TUTORIAL 3

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Linear Systems

* These take the form: $A\bar{x} = \bar{b}$
  where A is a matrix and $\bar{x}$ and $\bar{b}$ are vectors.
  We are solving for the unknown $\bar{x}$.

* An equivalent formulation is in terms of simultaneous equations:

m linear equations in n unknowns $x_1, \ldots, x_n$

$$\begin{cases} (1) & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\ (2) & a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\ \vdots & \\ (m) & a_{m1} x_1 + a_{m2} x_n + \ldots + a_{mn} x_n = b_m \end{cases}$$

* The matrix and simultaneous-equations formulations are entirely interchangeable!!

* Another matrix formulation is: $\bar{y} C = \bar{d}$
  where $C = A^T$, $\bar{y}$ is the row vector equivalent of column vector $\bar{x}$ (i.e. $\bar{y} = \bar{x}^T$) and $\bar{d}$ is likewise the row vector equivalent of column vector $\bar{b}$.
  We are solving for the unknown $\bar{y}$.

* $\bar{x}$ is <u>unique</u> $\iff$ A is non-singular
  $\iff$ $|A| \neq 0$

* $\bar{x} = 0$ (uniquely) $\iff$ $\bar{b} = 0$ <u>and</u> A is non-singular
  $\iff$ $\bar{b} = 0$ <u>and</u> $|A| \neq 0$

* Two methods for solving linear systems are Gaussian elimination and the inverse-matrix method (Gaussian elimination may be thought of conceptually equivalent to rearrangement of simultaneous equations)
  (NB) There are many other solution methods.

Relating Linear Systems and Eigen systems

* eigensystems take the form: $C\bar{x} = \lambda \bar{x}$
where C is a matrix, $\bar{x}$ is a vector and λ is a scalar.
we are solving for the unknowns λ and $\bar{x}$, where $\bar{x} \neq 0$.

* Note:

$$
\begin{aligned}
& C\bar{x} = \lambda \bar{x} \\
\iff & C\bar{x} = \lambda I \bar{x} \\
\iff & C\bar{x} - \lambda I \bar{x} = 0 \\
\iff & (C - \lambda I)\bar{x} = 0
\end{aligned}
$$

The first line in the note is in eigensystem form
The last lines is in linear-system form.
So, we can use the work done on linear systems to understand the solution method for eigensystems.

* we recall from the linear system write up that

$$
\begin{aligned}
\bar{x} \neq 0 \iff & |C - \lambda I| \neq 0 \\
\iff & (C - \lambda I) \text{ is non-singular.}
\end{aligned}
$$

So, this gives us a method for solving for λ.

* To solve for the $\bar{x}$ vectors (there may be many of these) corresponding to a particular λ, we set $A := C - \lambda I$ and use Gaussian elimination or the inverse matrix method to solve for $\bar{x}$.

* NB The description above is actually of <u>right</u>-eigenvectors. We solve for left-eigenvectors in the same way $\bar{y} C = \bar{y}\lambda$. The λ values do <u>not</u> depend on left or right, <u>but</u> the eigenvectors do! The left ones are <u>not</u> necessarily equivalent to the right ones.

Two Different Settings For Matrices

* Linear Transformations

The examples you consider take a graphical form. Consider rotations and reflections. Also, if in 2d settings, consider singular matrices which map planes to lines or points. If in 3d settings, consider singular matrices which map 3d spaces to planes or lines or points.

* Markov Probability Transition Matrices

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