TUTORIAL 5

SEQUENCES

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dvd 03

* Bounds

Lower bounds (in particular, infimum)
Upper bounds (in particular, supremum)

* Sample Convergence Proofs

E.g. 1 $1+\frac{1}{n} \to 1$ as $n \to \infty$

Proof:

Choose any $\varepsilon > 0$
Aim to find some N s.t. $\forall n > N$, $\left|\left(1+\frac{1}{n}\right) - 1\right| < \varepsilon$

Now, $\left|\left(1+\frac{1}{n}\right) - 1\right| < \varepsilon \Leftarrow \left|\frac{1}{n}\right| < \varepsilon$
$\Leftarrow |n| > \frac{1}{\varepsilon}$

So, if we choose $N := \frac{1}{\varepsilon}$, then $\forall n > N$,
we have that $\left|\left(1+\frac{1}{n}\right) - 1\right| < \varepsilon$ $\blacksquare$

E.g. 2 $\frac{n^2-1}{n^2+1} \to 1$ as $n \to \infty$

Proof:

Choose any $\varepsilon > 0$
Aim to find some N s.t. $\forall n > N$, $\left|\frac{n^2-1}{n^2+1} - 1\right| < \varepsilon$

Now, $\left|\frac{n^2-1}{n^2+1} - 1\right| < \varepsilon \Leftarrow \left|\frac{(n^2-1)-(n^2+1)}{n^2+1}\right| < \varepsilon$

$\Leftarrow \left|\frac{-2}{n^2+1}\right| < \varepsilon$

$\Leftarrow \left|\frac{n^2+1}{2}\right| > \frac{1}{\varepsilon}$

$$\Leftarrow \quad n^2+1 > \frac{2}{\varepsilon} \qquad (\text{as } n \geq 0)$$

$$\Leftarrow \quad n^2 > \frac{2}{\varepsilon}$$

$$\Leftarrow \quad n > \sqrt{\tfrac{2}{\varepsilon}}$$

So, if we choose $N := \sqrt{2/\varepsilon}$, then $\forall n > N$, we have that $\left| \frac{n^2-1}{n^2+1} - 1 \right| < \varepsilon$ ■

E.g. 3 A sequence can have at most one limit.

<u>Proof</u>:

Suppose $x_n \to l$ as $n \to \infty$

Suppose $x_n \to m$ as $n \to \infty$

Choose <u>any</u> $\varepsilon > 0$

By definition of convergence we have that:

$$\begin{aligned} |l-m| &= |l - x_n + x_n - m| \\ &\leq |l - x_n| + |x_n - m| \quad \longleftarrow \left\{ \text{By the triangle inequality} \right. \\ &< \varepsilon + \varepsilon \\ &= \varepsilon \end{aligned}$$

But, if <u>any</u> $\varepsilon > 0$ can be chosen, it must be that $|l-m| = 0$.

That is to say, $l = m$. □

(N) <u>$\triangle$ Inequality</u>: $\forall$ real a and b, $\begin{cases} |a+b| \leq |a| + |b| \\ |a-b| \geq |a| - |b| \end{cases}$

Can be very useful

E.g. 4

$x^n \to 0$ as $n \to \infty$ (if $|x| < 1$)

Proof

Choose any $\varepsilon > 0$

Aim to find some N such that $\forall n > N$

$$|x^n - 0| < \varepsilon$$

$$\Leftarrow |x|^n < \varepsilon$$

$$\Leftarrow \frac{1}{(1+h)^n} < \varepsilon \quad \text{where } h > 0$$

$$\Leftarrow \frac{1}{\sum_{i=0}^{n} \binom{n}{i} h^i} < \varepsilon$$

$$\Leftarrow \frac{1}{1 + nh + \frac{1}{2}n(n-1)h^2 + \ldots + h^n} < \varepsilon$$

$$\Leftarrow \frac{1}{nh} < \varepsilon$$

$$\Leftarrow \frac{1}{n} < h\varepsilon$$

$$\Leftarrow n > \frac{1}{h\varepsilon}$$

So, if we choose $N := \frac{1}{h\varepsilon}$ then $\forall n > N$

we have that $|x^n - 0| < \varepsilon$ $\square$

Convergence & Boundedness

(I) Every convergent sequence is bounded.

(II) If a sequence is increasing and bounded above then it converges to its smallest upper bound.

(III) If a sequence is decreasing and bounded below then it converges to its largest lower bound.

Proof:

(I) Let $x_n \to l$ as $n \to \infty$

Aim to find a K such that $|x_n| \leq K$, $\forall n$

Choose any $\varepsilon > 0$, say $\varepsilon = 1$, perhaps.

By definition of sequence convergence we have that

$\forall n > N \quad |x_n - l| < \varepsilon$

Now, by Δ inequality, $|x_n| - |l| \leq |x_n - l|$.

So, $\forall n > N$ $|x_n| < |l| + \varepsilon$

Let $K = \max \{ |x_1|, |x_2|, \ldots, |x_N|, |l| + \varepsilon \}$

Then $|x_n| \leq K$, $\forall n$ $\square$

(II) Let x_n be such that $\forall n \quad x_{n+1} \geqslant x_n$ and B is the supremum of x_n.

Choose any $\varepsilon > 0$.

Aim to find some N such that $\forall n > N$

$$|x_n - B| < \varepsilon$$

$$\Leftarrow B - \varepsilon < x_n < B + \varepsilon$$

Now, $\forall n$, $x_n \leq B$, so $x_n < B + \varepsilon$

Also, there is certainly some N such that $x_N > B - \varepsilon$ — else, $B - \varepsilon$ would be an upper bound smaller than the smallest upper bound.

Given that $\forall n \quad x_{n+1} \geqslant x_n \geqslant x_N$, the theorem follows.

(III) This follows from (II) and the subsequent combination theorem.

Combination Theorem

Let $x_n \to l$ as $n \to \infty$

Let $y_n \to m$ as $n \to \infty$

Let λ, μ be any reals.

Then,

(I) $\lambda x_n + \mu y_n \to \lambda l + \mu m$ as $n \to \infty$

(II) $x_n y_n \to lm$ as $n \to \infty$

(III) $\dfrac{x_n}{y_n} \to \dfrac{l}{m}$ as $n \to \infty$ (provided $m \neq 0$)

Proof:

(I) Let $z_n \to \rho$ as $n \to \infty$ and let α be any real

We first prove that $\alpha z_n \to \alpha \rho$ as $n \to \infty$

Choose any $\varepsilon > 0$

Aim to find some N s.t. $\forall n > N$,

$$|\alpha z_n - \alpha \rho| < \varepsilon$$

$$\Leftarrow \alpha |z_n - \rho| < \varepsilon$$

$$\Leftarrow |z_n - \rho| < \frac{\varepsilon}{\alpha}$$

(if $\alpha \neq 0$. The claim is trivial where $\alpha = 0$)

Now, by definition of sequence convergence we know that for $\varepsilon' := \frac{\varepsilon}{\alpha}$ there exists an N such that for any $n > N$, $|z_n - \rho| < \varepsilon'$, which is to say $|\alpha z_n - \alpha \rho| < \varepsilon$ — by the implication chain above. $\blacksquare$

It remains to show that $x_n + y_n \to l + m$ as $n \to \infty$.

Again, choose any $\varepsilon > 0$.

Aim to find some N such that $\forall n > N$,

$$\left| (x_n + y_n) - (l+m) \right| < \varepsilon$$

$$\Leftarrow \left| (x_n - l) + (y_n - m) \right| < \varepsilon$$

$$\Leftarrow |x_n - l| + |y_n - m| < \varepsilon \quad (\Delta \text{Inequality})$$

Now, by definition of sequence convergence we know that for $\varepsilon' := \varepsilon/2$ there is some N_1 such that $\forall n > N_1$, $|x_n - l| < \varepsilon'$. Also, there is some N_2 such that $\forall n > N_2$, $|y_n - m| < \varepsilon'$.

If we choose N to be the largest of N_1 and N_2 the $|x_n - l| + |y_n - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for any $n > N$ and the result follows.

(II) This is proved in lectures

(III) Choose any $\varepsilon > 0$
Aim to find some N s.t. $\forall n > N$

$$\left| \frac{x_n}{y_n} - \frac{l}{m} \right| < \varepsilon$$

$$\Leftarrow \left| \frac{m x_n - l y_n}{m y_n} \right| < \varepsilon$$

$$\Leftarrow \left| \frac{1}{m y_n} \right| \left| m x_n - l y_n \right| < \varepsilon$$

Recall that every convergent sequence, e.g. y_n, is bounded. Let $\left| \frac{1}{m y_n} \right| \leq B$

Now, by (I) we have that $|m x_n - l y_n| \to lm - lm = 0$ as $n \to \infty$.

So, for $\varepsilon' := B\varepsilon$ we can find an N such that $|m x_n - l y_n| < \varepsilon'$, which is to say $\left| \frac{x_n}{y_n} - \frac{l}{m} \right| < \varepsilon$ — by the chain of implications above.

Theorems to Prove / Test for Convergence

1. Sandwich
2. Ratio ———— Convergence & Divergence
3. Cauchy

As an exercise try to prove Cauchy theorem — i.e. Cauchy sequences are necessarily convergent <u>AND</u> convergent sequences are necessarily Cauchy.