–1 map from (the domain of) the structure into itself that preserves all relations both ways.

[2] Essentially by the downward Löwenheim–Skolem–Tarski theorem — a step-by-step argument, but this move is for convenience only.

**Claim.** There is an $L$-structure $K^+ \supseteq K$ such that any finite partial isomorphism of $K$ is induced by an automorphism of $K^+$.

**Proof of claim.** Let $\mathfrak{A}$ be the class of all finite $L$-structures $A$ such that for any $\bar{a} \in {}^nA$ there is at most one $\Gamma \in L$ such that $A \models \Gamma(\bar{a})$. Then $\mathfrak{A}$ is countable (up to isomorphism), is closed under substructures, and has the joint embedding and amalgamation properties. So by Fraïssé's theorem[3] there is a countable homogeneous $L$-structure $K^+$ with age $\mathfrak{A}$. Every finite partial isomorphism of $K^+$ extends to an automorphism of $K^+$. As $\mathrm{Age}(K) \subseteq \mathfrak{A}$, a standard argument [C, exercise 2.6.5(ii)] shows that, up to isomorphism, $K \subseteq K^+$. This proves the claim.

To prove the lemma, we need to exhibit an LSM for $\Sigma$. Let $G$ be the automorphism group of $K^+$. Define an LSM

$$(I \times G, M_{ig}, \nu_{ig,jh} : (i,g),(j,h) \in I \times G)$$

as follows.

- $M_{ig} = (X_i, \ell_i)$ — i.e., '$g$' is ignored;
- $\nu_{ig,jh} = gh^{-1} \cap (X_i \times X_j)$.

Let us check the definition for an LSM.

1. $\nu_{ig,ig} = Id_{X_i}$ for each $i, g$ — clear.
2. $\nu_{jh,ig} = \nu^{-1}_{ig,jh}$ for all $i, j \in I, g, h \in G$.
   This holds because $\nu^{-1}_{ig,jh} = (gh^{-1} \cap (X_i \times X_j))^{-1} = hg^{-1} \cap (X_j \times X_i) = \nu_{jh,ig}$.
3. $\nu_{ig,jh} \circ \nu_{jh,kf} \subseteq \nu_{ig,kf}$ for all $i, j, k \in I$ and all $f, g, h \in G$.
   We reason as follows. If $(x,y) \in \nu_{ig,jh} \circ \nu_{jh,kf}$ then there is a $z$ such that $(x,z) \in \nu_{ig,jh}$ and $(z,y) \in \nu_{jh,kf}$. That is, $(x,z) \in gh^{-1} \cap (X_i \times X_j)$ and $(z,y) \in hf^{-1} \cap (X_j \times X_k)$. So $(x,y) \in gf^{-1}$, and obviously $(x,y) \in X_i \times X_k$. So $(x,y) \in \nu_{ig,kf}$.
4. If $i, j \in I$, $g, h \in G$, and $\bar{s} \in {}^n(dom(\nu_{ig,jh}))$, then $\ell_i(\bar{s}) = \ell_j(\bar{s}\nu_{ig,jh})$. (We can write $\ell_i$ rather than $\ell_{ig}$ here.)
   Suppose $\ell_i(\bar{s}) = \Gamma$. Then $K \models \Gamma(\bar{s})$. $K$ is a substructure of $K^+$, so $K^+ \models \Gamma(\bar{s})$, too. As $\nu_{ig,jh}$ is induced by an automorphism ($gh^{-1}$) of $K^+$, we get $K^+ \models \Gamma(\bar{s}\nu_{ig,jh})$. But $K \subseteq K^+$ and $\bar{s}\nu_{ig,jh} \in {}^nK$, so $K \models \Gamma(\bar{s}\nu_{ig,jh})$ too. That is, $\ell_j(\bar{s}\nu_{ig,jh}) = \Gamma$.
5. (no defects) If $i \in I$, $g \in G$, $\bar{s} \in {}^nX_i$, $q < n$, and $\Diamond_q\varphi \in \ell_i(\bar{s})$, then there are $j \in I$, $h \in G$, and $\bar{t} \in {}^nX_j$ such that $s_p \in dom(\nu_{ig,jh})$ and $s_p\nu_{ig,jh} = t_p$ for all $p \neq q$, and $\varphi \in \ell_j(\bar{t})$.
   By property (†), there are $j \in I$, $\bar{t} \in {}^nX_j$, and a partial isomorphism $\alpha : X_i \to X_j$ of $K$ 'resolving the defect': $s_p \in dom(\alpha)$ and $s_p\alpha = t_p$

[3] See [F] or [Ho, theorem 7.1.2]; the argument is a step-by-step, but it is rather an old one (1954).

for all $p \neq q$, and $\varphi \in \ell_j(\bar{t})$. Now $\alpha$ is induced by an automorphism $f \in G$, say. Choose $h \in G$ such that $gh^{-1} = f$ — so $h = f^{-1}g$. Then $\nu_{ig,jh} = f \cap (X_i \times X_j)$, and the required property follows.

By property $(*)$, there are $i \in I$, $g \in G$, and $\bar{s} \in {}^nX_i$ such that $\Sigma \subseteq \ell_i(\bar{s})$. So we have exhibited an LSM for $\Sigma$. ■

These two lemmas suffice to prove the completeness theorem [VM, theorem 4.2] for the given axioms (definition 4.1) with respect to locally cubic models.

[VM] also prove decidability of the logic by finitizing their constructions. We can do the same here, and we obtain as a bonus the finite-base result (corollary 8). Fix a single formula, $\xi$. The notions of '$\xi$-mosaic' and '$\xi$-SSM' were defined in [VM, section 5]; we define '$\xi$-LSM' analogously. Formally, the definition of $\xi$-LSM is obtained from definition 1 above by replacing 'mosaic' by '$\xi$-mosaic', and adding a clause saying that $\xi \in \ell_i(\bar{s})$ for some $i \in I$ and $\bar{s} \in {}^nX_i$. We may now repeat lemma 2 above using $\xi$-LSMs, to get:

**LEMMA 4 (cf. [VM, §5, claim 2])**

1. *$\xi$ is satisfiable in a locally cubic model iff there exists a $\xi$-LSM.*
2. *$\xi$ is satisfiable in a locally cubic model with finite base iff there exists a finite $\xi$-LSM (i.e., '$I$' is finite).*

**Proof.**

As before; the second part is clear by construction. As in [VM], the 'finitized truth lemma' will take the form '$\mathfrak{N}, \bar{s} \Vdash \varphi \iff \varphi \in \Gamma_{\bar{s}}$ for all $\bar{s}$ and all formulas $\varphi$ *of complexity at most that of* $\xi$'; the proof is nearly identical to the original version. ■

However, it is important to note that the two parts of lemma 4 are in fact equivalent:

**LEMMA 5** *For any formula $\xi$, if there exists a $\xi$-LSM then there exists a* finite *$\xi$-LSM.*

**Proof.**

Observe that up to isomorphism there exist only finitely many $\xi$-mosaics. Hence, if $\mathcal{M} = (J, M_j, \nu_{jk} : j, k \in J)$ is a $\xi$-LSM, we may take a finite set $I \subseteq J$ such that, up to isomorphism, $\{M_i : i \in I\} = \{M_j : j \in J\}$. Forming $K$ from $I$ as in the proof of lemma 3, we see that $K$ has the properties $(*, \dagger)$ (for $\Sigma = \{\xi\}$). So we can continue with that proof, omitting the use of the downward Löwenheim–Skolem–Tarski theorem (it is not needed, as $K$ is already finite), and replacing the claim by the following theorem.

**THEOREM 6 (Herwig, [He])** *Let $K$ be a finite structure in a finite relational language $L$. There is a finite $L$-structure $K^+ \supseteq K$ such that any partial isomorphism of $K$ is induced by an automorphism of $K^+$.*

We obtain a finite $\xi$-LSM, as required. ∎

**COROLLARY 7** *The logic $\Theta_{MLR_n}(\mathsf{LC}_n)$ is decidable.*

**Proof.**

Let $\xi$ be any formula. It is clear that $\xi$ is satisfiable in a locally cubic model iff there exists a structure $K$ with properties $(*, \dagger)$ as in the proof of lemma 5. $K$ is finite, and a bound on its size can be calculated from $\xi$. Thus, the existence of $K$ can be checked algorithmically. ∎

Combining lemmas 4 and 5 also gives the following corollary, which was already known [AN,N3]:

**COROLLARY 8 (Andréka, Németi)** *Any satisfiable formula $\xi$ is satisfiable in a locally cubic model with finite base.*

**REMARKS 9** These methods can be extended to other logics: for example, to prove the finite base property for $\mathsf{Crs}_n$, with semantics '$W$ an arbitrary subset of ${}^nU$'; again, this result was already known [AN,N3].

Herwig's (and Hrushovski's) proof is similar to the proofs of lemmas 2 and 3. It does not use step-by-step. The proof also gives a bound on the cardinality of $K^+$, in terms of $|K|$ and $n$. This can be used to give a bound on the size of the base set in corollary 8 in terms of $|\xi|$, and hence an alternative proof of decidability. Possibly a better bound can be obtained from a more recent alternative proof of theorem 6 by Lascar.

The model $\mathfrak{N}$ of $\Sigma$ obtained from the lemmas has nice 'homogeneity' properties: for example, for any two worlds $\bar{s}, \bar{t}$ satisfying the same modal formulas, there is an automorphism $g$ of $\mathfrak{N}$ with $\bar{s}g = \bar{t}$.

**REMARK 10** Andréka and Németi observe that the structure $K^+$ in the proofs of lemmas 3 and 5 is essentially already a model of $\Sigma$. This permits a completeness and decidability proof avoiding the use of LSMs, and also a proof of the finite base property directly from the semantics, using Herwig's theorem but no mosaics or completeness.

The idea is as follows. Let $K, K^+, L$ be as in lemmas 3 and 5. Let $W = \{\bar{s} \in {}^nK^+ : K^+ \models \Gamma(\bar{s}) \text{ for some } \Gamma \in L\}$. In fact, one may choose $K^+$ so that for any $\bar{s} \in W$ there is an automorphism of $K^+$ mapping $\bar{s}$ into ${}^nK$. Hence, for any $\bar{s} \in W$ there will be a unique $\Gamma_{\bar{s}} \in L$ such that $K^+ \models \Gamma_{\bar{s}}(\bar{s})$. Also, $W$ will be locally cubic. For atoms $r$, set $V(r) = \{\bar{s} \in W : r \in \Gamma_{\bar{s}}\}$. We can now define a model $\mathfrak{K} = (\mathfrak{C}_n^W(K^+), V)$. One can prove a truth lemma: for all $\bar{s} \in W$ and all formulas $\varphi$ (all $\varphi \in For_{(\xi)}$, in the 'finitized' case), $\mathfrak{K}, \bar{s} \Vdash \varphi \iff \varphi \in \Gamma_{\bar{s}}$. So $\mathfrak{K}$ will be a model of $\Sigma$.

This approach certainly allows a shorter treatment than the one we gave. However, we have decided to retain LSMs, for two reasons. First, the LSM is a notion rather between syntax and semantics, as was the SSM of [VM]; as with SSMs, it may gain independent interest from this. Second, the two approaches are essentially equivalent. Lemma 3 (or 5) builds an LSM for $\Sigma$, and the construction of lemma 2 (or 4) then yields a model $\mathfrak{N}$ of $\Sigma$. We can also build a model $\mathfrak{K}$ of $\Sigma$ directly from $K^+$, as above. These two models are isomorphic. Following through the proofs of the lemmas shows that $\mathfrak{N}$ essentially has base set $B = (K \times G)/\approx$, where $G = Aut(K^+)$ and $\approx$ is defined by $(x, g) \approx (y, h)$ iff $xg = yh$. We may define $\theta : B \to K^+$ by $((x, g)/\approx)\theta = xg$. Clearly, $\theta$ is a well-defined injection from $B$ to $K^+$ (for suitable $K^+$, a *bijection*), and an isomorphism in the sense that $\mathfrak{N}, \bar{s} \Vdash \Gamma$ iff $K^+ \models \Gamma(\bar{s}\theta)$ for all worlds $\bar{s}$ of $\mathfrak{N}$ and all $\Gamma \in L$.

## References

**AN** H. Andréka, I. Németi, Finite base property of cylindric-relativized set algebras, Math. Institute, Budapest, 1995, manuscript.

**C** P.J. Cameron, Oligomorphic Permutation Groups, London Mathematical Society Lecture Notes 152, Cambridge University Press, 1990.

**F** R. Fraïssé, Sur l'extension aux relations de quelques propriétés des ordres, Ann. Sci. École Norm. Sup. 71 (1954) 363–388.

**He** B. Herwig, Extending partial isomorphisms on finite structures, Combinatorica 15 (1995) 365–371.

**Ho** W. Hodges, Model Theory, Cambridge University Press, 1993.

**Hr** E. Hrushovski, Extending Partial Isomorphisms of Graphs, Combinatorica 12 (1992) 411–416.

**N1** I. Németi, Free algebras and decidability in algebraic logic. Second doctoral dissertation, Hungar. Acad. Sci., 1986 [in Hungarian].

**N2** I. Németi, Decidable versions of first order logic and cylindric- relativized set algebras. In: Logic Colloquium '92 (Proc. Veszprem, Hungary, 1992), L. Csirmaz, D. Gabbay, M. de Rijke (eds.), Studies in Logic, Language and Computation, CSLI Publications, 1995, pp.177–241.

**N3** I. Németi, Fine structure analysis of first order logic. In: Arrow Logic and Multi-Modal Logic, M. Marx, L. Pólos and M. Masuch (eds.), CSLI Publications, 1996.

**VM** Y. Venema, M. Marx, A Modal Logic of Relations, this volume (Report no. IR-396, Faculteit der Wiskunde en Informatica, Vrije Universiteit Amsterdam, 1995).

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