Mathematical Methods

for Computer Science

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Vectors

- Used in (amongst others):
  - Computational Techniques (2nd Year)
  - Graphics (3rd Year)
  - Computational Finance (3rd Year)
  - Modelling and Simulation (3rd Year)
  - Performance Analysis (3rd Year)
  - Digital Libraries and Search Engines (3rd Year)
  - Computer Vision (4th Year)

Vector Contents

- What is a vector?
- Useful vector tools:
  - Vector magnitude
  - Vector addition
  - Scalar multiplication
  - Dot product
  - Cross product
- Useful results – finding the intersection of:
  - a line with a line
  - a line with a plane
  - a plane with a plane

What is a vector?

- A vector is used :
  - **to convey *both* direction and magnitude**
  - to store data (usually numbers) in an ordered form
- $\vec{p} = (10, 5, 7)$ is a *row* vector
- $\vec{p} = \begin{pmatrix} 10 \\ 5 \\ 7 \end{pmatrix}$ is a *column* vector
- A vector is used in computer graphics to represent the position coordinates for a point

What is a vector?

- The dimension of a vector is given by the number of elements it contains. e.g.
  - $(-2.4, 5.1)$ is a 2-dimensional real vector
  - $(-2.4, 5.1)$ comes from set $\mathbb{R}^2$ (or $\mathbb{R} \times \mathbb{R}$)
  - $\begin{pmatrix} -2 \\ 5 \\ 7 \\ 0 \end{pmatrix}$ is a 4-dimensional integer vector
    (comes from set $\mathbb{Z}^4$ or $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$)

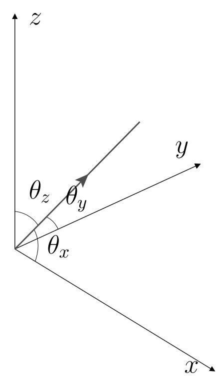
Vector Magnitude

- The size or magnitude of a vector $\vec{p} = (p_1, p_2, p_3)$ is defined as its length:

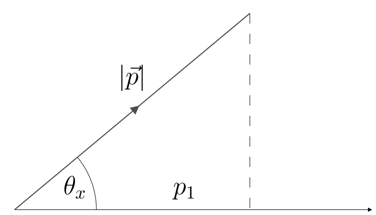
$$|\vec{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2} = \sqrt{\sum_{i=1}^{3} p_i^2}$$

- e.g. $\left| \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$
- For an n-dimensional vector, $\vec{p} = (p_1, p_2, \ldots, p_n)$, $|\vec{p}| = \sqrt{\sum_{i=1}^{n} p_i^2}$

Vector Direction



Vector Angles



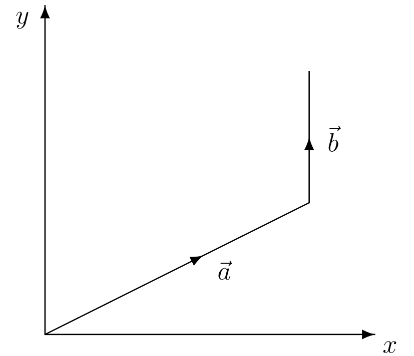
- For a vector, $\vec{p} = (p_1, p_2, p_3)$:
  - $\cos(\theta_x) = p_1/|\vec{p}|$
  - $\cos(\theta_y) = p_2/|\vec{p}|$
  - $\cos(\theta_z) = p_3/|\vec{p}|$

Vector addition

- Two vectors (of the same dimension) can be added together:
- e.g. $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
- So if $\vec{p} = (p_1, p_2, p_3)$ and $\vec{q} = (q_1, q_2, q_3)$ then:

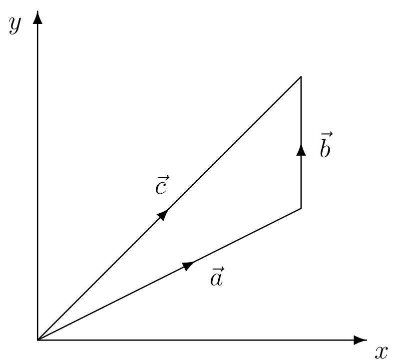
$$\vec{p} + \vec{q} = (p_1 + q_1, p_2 + q_2, p_3 + q_3)$$

Vector addition



$\vec{a} + \vec{b}$

Vector addition



$\vec{a} + \vec{b} = \vec{c}$

Scalar Multiplication

- A scalar is just a number, e.g. 3. Unlike a vector, it has no direction.
- Multiplication of a vector $\vec{p}$ by a scalar λ means that each element of the vector is multiplied by the scalar
- So if $\vec{p} = (p_1, p_2, p_3)$ then:

$$\lambda\vec{p} = (\lambda p_1, \lambda p_2, \lambda p_3)$$

3D Unit vectors

- We use $\vec{i}$, $\vec{j}$, $\vec{k}$ to define the 3 unit vectors in 3 dimensions
- They convey the basic directions along x, y and z axes.
- So: $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- All unit vectors have magnitude 1; i.e. $|\vec{i}| = 1$

Vector notation

- All vectors in 3D (or $\mathbb{R}^3$) can be expressed as weighted sums of $\vec{i}, \vec{j}, \vec{k}$
- i.e. $\vec{p} = (10, 5, 7) \equiv \begin{pmatrix} 10 \\ 5 \\ 7 \end{pmatrix} \equiv 10\vec{i} + 5\vec{j} + 7\vec{k}$
- $|p_1\vec{i} + p_2\vec{j} + p_3\vec{k}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$

Dot Product

- Also known as: *scalar product*
- Used to determine how close 2 vectors are to being parallel/perpendicular
- The dot product of two vectors $\vec{p}$ and $\vec{q}$ is:

$$\vec{p} \cdot \vec{q} = |\vec{p}|\, |\vec{q}| \cos\theta$$

- where θ is angle between the vectors $\vec{p}$ and $\vec{q}$
- For $\vec{p} = (p_1, p_2, p_3)$ and $\vec{q} = (q_1, q_2, q_3)$ then:

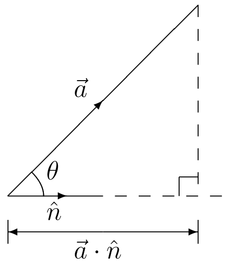
$$\vec{p} \cdot \vec{q} = p_1q_1 + p_2q_2 + p_3q_3$$

Properties of the Dot Product

- $\vec{p} \cdot \vec{p} = |\vec{p}|^2$
- $\vec{p} \cdot \vec{q} = 0$ if $\vec{p}$ and $\vec{q}$ are perpendicular (at right angles)
- Commutative: $\vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{p}$
- Linearity: $\vec{p} \cdot (\lambda\vec{q}) = \lambda(\vec{p} \cdot \vec{q})$
- Distributive over addition:

$$\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$$

Vector Projection



- $\hat{n}$ is a unit vector, i.e. $|\hat{n}| = 1$
- $\vec{a} \cdot \hat{n} = |\vec{a}| \cos\theta$ represents the *amount* of $\vec{a}$ that points in the $\hat{n}$ direction

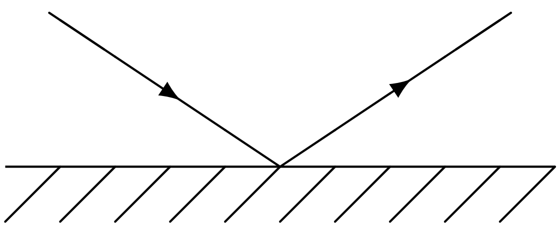
What can't you do with a vector...

The following are **classic mistakes** – $\vec{u}$ and $\vec{v}$ are vectors, and λ is a scalar:

- **Don't do it!**
  - Vector division: $\frac{\vec{u}}{\vec{v}}$
  - Divide a scalar by a vector: $\frac{\lambda}{\vec{u}}$
  - Add a scalar to a vector: $\lambda + \vec{u}$
  - Subtract a scalar from a vector: $\vec{u} - \lambda$
  - Cancel a vector in a dot product with vector:

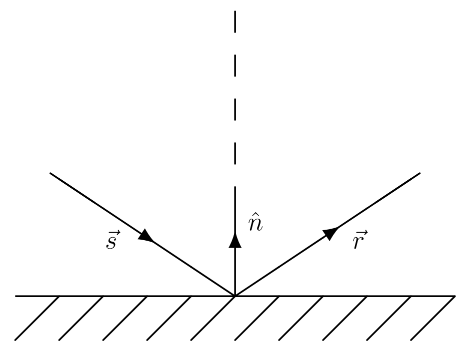
$$\frac{1}{\vec{a} \cdot \cancel{\vec{n}}} \cancel{\vec{n}} = \frac{1}{\vec{a}}$$

Example: Rays of light

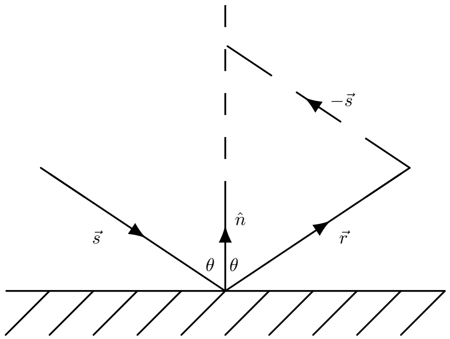


- A ray of light strikes a reflective surface...
- Question: in what direction does the reflection travel?

Rays of light

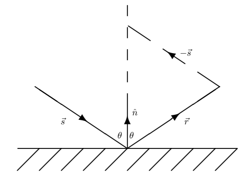


Rays of light



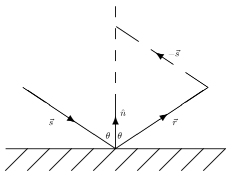
- Problem: find $\vec{r}$, given $\vec{s}$ and $\hat{n}$?

Rays of light



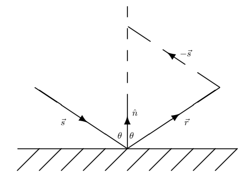
- angle of incidence = angle of reflection
  $\Rightarrow -\vec{s}\cdot\hat{n} = \vec{r}\cdot\hat{n}$
- Also: $\vec{r} + (-\vec{s}) = \lambda\hat{n}$ thus $\lambda\hat{n} = \vec{r} - \vec{s}$
- Taking the dot product of both sides:
  $\Rightarrow \lambda|\hat{n}|^2 = \vec{r}\cdot\hat{n} - \vec{s}\cdot\hat{n}$

Rays of light



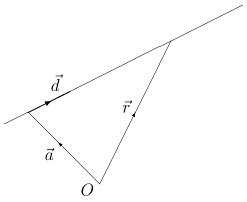
- But $\hat{n}$ is a unit vector, so $|\hat{n}|^2 = 1$
  $\Rightarrow \lambda = \vec{r}\cdot\hat{n} - \vec{s}\cdot\hat{n}$
- ...and $\vec{r}\cdot\hat{n} = -\vec{s}\cdot\hat{n}$
  $\Rightarrow \lambda = -2\vec{s}\cdot\hat{n}$

Rays of light



- Finally, we know that: $\vec{r} + (-\vec{s}) = \lambda\hat{n}$
  $\Rightarrow \vec{r} = \lambda\hat{n} + \vec{s}$
  $\Rightarrow \vec{r} = \vec{s} - 2(\vec{s}\cdot\hat{n})\hat{n}$

Equation of a line



- For a general point, $\vec{r}$, on the line:

$$\vec{r} = \vec{a} + \lambda\vec{d}$$

- where: $\vec{a}$ is a point on the line and $\vec{d}$ is a vector parallel to the line

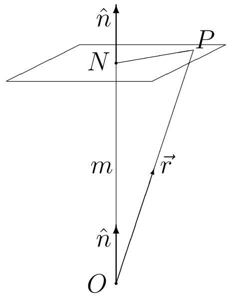
Equation of a plane

- Equation of a plane. For a general point, $\vec{r}$, in the plane, $\vec{r}$ has the property that:

$$\vec{r}.\hat{n} = m$$

- where:
  - $\hat{n}$ is the unit vector perpendicular to the plane
  - $|m|$ is the distance from the plane to the origin (at its closest point)

Equation of a plane



- Equation of a plane (why?):

$$\vec{r}.\hat{n} = m$$

How to solve Vector Problems

1. IMPORTANT: Draw a diagram!
2. Write down the equations that you are given/apply to the situation
3. Write down what you are trying to find?

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4. Try variable substitution
5. Try taking the dot product of one or more equations
   - What vector to dot with?

Answer: if eqn (1) has term $\vec{r}$ in and eqn (2) has term $\vec{r} \cdot \vec{s}$ in: *dot eqn (1) with* $\vec{s}$.

Two intersecting lines

- Application: *projectile interception*
- Problem — given two lines:
  - Line 1: $\vec{r_1} = \vec{a}_1 + t_1\vec{d}_1$
  - Line 2: $\vec{r_2} = \vec{a}_2 + t_2\vec{d}_2$
- Do they intersect? If so, at what point?
- This is the same problem as: find the values t_1 and t_2 at which $\vec{r_1} = \vec{r_2}$ or:

$$\vec{a}_1 + t_1\vec{d}_1 = \vec{a}_2 + t_2\vec{d}_2$$

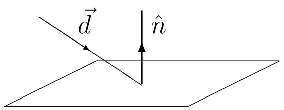
How to solve: 2 intersecting lines

- Separate $\vec{i}$, $\vec{j}$, $\vec{k}$ components of equation:

$$\vec{a}_1 + t_1\vec{d}_1 = \vec{a}_2 + t_2\vec{d}_2$$

- ...to get **3** equations in t_1 and t_2
- If the 3 equations:
  - contradict each other then **the lines do not intersect**
  - produce a single solution then **the lines do intersect**
  - are all the same (or multiples of each other) then **the lines are identical** (and always intersect)

Intersection of a line and plane



- Application: *ray tracing, particle tracing, projectile tracking*
- Problem — given one line/one plane:
  - Line: $\vec{r} = \vec{a} + t\vec{d}$
  - Plane: $\vec{r} \cdot \hat{n} = s$
- Take dot product of line equation with $\hat{n}$ to get:

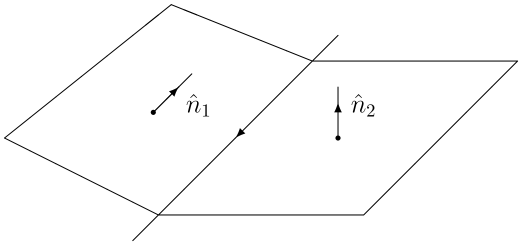
$$\vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n} + t(\vec{d} \cdot \hat{n})$$

Intersection of a line and plane

- With $\vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n} + t(\vec{d} \cdot \hat{n})$ — what are we trying to find?
  - We are trying to find a specific value of t that corresponds to the point of intersection
- Since $\vec{r} \cdot \hat{n} = s$ at intersection, we get: $t = \frac{s - \vec{a}\cdot\hat{n}}{\vec{d}\cdot\hat{n}}$
- So using line equation we get our point of intersection, $\vec{r'}$:

$$\vec{r'} = \vec{a} + \frac{s - \vec{a}\cdot\hat{n}}{\vec{d}\cdot\hat{n}}\vec{d}$$

Example: intersecting planes



- Problem: find the line that represents the intersection of two planes

Intersecting planes

- Application: *edge detection*
- Equations of planes:
  - Plane 1: $\vec{r} \cdot \hat{n}_1 = s_1$
  - Plane 2: $\vec{r} \cdot \hat{n}_2 = s_2$
- We want to find the line of intesection, i.e. find $\vec{a}$ and $\vec{d}$ in: $\vec{s} = \vec{a} + \lambda \vec{d}$
- If $\vec{s} = x\vec{i} + y\vec{j} + z\vec{k}$ is on the intersection line:

  $\Rightarrow$ it also lies in both planes 1 and 2

  $\Rightarrow$ $\vec{s} \cdot \hat{n}_1 = s_1$ and $\vec{s} \cdot \hat{n}_2 = s_2$
  - Can use these two equations to generate equation of line

Example: Intersecting planes

- Equations of planes:
  - Plane 1: $\vec{r} \cdot (2\vec{i} - \vec{j} + 2\vec{k}) = 3$
  - Plane 2: $\vec{r} \cdot \vec{k} = 4$
- Pick point $\vec{s} = x\vec{i} + y\vec{j} + z\vec{k}$
  - From plane 1: $2x - y + 2z = 3$
  - From plane 2: $z = 4$
- We have two equations in 3 unknowns – not enough to solve the system
- But... we can express all three variables in terms of one of the other variables

Example: Intersecting planes

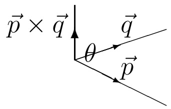
- From plane 1: $2x - y + 2z = 3$
- From plane 2: $z = 4$

- Substituting (Eqn. 2) $\rightarrow$ (Eqn. 1) gives:

  $\Rightarrow 2x = y - 5$
- Also trivially: $y = y$ and $z = 4$
- Line: $\vec{s} = ((y-5)/2)\vec{i} + y\vec{j} + 4\vec{k}$

  $\Rightarrow \vec{s} = -\frac{5}{2}\vec{i} + 4\vec{k} + y(\frac{1}{2}\vec{i} + \vec{j})$
- ...which is the equation of a line

Cross Product



- Also known as: *Vector Product*
- Used to produce a 3rd vector that is perpendicular to the original two vectors
- Written as $\vec{p} \times \vec{q}$ (or sometimes $\vec{p} \wedge \vec{q}$)
- Formally: $\vec{p} \times \vec{q} = (|\vec{p}|\,|\vec{q}| \sin\theta)\hat{n}$
  - where $\hat{n}$ is the unit vector perpendicular to $\vec{p}$ and $\vec{q}$; θ is the angle between $\vec{p}$ and $\vec{q}$

Cross Product

- From definition: $|\vec{p} \times \vec{q}| = |\vec{p}|\,|\vec{q}| \sin\theta$
- In coordinate form: $\vec{a} \times \vec{b} = \left| \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \right|$

  $\Rightarrow \vec{a} \times \vec{b} =$
  $(a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$
- Useful for: e.g. given 2 lines in a plane, write down the equation of the plane

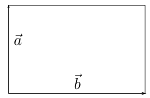
Properties of Cross Product

- $\vec{p} \times \vec{q}$ is itself a vector that is perpendicular to both $\vec{p}$ and $\vec{q}$, so:
  - $\vec{p} \cdot (\vec{p} \times \vec{q}) = 0$ and $\vec{q} \cdot (\vec{p} \times \vec{q}) = 0$
- If $\vec{p}$ is parallel to $\vec{q}$ then $\vec{p} \times \vec{q} = \vec{0}$
  - where $\vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k}$
- **NOT** commutative: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$
  - In fact: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- **NOT** associative: $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$
- Left distributive: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- Right distributive: $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

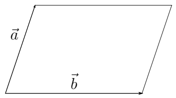
Properties of Cross Product

- Final important vector product identity:
  - $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
  - which says that: $\vec{a} \times (\vec{b} \times \vec{c}) = \lambda\vec{b} + \mu\vec{c}$
  - i.e. the vector $\vec{a} \times (\vec{b} \times \vec{c})$ lies in the plane created by $\vec{b}$ and $\vec{c}$

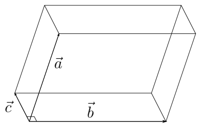
Examples of Cross Product



Area of rectangle
$= |\vec{a}|\,|\vec{b}|$

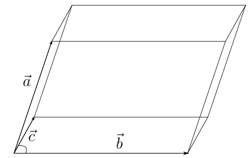


Area of parallelogram
$= |\vec{a} \times \vec{b}|$

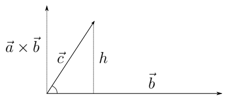


Volume of prism
$= |\vec{a} \times \vec{b}||\vec{c}|$

Examples of Cross Product



Volume of parallelepiped
$= (\vec{a} \times \vec{b}) \cdot \vec{c}$



View from above:
$h = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} \cdot \vec{c}$