TUTORIAL 6

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SERIES

* *Series* take the form: $\sum_{n=1}^{\infty} a_n$ (where $a_n \in \mathbb{R}, \forall n$)

* Given a series $\sum_{n=1}^{\infty} a_n$, we can define a sequence of reals $\langle S_N \rangle$ by:

$$S_N := \sum_{n=1}^{N} a_n .$$

We term this sequence the sequence of *partial sums* of the series.

* *Series Convergence*:
If there is some S such that $S_N \to S$ as $N \to \infty$, then the corresponding series is said to converge.
If this is so, we write: $S = \sum_{n=1}^{\infty} a_n$.

* Tests for Series Convergence

(1) Note: partial sums are a special type of sequence, so all the sequence-convergence tests may be used — e.g. Cauchy theorem; and increasing, bounded sequences converge; etc.

(2) **Comparison Test**

Let $\sum_{n=1}^{\infty} b_n$ be a convergent series of positive real numbers.

If $|a_n| \leq b_n$ $(n = 1, 2, 3, \ldots)$ then series $\sum_{n=1}^{\infty} a_n$ converges.

NB Comparison Test Hypothesis is NOT:

$$\left| \sum_{k=1}^{N} a_k \right| \leq \sum_{k=1}^{N} b_k$$

(3) **Ratio Test for Series**

Let $\sum_{n=1}^{\infty} a_n$ be a series which satisfies:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = l$$

Then, (i) If $l > 1$, then the series DIVERGES
(ii) If $l < 1$, then the series CONVERGES

Proofs:

(2) Let $\sum_{k=1}^{\infty} b_k$ be a convergent series of positive reals.

Let $|a_k| \leq b_k \quad \forall k$

Choose any $\varepsilon > 0$.

By definition of convergence, we know that there is some N such that $\forall n > N$

$$\sum_{k=n+1}^{\infty} b_k = \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{n} b_k < \varepsilon$$

(∧) Consider examples later. This fact is often termed: "tail of convergent series tends to 0"

Let the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$ be denoted $\langle s_k \rangle$.

Then, for any n and m such that $n > m > N$,

$$|s_n - s_m| = |(a_1 + a_2 + \dots a_n) - (a_1 + a_2 + \dots a_m)|$$
$$= |a_{m+1} + a_{m+2} + \dots + a_n|$$
$$\leq |a_{m+1}| + |a_{m+2}| + \dots + |a_n| \quad \text{— } \triangle \text{ Inequality}$$
$$\leq b_{m+1} + b_{m+2} + \dots b_n$$
$$\leq \sum_{k=m+1}^{\infty} b_k < \varepsilon \quad \text{— by } (\wedge)$$

Thus, $\langle s_n \rangle$ is a Cauchy sequence.

So, by Cauchy theorem, $\sum_{k=1}^{\infty} a_k$ is convergent □

③ Let $\sum_{k=1}^{\infty} a_k$ be a series such that $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = l$

We first prove that if $l < 1$, then the series converges.

If $l < 1$, we may choose $\varepsilon > 0$ so small that $l + \varepsilon < 1$.

Then, for sufficiently large N,

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+2}}{a_{N+1}}\right| |a_{N+1}|$$

$$< (1+\varepsilon)^{n-N-1} |a_{N+1}|$$

Now, using the comparison test with the geometric series $\sum_{k=1}^{\infty} (l+\varepsilon)^n$, [in an example later on]

we see that $\sum_{k=}^{\infty} a_n$ converges.

We now prove that if $l > 1$, then the series diverges.

If $l > 1$, we may choose $\varepsilon > 0$ so small that $l - \varepsilon > 1$.

Then, for sufficiently large N,

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdots \left|\frac{a_{N+2}}{a_{N+1}}\right| \cdot |a_{N+1}|$$

$$> (1-\varepsilon)^{n-N+1} |a_{N+1}|$$

$$\to \infty \text{ as } n \to \infty$$

Hence, $a_n \not\to 0$ as $n \to \infty$

So, $\sum_{k=1}^{\infty} a_k$ diverges

$\triangle$

* Classifying Series Convergence

- A series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

- A series which converges, but does not converge absolutely is said to be conditionally convergent.

* Some Series Limit Properties

(I) Suppose that $\sum_{k=1}^{\infty} a_k = \alpha$ and $\sum_{k=1}^{\infty} a_k = \beta$.
Then, for any reals λ and μ, we have that $\sum_{k=1}^{\infty} (\lambda a_k + \mu b_k)$ converges to $\lambda\alpha + \mu\beta$

Proof: $\sum_{n=1}^{N}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{N} a_n + \mu \sum_{n=1}^{N} b_n$

$\rightarrow \lambda\alpha + \mu\beta$ as $N \to \infty$

By combination theorem for sequence limits.

(II) Cauchy Product

See your notes

Some Examples

1. Geometric series

- If $|x| < 1$, then $\sum_{n=1}^{\infty} x^n$ converges, and
$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

- If $|x| \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges

- If $|x| < 1$, then $\sum_{n=0}^{\infty} x^n$ converges, and
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Starts with 1 here

- If $|x| \geq 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

Proof:

See your notes about first and second claims. I consider here the third.

Suppose $|x| < 1$.

$$S_N := \sum_{n=0}^{N} x^n = 1 + x + \ldots + x^N$$

$$x S_N = x + x^n + \ldots + x^N + x^{N+1}$$

$$\Rightarrow S_N - x S_N = 1 - x^{N+1}$$

$$\Rightarrow S_N = \frac{1 - x^{N+1}}{1-x}$$

If $|x| < 1$, $x^{N+1} \to 0$ as $N \to \infty$, so

$$S_N \to \frac{1}{1-x} \text{ as } N \to \infty.$$

2 Suppose $\sum_{n=1}^{\infty} a_n$ converges

Then,

(i) for each natural N, $\sum_{n=N}^{\infty} a_n$ converges, and

(ii) $\sum_{n=N}^{\infty} a_n \to 0$ as $N \to \infty$

The tail of convergent series $\sum_{n=1}^{\infty} a_n$ tends to 0

<u>Proof</u>:
This follows easily from definition of sequence convergence. See Comparison Test proof for how.

3 $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u> to $+\infty$

(N) Recall that the sequence $\frac{1}{n} \to 0$ [redacted], so if the terms of series $\to 0$, it does <u>not</u> follow that the series converges. We may say that the terms don't tend to 0 "fast enough".

<u>Proof</u>:

Set $M := 2^N$

Then, $S_M = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^N}$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{N-1}+1} + \dots + \frac{1}{2^N}\right)$$

See notes for bracketing conventions

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^N} + \dots + \frac{1}{2^N}\right)$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots + \frac{2^{N-1}}{2^N}$$

$$= 1 + \frac{N}{2} \to +\infty$$ as N and accordingly M tend to ∞.

* Using Series (Series Approx. to Functions)

① Maclaurin

② Taylor

Suppose $f^{(n)}$ exists for $x \in [a, a+h]$.
Then, there is at least one α in $[a, a+h]$
such that:

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + E_n$$

where $E_n := \dfrac{h^n}{n!} f^{(n)}(\alpha)$