TUTORIAL 4

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Differential Equations

1st Order

Separable Variables

These equations take the form: $f(y)\frac{dy}{dx} = g(x)$

where f is a function just of y and g is a function just of x.

Integrating both sides with respect to x gives:

$$\int f(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

$$\Rightarrow \int f(y)\,dy = \int g(x)\,dx$$

This last line gives us the general solution.

E.g. 1

$$5y\frac{dy}{dx} = 5x^2$$

$$\Rightarrow \int 5y\frac{dy}{dx}\,dx = \int 5x^2\,dx$$

$$\Rightarrow \int 5y\,dy = \int 5x^2\,dx$$

$$\Rightarrow \underline{\frac{5}{2}y^2 = 2x^3 + A}$$

E.g. 2

$$2\frac{dy}{dx} + y = 0$$

Constant coefficient example from notes!

$$\Rightarrow 2\frac{dy}{dx} = -y$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{2}$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int -\frac{1}{2}\,dx$$

$$\Rightarrow \ln|y| = -\frac{x}{2} + A$$

$$\Rightarrow \underline{y = Be^{-\frac{x}{2}}}$$

Exact 1^{st} Order D.E.

Let $v(x)$ and $w(x)$ be functions of x.
Let $u(x,y)$ be a function of x and y defined as follows: $u(x,y) := w(x) * y$,
that is, y times $w(x)$.

Now, recall the product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

We term an exact 1^{st} order d.e. an equation of the form:

$$f(x)\frac{dy}{dx} + g(x)\,y = h(x)$$

such that there is some function product $u(x,y)\,v(x)$ given which

$$\frac{d}{dx}(uv) = f(x)\frac{dy}{dx} + g(x)\,y$$

That is to say, some function product uv which has as its derivative the LHS of the d.e.

Using the anti-derivative rule, we can integrate an exact d.e. with respect to x to give general solution:

$$\underline{u(x,y)\,v(x) = \int h(x)\,dx}$$

$\boxed{E.g.}$

$$x\frac{dy}{dx} + y = e^{x}$$

LHS is the derivative of uv where $\begin{cases} u := y \\ v := x \end{cases}$

$$\Rightarrow \quad xy = \int e^{x}$$

$$\Rightarrow \quad \underline{xy = e^{x} + k}$$

Inexact (Integrating Factor)

Consider an inexact equation of the form : $\frac{dy}{dx} + g(x)y = h(x)$

Let $I(x)$ be a function of x such that $I(x)\frac{dy}{dx} + I(x)g(x)y = I(x)h(x)$ is exact.

We term such an $I(x)$ an <u>integrating factor</u>.

Now, integrating factors and the according general solution of inexact d.e.s happen to take particular forms. Consider :

$\boxed{\text{I.F.}}$ Compare $I\frac{dy}{dx} + Igy$ with $v\frac{du}{dx} + u\frac{dv}{dx}$

This gives: $\begin{cases} v = I & \text{———— } * \\ \frac{du}{dx} = \frac{dy}{dx} \\ u = y \\ \frac{dv}{dx} = gI & \text{———— } ** \end{cases}$

$*$ and $**$ give us that

$$\frac{dI}{dx} = gI \quad \text{———— } ***$$

Solving $***$ as a 1^{st} order d.e. with separable variables gives :

$$\int \frac{1}{I}\, dI = \int g\, dx$$

$$\Rightarrow \ln|I| = \int g\, dx$$

$$\Rightarrow \underline{\underline{I = e^{\int g\, dx}}} \longleftarrow \left\{ \text{Form taken by integrating factor} \right.$$

$\boxed{\text{Gen. Sol.}}$ Using the assignment of $u, v, \frac{du}{dx}, \frac{dv}{dx}$ above (between $*$ and $**$), and employing the methodology used for exact equations, we get :

$$I\frac{dy}{dx} + Igy = Ih$$

$$\Rightarrow \int \left(I\frac{dy}{dx} + Igy\right) dx = \int Ih\, dx$$

$$\Rightarrow \underline{\underline{Iy = \int Ih\, dx}} \longleftarrow \left\{ \text{This is the gen. sol} \right.$$

2^{nd} Order

(NB) 3^{rd} not covered in lectures.

We'll consider 3 cases:

Case (1)

Suppose $y := Ae^{\alpha x} + Be^{\beta x}$ ——— $*$

where A, α, B, β are arbitrary real constants, and α and β are distinct ($\alpha \neq \beta$).

$$\Rightarrow \frac{dy}{dx} = A\alpha e^{\alpha x} + B\beta e^{\beta x} \text{ ——— } **$$

$$\Rightarrow \frac{d^2y}{dx^2} = A\alpha^2 e^{\alpha x} + B\beta^2 e^{\beta x} \text{ ——— } ***$$

Substituting $*$ and $**$ into $***$ to remove A and B yields:

$****$

$$\frac{d^2y}{dx^2} - (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y = 0$$

(NB) we term this equation with 0 on RHS homogeneous. We can solve inhomogeneous equations, but this is beyond scope of course.

Compare $****$ with the quadratic equation : $u^2 - (\alpha+\beta)u + \alpha\beta = 0$

We term this the auxiliary quadratic equation and use it to recognise the gen. sol. of equations of the form:

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0} \text{ ——— } *****$$

If the auxiliary equation $au^2 + bu + c = 0$ has real, distinct roots (i.e. $\Delta := b^2 - 4ac > 0$), then we can quote (by way of the anti-derivative rule) that

$$\boxed{y = Ae^{\alpha x} + Be^{\beta x}}$$ ——— where α, β are the real, distinct roots

is a solution.

It is beyond the scope of the course to show that this solution, with arb A and B is the <u>general solution</u> of *****.

The reader will need either just to accept the fact, or, if interested, to consider the <u>principle of superposition</u>:

Let y_1 and y_2 be two solutions of *****.

(i) For any constants c_1 and c_2, $c_1 y_1 + c_2 y_2$ is also a solution of *****.

(ii) Let y be any other solution of *****. If y_2 is <u>not</u> a constant multiple of y_1 and $y_1 \neq 0$, then there exists some contant k_1 and some constant k_2 such that $y = k_1 y_1 + k_2 y_2$.

Case (2)

Suppose $y := e^{\alpha x}(A + Bx)$ ———————— $*$

where A, B, α are arbitrary, <u>real</u> constants.

$$\Rightarrow \frac{dy}{dx} = \alpha y + B e^{\alpha x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \alpha \frac{dy}{dx} + B\alpha e^{\alpha x}$$

$$= \alpha \frac{dy}{dx} + \alpha\left(\frac{dy}{dx} - \alpha y\right)$$

$$= 2\alpha \frac{dy}{dx} - \alpha^2 y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\alpha \frac{dy}{dx} + \alpha^2 y = 0$$

So, by reasoning similarly to case (1),

$$\boxed{y = e^{\alpha x}(A + Bx)}$$

gives us the general solution of the differential equation

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0}$$

provided the auxiliary equation $au^2 + b^2 + c = 0$ (i.e. $b^2 - 4ac = 0$) has equal roots, namely α

Not covered in lectures.

Case (3):

For the interested reader, consider $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of

$$y := Ae^{px}\cos(qx+r)$$
$$= e^{px}(B\cos qx + C\sin qx)$$

By compound angle formula

By proceeding analogously to earlier cases, it can be shown that

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0}$$

has general solution

$$\boxed{y = Ae^{px}\cos(qx+r)}$$

if $b^2 - 4ac < 0$ (auxiliary equation has complex roots)