1

Normative Positions

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ABSTRACT. The Kanger-Lindahl theory of normative positions is an attempt to apply the tools of modal logic to the formalisation of Hohfeld's 'fundamental legal conceptions', to the construction of a formal theory of duties and rights, and to the formal characterisation generally of complex normative relations that can hold between (pairs of) agents with regard to an action by one or other of them. The theory employs a standard deontic logic, a logic of action/agency of the 'brings it about' or 'sees to it' kind, and a method of mapping out in a systematic and exhaustive fashion the complete space of all logically possible normative relations—or 'positions'—of some given type. The article presents a generalised version of the methods and a brief dicussion of its limitations as a comprehensive theory of duty and right.

1 Introduction

The theory of normative positions is an attempt to apply the tools of modal logic to the formalisation of the 'fundamental legal conceptions' (duty, right, privilege, power, immunity, etc.) most closely associated with the American jurist W.N. Hohfeld [1913], to the construction of a formal theory of duties and rights, and to the formal characterisation generally of complex normative relations that can hold between (pairs of) agents with regard to an action by one or other of them. The development was initiated by Stig Kanger and subsequently extended and refined, most notably by Lars Lindahl. Ingmar Pörn applied similar techniques to the study of 'control and influence' relations in social interactions.

The theory employs a standard deontic logic, a logic of action/agency of the 'brings it about' or 'sees to it' kind, and a method for mapping out in a systematic and exhaustive fashion the complete space of all logically possible normative relations between two agents with respect to some given act type. Kanger called these relations the 'atomic types of rights relation'; we will follow later usage and refer to them generally as normative 'positions'. The methods are presented in [Kanger, 1971; Kanger, 1985; Kanger and Kanger, 1966] with a more general account of related issues in [Kanger, 1972]. As described later in the article, Lars Lindahl [1977]

¡book title¿, 1–54.

developed Kanger's account in several important respects, providing also a commentary on the relationships to Hohfeld's work and the jurisprudential tradition, of Jeremy Bentham and John Austin, within which it falls. Ingmar Pörn [1977] applied similar techniques to the study of what he called 'control' and 'influence' relations in which there are iterations of the action/agency modalities in place of the deontic logic component. For further discussion of the theory and some of its features and possible applications, see e.g. [Talja, 1980; Makinson, 1986; Lindahl, 1994; Jones and Sergot, 1992; Jones and Sergot, 1993; Herrestad and Krogh, 1995; Herrestad, 1996; Krogh, 1997; Sergot and Richards, 2000; Jones and Parent, 2008]. The technical account presented in this article is extracted from [Sergot, 2001].

The concepts treated by the theory of normative positions are usually discussed within the context of law and legal relations. Hohfeld himself referred to them as the 'fundamental legal conceptions'. These are not exclusively *legal* concepts, however, but characteristic of all forms of regulated and organised agent interaction. Although the theory does address fundamental issues in the formal representation of laws and regulations and legal contracts—Allen and Saxon [1986; 1993] for example long argued that proper attention to the Hohfeldian concepts is essential for legal knowledge representation—it also finds applications in other areas, such as the specification of aspects of computer systems (see e.g. [Jones and Sergot, 1993; Krogh, 1997; Jones and Parent, 2008]), as a contribution to the formal theory of organisations in the analysis of notions such as responsibility, entitlement, authorisation and delegation, and in the field of multi-agent systems, where the notion of commitment in particular, in the sense of a directed obligation of an agent a to another agent b, features prominently in the literature on co-ordinated action, joint planning, and agent communication languages. (See e.g. [Jennings, 1993; Shoham, 1991; Shoham, 1993; Singh, 1998; Singh, 1999; Colombetti, 1999; Colombetti, 2000] for some early references.)

The theory of normative positions has a number of important and well-documented limitations. As a theory of rights, it lacks a treatment of the role of *counterparty*, the agent who is the beneficiary of a right relation or to whom a duty is owed. As a formalisation of the Hohfeldian framework, it does not deal with the feature Hohfeld called '(legal) power', also referred to sometimes as 'legal capacity' or 'competence'. See e.g. [Makinson, 1986; Lindahl, 1994] for some of these points, and the discussion that follows in Section 8 below. The theory of normative positions is therefore best seen as a *component* of a formal theory of duty and right, and not as a complete theory of all aspects of these complex concepts. Its methods need to be augmented: with a treatment of 'power', with temporal constructs, and with a richer set of action concepts, at the very least.

Nevertheless, the Kanger-Lindahl theory is generally regarded as the most comprehensive and best developed attempt to formalize distinctions such as Hohfeld's. For example, Hohfeld identified four distinct legal/normative relations that could hold between any two agents with respect to some given act type. Some examples are given later in Section 3. Kanger's systematic, formal analysis yielded 26 distinct 'atomic types of rights relations' or 'normative positions' as a refinement of Hohfeld's four. Lindahl's subsequent analysis produced 35 of the same basic kind as Kanger's and 127 if a more precise set of possible relationships is considered instead. Section 4 discusses the methods in more detail. It also explains why there are more possibilities still than are accounted for in Lindahl's version: employing the same logics, 255 distinct relationships can be generated refining Kanger's 26 and Lindahl's 127, and many more if we include more complex act types and more agents than two.

This article follows the formal treatment presented in [Sergot, 2001] which generalised the Kanger-Lindahl accounts in the following respects. (1) The generalised theory deals with interaction between any number of agents, not just two, including 'ought-to-be' statements where no agent is specified. (2) The Kanger-Lindahl-Pörn theories deal with act expressions of the form 'agent x brings it about that F'. The generalised theory allows any number of such act expressions in any combination, and allows compound acts, that is to say, boolean compounds of propositions in the scope of the 'brings it about' operator. (3) Building on a suggestion by David Makinson it is possible to give an abstract characterisation of classes of 'positions' and relationships between them, and a complete separation of the method of generating the space of 'positions' from properties of the underlying modal logics. The generalised theory does not rely on any in-built assumptions about the specific deontic or action logics employed. It also means that, in principle at least, a richer combination of modalities could be used to represent more complex notions.

[Sergot, 2001] also shows how the methods for generating 'positions' can be automated without the need for theorem provers for the modal logics, and presents an automated computer system intended to facilitate application of the theory to the analysis of practical problems. Those methods will not be covered in this article.

2 Preliminary discussion

Hohfeld's seminal work [1913] is still often taken as the starting point for much that is written in this field. It identified two groups of four concepts

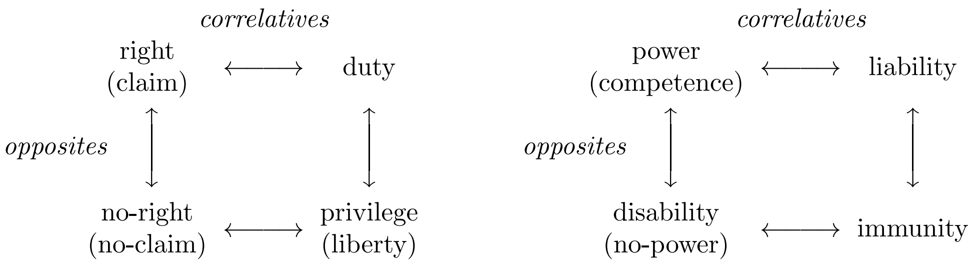

Figure 1. Hohfeld's 'fundamental legal conceptions'

with various relationships between them, as summarized in Figure 1. Right and duty are 'correlatives' in the sense that when x has a right (a 'claim-right') against y that F (be done by y) then y owes a duty to x that F (be done by y); and conversely. The relationships may be summarised semi-formally by the following scheme, adapted from [Lindahl, 1977]:

$$\mathit{Right}(x, y, F) \leftrightarrow \mathit{Duty}(y, x, F)$$
$$\mathit{Right}(x, y, \text{not-}F) \leftrightarrow \mathit{Duty}(y, x, \text{not-}F)$$

Here not-F is intended to stand for y's refraining from doing F. Of course it remains to explain how this notion of refraining is to be represented formally; this is one of the features of Kanger's framework.

Duty and privilege (some authors prefer 'liberty') are 'opposites' in the Hohfeldian scheme in the sense that x has a privilege/liberty from y with respect to F when x does not owe a duty to y to refrain from F; x has a privilege/liberty from y to refrain from F when x does not owe a duty to y that F (be done by x). In the semi-formal notation these relationships may be summarised as follows:

$$\mathit{Privilege}(x, y, F) \leftrightarrow \neg \mathit{Duty}(x, y, \text{not-}F)$$
$$\mathit{Privilege}(x, y, \text{not-}F) \leftrightarrow \neg \mathit{Duty}(x, y, F)$$

Similarly, right/no-right and no-right/privilege are also opposite and correlative pairs in the Hohfeldian scheme, in the following sense:

$$\mathit{Right}(x, y, F) \leftrightarrow \neg \mathit{No\text{-}right}(x, y, F)$$
$$\mathit{Right}(x, y, \text{not-}F) \leftrightarrow \neg \mathit{No\text{-}right}(x, y, \text{not-}F)$$
$$\mathit{No\text{-}right}(x, y, F) \leftrightarrow \mathit{Privilege}(y, x, \text{not-}F)$$
$$\mathit{No\text{-}right}(x, y, \text{not-}F) \leftrightarrow \mathit{Privilege}(y, x, F)$$

One can see already, however, as pointed out in [Lindahl, 1977, pp26–27] and in [Kanger and Kanger, 1966], that there are discrepancies in Hohfeld's account: the right/duty and no-right/privilege correlative pairs are not exactly of the same form, and nor are the right/no-right and duty/privilege opposites.

There is further inexactitude in Hohfeld's scheme for his second group of concepts, those on the right of the diagram in Figure 1. This second group is concerned with *changes* of legal/normative relations, as when it is said, for example, that x has power (competence) to impose a duty on y that such-and-such or to grant a privilege or right to z that such-and-such. Discussion of this second set of concepts raises a new set of questions however and is beyond the scope of this article. The second part of [Lindahl, 1977] is concerned with this group of concepts. See [Jones and Sergot, 1996] for an alternative account of power/competence.

For present purposes, the point is that Hohfeld's writings, and much else that has been written on these topics in legal theory, provide a wealth of examples and the beginnings of a systematic account, but are not precise enough to give a formal theory. Kanger attempted to provide such a theory by applying the formal tools of modal logic to this task.

The Kanger-Lindahl theory has a deontic logic component, an action logic component, and a method for generating the space of all logically possible positions. The language is that of propositional logic augmented with modal operators O (for 'obligation') and its dual P (for 'permission'), and relativised modal operators $\mathrm{E}_a, \mathrm{E}_b, \ldots$ for act expressions, where $a, b, \ldots$ are the names of individual agents. (This notation is slightly different from Kanger and Lindahl's, who use *Shall* and *May* for O and P, and *Do* for act expressions. The alternative notation is chosen simply because it is more concise and reduces the size of the formal expressions to be manipulated.)

An expression of the form $\mathrm{O}A$ may be read as 'it is obligatory that A' or 'it ought to be the case that A'. P is the dual of O: $\mathrm{P}A =_{\mathrm{def}} \neg\mathrm{O}\neg A$. The expression $\mathrm{P}A$ may be read as 'it is permissible that A'. We will also say 'permitted'. The deontic logic employed by Kanger and Lindahl is—for all intents and purposes—the system usually referred to as Standard Deontic Logic (SDL). Specifically, the deontic logic employed is the smallest system containing propositional logic (*PL*) and the following axiom schemas and rules:

O.RE	$\dfrac{A \leftrightarrow B}{\mathrm{O}A \leftrightarrow \mathrm{O}B}$
O.M	$\mathrm{O}(A \wedge B) \to (\mathrm{O}A \wedge \mathrm{O}B)$
O.C	$(\mathrm{O}A \wedge \mathrm{O}B) \to \mathrm{O}(A \wedge B)$
O.P	$\neg \mathrm{O}\bot$

The names of axiom schemas and rules in this article are based on those of [Chellas, 1980]: the logic of O is a classical modal logic of type *EMCP*. For comparison, Standard Deontic Logic (SDL) is a normal modal logic of type *KD*, which is type *EMCP* together with the additional rule of necessitation

O.RN $\quad \dfrac{A}{\mathrm{O}A}$

or, equivalently, the axiom schema $\mathrm{O}\top$ ($\top$ any tautology). The absence or presence of rule O.RN plays no role in the generation of normative positions: this is why we say that Kanger's choice of deontic logic is to all intents and purposes Standard Deontic Logic. The 'deontic axiom' of Standard Deontic Logic

O.D $\quad \mathrm{O}A \to \mathrm{P}A$

follows from O.C and O.P.

Of course Standard Deontic Logic (of type *KD* or *EMCP*) has many well-known limitations and its inadequacies are taken as the starting point for many of the developments in the field. Both axioms O.M and O.C can be criticised as simplistic, for example. However, in combination with the logic of action, and in the restricted ways it is employed in the generation of normative positions, these inadequacies are relatively benign. In any case, the extended theory of normative positions to be presented in later sections is not dependent on specific choices for the deontic and action logics employed. These can be changed, as explained below.

As regards the action component, expressions of the form $\mathrm{E}_x A$ stand for 'agent x sees to it that, or brings it about that, A'. This approach to the logic of action has been extensively studied in analytical philosophy and philosophical logic though is perhaps not so familiar in Computer Science. The *stit* operator of [Belnap and Perloff, 1988; Belnap and Perloff, 1992] and *dstit* of [Horty and Belnap, 1995] are instances of the general approach that have had some exposure in the AI literature. The focus of attention is not on transitions and state changes as in most treatments of action in AI and Computer Science, but rather on the end result A and the agent x

whose actions are responsible, in some appropriate sense, for this end result; the specific means or actions employed by agent x to bring about A are not expressed.

The logic of each E_x is that of a (relativised) classical modal system of type *ET* in the Chellas classification, i.e. the smallest system containing *PL*, closed under the rule E.RE:

$$\text{E.RE} \qquad \frac{A \leftrightarrow B}{\mathrm{E}_x A \leftrightarrow \mathrm{E}_x B}$$

and containing the axiom schema

$$\text{E.T} \qquad \mathrm{E}_x A \to A$$

The schema E.T indicates that this is a notion of *successful* action. It does not matter, for the purposes of this article, whether x brings about A intentionally or unintentionally, knowingly or unknowingly.

The E_x notation is from [Pörn, 1977]. For present purposes, however, the (relativised) operators E_x should be regarded as standing for one of a range of possible action modalities rather than any one of them specifically. For a discussion of some candidates and their relative merits see e.g. [Chellas, 1969; Pörn, 1970; Pörn, 1974; Pörn, 1977; Pörn, 1989; Åqvist, 1974; Segerberg, 1985; Segerberg, 1989; Segerberg, 1992; Belnap and Perloff, 1988; Belnap and Perloff, 1992; Perloff, 1991; Horty and Belnap, 1995; Elgesem, 1992; Hilpinen, 1997; Horty, 2001] as well as more recent works on 'stit' logics in particular. It is likely that a comprehensive theory of rights and/or organisations would require several different notions of action and agency. In [Santos and Carmo, 1996; Santos *et al.*, 1997], for instance, it is suggested that distinguishing between direct and indirect action may be important for describing certain organisational structures. Nothing in the present account depends on such detailed choices. As in the Kanger-Lindahl framework, the only properties assumed for the action modalities E_x are the schema E.T and closure under logical equivalence, E.RE.

3 Motivating examples

We conclude this introductory discussion with some brief examples to illustrate the expressive power of the language and to motivate the formal development to be undertaken in the remainder of the article. These examples are intended to be simple and familiar. They are the same as those used in [Sergot, 2001].

Example 3.1 (Library book) *Let b name a borrower in a library who has some book out on loan. Let R represent that this book is returned to the library by the date due. b has an obligation to return the book by date due. In the Kanger framework this obligation on b can be represented by the following expression.*

(1) $\mathrm{OE}_b R$

Expression (1) is not the only, nor perhaps even an adequate, representation of what we mean by saying that b has an obligation to return the book. It employs what some authors refer to as the Meinong-Chisholm analysis, whereby 'x ought to bring it about that F' is taken to mean 'it ought to be that x brings it about that F'. It is possible to question whether these expressions are in fact equivalent. See e.g. the discussions in [Horty, 1996; Horty, 2001; Sergot and Richards, 2000; Brown, 2000] among others. There are also some senses of 'obligation'—as when we say e.g. 'x is responsible for, or held accountable for, ensuring that F is the case'—which are not adequately represented by this construction. Possible formalisations of these other senses will not be discussed in this article.

Studies of duty and right, such as Hohfeld's, adopt a relational perspective: the focus is on relationships between pairs of agents. So, given the truth of e.g. $\mathrm{OE}_b R$, one is led to ask about the obligations and permissions of other agents, a say, with respect to the returning of the book. One can see that, according to the logics employed, the following three possibilities are all consistent with $\mathrm{OE}_b R$:

1. a is obliged to return the book: $\mathrm{OE}_a R$;

2. a is permitted but not obliged to return the book:

$$(\mathrm{PE}_a R \land \neg\mathrm{OE}_a R) = (\mathrm{PE}_a R \land \mathrm{P}\neg\mathrm{E}_a R);$$

3. a is not permitted to return the book: $\neg\mathrm{PE}_a R$.

Note that the first of these is logically possible given (1): the expression $\mathrm{OE}_a R \land \mathrm{OE}_b R$ is not inconsistent. In the logics employed, it is equivalent to $\mathrm{O}(\mathrm{E}_a R \land \mathrm{E}_b R)$, but there is no principle in the logic of action to say that a and b could not both act in such a way that they both see to it that R.

Are there any other possibilities besides the three listed above? It is the systematic exploration of all such possible relations that motivates in large part the construction of the Kanger-Lindahl theories.

Notice that the three possibilities above may be distinguished by asking in turn whether $\mathrm{PE}_a R$ is true, and if so, whether $\mathrm{P}\neg\mathrm{E}_a R$ is true. This is the kind of analysis that the automated system described in [Sergot, 2001] is designed to support.

Example 3.2 (Fence) *The following example is adapted from [Lindahl, 1977]. Again, no claim is made here for the completeness or adequacy of the representation. The aim is merely to illustrate some of the distinctions and nuances that can be expressed with the resources available.*

Suppose a and b are neighbours, and let F represent that there is a fence on the boundary between their adjoining properties. We want to say that a has a 'right' to erect such a fence, or more generally, that a has a 'right' to see to it that there is such a fence.

We build up a (partial) representation in stages. In the first instance it seems reasonable to assert that the following is true:

(2) $\mathrm{PE}_a F \wedge \neg\mathrm{PE}_b\neg F$

The second conjunct of expression (2) captures something of the idea that the neighbour b is not permitted to *prevent* a from seeing to it that F. One could also add a conjunct $\neg\mathrm{PE}_b\neg\mathrm{E}_a F$ to cover a different sense in which b is forbidden to prevent a from seeing to it that F. The ability to iterate action operators in this fashion has been seen as one of the main advantages of using the E_x device in the treatment of action. 'x refrains from seeing to it that F' can be represented as $\mathrm{E}_x\neg\mathrm{E}_x F$, for example. We shall not study iterated act expressions in any detail in this article, however. Some examples and some possible lines of development are discussed briefly in Section 8. Iterated act expressions are the basis of the 'control' and 'influence' positions examined in [Pörn, 1977].

Of course a is not *obliged* to see to it that F, so also $\neg\mathrm{OE}_a F$ is true in the example. Furthermore, a's permission to see to it that F does not depend on b's actions, in the sense that the following is also true: $\mathrm{P}(\mathrm{E}_a F \wedge \neg\mathrm{E}_b F)$. Putting these together:

(3) $\mathrm{PE}_a F \wedge \neg\mathrm{PE}_b\neg F \wedge \neg\mathrm{OE}_a F \wedge \mathrm{P}(\mathrm{E}_a F \wedge \neg\mathrm{E}_b F)$

Expression (3) is an approximation to the concept of a 'vested right'. It is only an approximation because as already observed there are other possible ways in which b can be said to 'prevent' a's seeing to it that F, e.g. as expressed by $\mathrm{E}_b\neg\mathrm{E}_a F$. It also fails to capture the idea that a's rights may already be infringed by *unsuccessful* attempts by b to *interfere* with a's actions [Makinson, 1986]. Moreover (3) does not say what further rights and obligations are created if b should so interfere.

In this example b's normative status in relation to F is clearly symmetrical to a's and so we may add also:

(4) $\mathrm{PE}_b F \wedge \neg \mathrm{PE}_a \neg F \wedge \neg \mathrm{OE}_b F \wedge \mathrm{P}(\mathrm{E}_b F \wedge \neg \mathrm{E}_a F)$

Still there are a number of unresolved questions. Is it the case that $\mathrm{P}\neg F$, or is it obligatory, $\neg\mathrm{P}\neg F$, that there is a fence? Is it the case that $\mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F \wedge \neg \mathrm{E}_b \neg F)$: is it permitted that there is no fence when neither a nor b brought this about? As a matter of fact, in the logics employed (3) and (4) together imply

(5) $\mathrm{P}\neg F \ \leftrightarrow \ \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F \wedge \neg \mathrm{E}_b \neg F)$

i.e., it is obligatory that there is a fence iff $\mathrm{O}(\neg F \rightarrow (\mathrm{E}_a \neg F \vee \mathrm{E}_b \neg F))$ is true. On the other hand, (3) and (4) together do not imply $\mathrm{P}(F \wedge \neg \mathrm{E}_a F \wedge \neg \mathrm{E}_b F)$. That question remains unresolved. Perhaps some other agent, besides a and b, is permitted to see to it that there is a fence between their adjoining properties, perhaps not.

The example is intended in part to demonstrate why there is a need for automated support even for the analysis of simple examples. Questions such as those above can can be explored systematically by means of the automated inference methods described in [Sergot, 2001] and summarised in Section 7 below.

The fence example also demonstrates that there may be an obligation on a and b together, without there being an obligation on either of them individually: it is possible that $\mathrm{O}(\mathrm{E}_a F \vee \mathrm{E}_b F)$ is true while both $\mathrm{OE}_a F$ and $\mathrm{OE}_b F$ are false.

Example 3.3 (Car park) *Ronald Lee [1988] presents a rule-based language intended for specifying permitted, obligatory and forbidden actions. The example used for illustration concerns the rules governing a University car park. For simplicity "assume that administrators have unrestricted parking privileges. Faculty, however, must obtain a parking permit to park on campus. Students must park off campus." Lee represents such rules in the form of if/then rules whose antecedent ('body') is a conjunction of factual conditions ('is an administrator', 'has a parking permit', etc.) and whose consequent specifies an action (here, 'park') that can be permitted, obligatory, or prohibited.*

Leaving aside the details of the language, one might ask whether these primitives 'permitted', 'obligatory', 'prohibited' are enough, whether they cover all imaginable cases. Notice first that they are not mutually exclusive: an obligatory action is also (presumably) permitted. It may be that the primitive 'permitted' in Lee's language was intended to be understood in the sense of permissible but not obligatory, or what is sometimes referred to as 'facultative'. In the logic we are using, A is facultative when $\mathrm{P}A \wedge \neg \mathrm{O}A$ is true, or equivalently when $\mathrm{P}A \wedge \mathrm{P}\neg A$ is true.

One can see a very close connection between this rule based representation and the conception of a normative system as introduced and developed in Alchourrón and Bulygin's classical work [1971]. There a normative system $\mathcal{N}$ is defined in terms of a 'universe of cases'—these are all the possible fact combinations that can be expressed using some fixed set $\mathcal{P}rops$ of propositional atoms—and a set of actions. For each action and case (set of factual circumstances) a normative system assigns a 'solution' which specifies whether that action is obligatory, prohibited, or facultative in that factual circumstance. The normative system is consistent when no case is assigned different solutions for any given action, and complete when every case is assigned a solution for every action.

But again, taking a relational perspective, one is led to think in terms of interactions between the administrator who is permitted to park and other agents: other users of the car park, passers by, the gatekeepers who control access to the car park, the University who owns the car park and to whom the gatekeepers are responsible, and so on. An analysis based on the Hohfeldian scheme, for example, would ask not whether there is a permission to park *simpliciter* but whether the administrator has a 'privilege' to park or whether this is in fact a 'claim-right' (vis-à-vis, in turn, other users of the car park, the gatekeepers, the University). And likewise for other pairs of agents.

If in place of the informal Hohfeldian scheme, we employ the formal machinery offered by the Kanger-Lindahl theories or the extended scheme of [Sergot, 2001], the if/then rules of the representation language would take the form

if *conditions* then *normative-position*

where *normative-position* is one of some appropriately chosen class of normative positions. Lee's rule-based language, and solutions in Alchourrón and Bulygin's formalisation of a normative system, can be regarded as a special case where the class of candidate normative positions is a particularly simple one. For more precision, more complex classes of normative positions should be considered.

We will return to this point in Section 8 after the formal machinery has been introduced, and we will look again at the car park example in more detail in Section 7.

One might ask why anyone would be interested in representing the rules of a library or the rules of a car park at these levels of precision. One answer is that a precise specification may be essential if we were assigned the task of constructing a system that advises the employees and users of a library about their duties and rights, or if we were given the task of designing a system for controlling access to a car park. Or instead of controlling who may put cars in a car park, imagine for instance that the car park is a computer file of some kind, and that $p(x)$ represents not that car x is parked in the car park but that data entry x is stored in the file. The task is then to specify with precision which agents (computer agents or human) are to be permitted to insert and delete data entries in this file, in which circumstances and in which combinations. A gatekeeper agent g who controls access to a car park is not so different from a 'file monitor' (human or computer agent) which controls access to a computer file. And likewise for many of the other forms of interactions that take place in regulated human and electronic societies.

4 The Kanger-Lindahl theory

The focus in the Kanger-Lindahl theory is on mapping out the space of logically possible legal/normative relations of given forms that can hold between pairs of agents. In order to examine the possibilities systematically, Kanger considers first what he called the 'simple types of rights relations' of two agents a and b with respect to some state of affairs F. They are represented by the expressions falling under the scheme:

(6) $$\pm \mathrm{O} \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm F$$

The notation was suggested by David Makinson [1986]. $\pm$ stands for the two possibilities of affirmation and negation; the *choice-scheme* $\begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix}$ indicates the (here, two) alternatives E_a and E_b. There are thus sixteen expressions falling under the scheme (6), ranging from $\mathrm{OE}_a F$ to $\neg\mathrm{O}\neg\mathrm{E}_b\neg F$. The choice-scheme notation can be seen as shorthand for a *set* of expressions and so will be mixed freely with standard set notation.

The 'simple types' were given names by Kanger in addition to their symbolic explication. Following Lindahl's summary [1994], from the perspective of a's rights versus b, those in the scheme $\mathrm{O} \pm \mathrm{E}_b \pm F$ are called *Claim, Counter-claim, Immunity, Counter-immunity*; those in the scheme $\neg\mathrm{O} \pm \mathrm{E}_a \pm F$ (equivalently, $\mathrm{P} \pm \mathrm{E}_a \pm F$) are called *Power, Counter-power, Freedom, Counter-freedom.* The Appendix of Henning Herrestad's doctoral dissertation [1996] lists out the correspondence between names of the 'simple types' and their symbolic expression. We will not reproduce the details here since the naming scheme is of less importance than the symbolic scheme. We note only that the choice of some of these names is unfortunate, since they do not all correspond to Hohfeld's terminology. 'Power' in particular means something quite different in the Hohfeldian scheme (it is to do with the capacity or competence to effect *changes* in rights relations).

Of more interest than the 'simple types' are the various compounds that may be formed from them, or what Kanger called the 'atomic types of rights relation'. Makinson's observation [1986] was that Kanger's 'atomic types', for two agents a, b with respect to the bringing about of some state of affairs F, can be characterised as the expressions belonging to the set:

$$(7) \qquad \left[\!\left[\pm\, \mathrm{O} \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm F \right]\!\right]$$

The brackets denote *maxi-conjunctions*: where Φ is a choice-scheme (or set of sentences) $[\![\Phi]\!]$ stands for the set of *maxi-conjunctions* of Φ —the maximal consistent conjunctions of expressions belonging to Φ. 'Consistent' refers to some underlying logic, here the specific logics for O and E_x employed by Kanger and Lindahl. 'Conjunction' means a conjunction without repetitions, and with some standard order and association of conjuncts. A conjunction is 'maximal consistent' when addition of any other conjunct from Φ yields an inconsistent conjunction: in other words, a conjunction Γ is a maxi-conjunction of Φ if and only if Γ is consistent, and every expression of Φ either appears as a conjunct in Γ or is inconsistent with Γ. Note that maxi-conjunctions may contain logical redundancies (one or more conjuncts may be logically implied by the others). We shall occasionally abuse the notation and write also $[\![\Phi]\!]$ for the set of conjunctions obtained by removing all logical redundancies from the maxi-conjunctions of Φ. A justification for this practice will be provided in later sections.

As can readily be checked, and will be shown more generally later (Theorem 4.1), Kanger's 'atomic types' (7) can be written as conjunctions of two simpler expressions:

$$(8) \qquad \left[\!\left[\pm\, \mathrm{O} \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm F \right]\!\right] = [\![\pm\, \mathrm{O} \pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm\, \mathrm{O} \pm \mathrm{E}_b \pm F]\!]$$

Here the notation is as follows: when $\mathbf{P}$ and $\mathbf{Q}$ represent sets of expressions, $\mathbf{P} \cdot \mathbf{Q}$ stands for the set of all the *consistent* conjunctions that can be formed by conjoining an expression from set $\mathbf{P}$ with an expression from set $\mathbf{Q}$. (For technical reasons, it is convenient to take $\mathbf{P}\cdot\emptyset =_{\mathrm{def}} \emptyset\cdot\mathbf{P} =_{\mathrm{def}} \mathbf{P}$.) In order to reduce the need for parentheses, we adopt the convention that the $\cdot$

binds more tightly than other operators. So, for example, the choice-scheme expression $(\pm\,\mathrm{O} \pm \Phi_1{\cdot}\Phi_2)$ is to be read as $(\pm\,\mathrm{O} \pm (\Phi_1{\cdot}\Phi_2))$.

The maxi-conjunctions in

$$(9) \quad \left[\!\left[\pm\,\mathrm{O} \pm \mathrm{E}_a \pm F \right]\!\right]$$

are, in the terminology of [Jones and Sergot, 1993], Kanger's *normative one-agent act positions*. According to the logic employed by Kanger, there are six elements in (9). Following the numbering at [Lindahl, 1977, p100] and eliminating logical redundancies, they are:

(K_1) $\mathrm{PE}_a F \;\wedge\; \mathrm{PE}_a \neg F$

(K_2) $\mathrm{O}\neg\mathrm{E}_a F \;\wedge\; \mathrm{O}\neg\mathrm{E}_a \neg F$

(K_3) $\mathrm{OE}_a F$

(K_4) $\mathrm{PE}_a F \;\wedge\; \mathrm{P}\neg\mathrm{E}_a F \;\wedge\; \mathrm{O}\neg\mathrm{E}_a \neg F$

(K_5) $\mathrm{OE}_a \neg F$

(K_6) $\mathrm{O}\neg\mathrm{E}_a F \;\wedge\; \mathrm{PE}_a \neg F \;\wedge\; \mathrm{P}\neg\mathrm{E}_a \neg F$

These six expressions, by construction, are consistent, mutually exclusive, and their disjunction is a tautology. In any given situation precisely one of them must be true, according to the logical principles employed.

One can see that (K_1)–(K_6) are symmetric in F and $\neg F$ (as is obvious from the form of the expression (9)). (K_3) expresses an obligation on a, in the Meinong-Chisholm sense, to bring it about that F. In (K_1) a is permitted to bring it about that F and permitted to bring it about that $\neg F$. (K_2) can be written equivalently in a number of different ways.

(K_2') $\neg\mathrm{PE}_a F \;\wedge\; \neg\mathrm{PE}_a \neg F$

says that a is neither permitted to bring it about that F nor permitted to bring it about that $\neg F$. Following Lindahl, it is convenient to define the following abbreviation:

$$(10) \quad \mathrm{Pass}_a F \;=_{\mathrm{def}}\; \neg\mathrm{E}_a F \wedge \neg\mathrm{E}_a \neg F$$

$\mathrm{Pass}_a F$ represents a kind of 'passivity' of agent a with respect to state of affairs F. (K_2) can be written equivalently as:

(K_2'') $\mathrm{O}(\neg\mathrm{E}_a F \wedge \neg\mathrm{E}_a \neg F) \;=\; \mathrm{OPass}_a F$

and so expresses an obligation on a to remain 'passive' with respect to F. (K_4) is equivalent to

(K_4') $\mathrm{PE}_a F \;\wedge\; \mathrm{P}\neg\mathrm{E}_a F \;\wedge\; \neg\mathrm{PE}_a \neg F$

According to (K_4'), a is permitted to bring it about that F and permitted to refrain from bringing it about that F, but a is not permitted to bring it about that $\neg F$.

For Kanger's 'atomic types' for two agents, expression (8), there are $6 \times 6 = 36$ conjunctions to consider. Of these, 10 turn out to be logically inconsistent. On Kanger's analysis, therefore, there are 26 atomic types of right (for two agents with respect to the bringing about of some given state of affairs). Again, by construction these 26 'atomic types' are internally consistent, mutually exclusive, and their disjunction is a tautology. In any given situation precisely one of them must be true, according to the logics employed. It is in this sense that Kanger can be said to provide a complete and exhaustive analysis of all the logically possible normative positions.

Kanger's 26 'atomic types' are listed in full in [Kanger and Kanger, 1966, pp93–94] and [Lindahl, 1977, p56] and in several other works. In these works however each position (atomic type) is described by listing the names (i.e., claim, freedom, power, etc) of the constituent single-agent types rather than the symbolic expressions.

For example, the first of the 26 atomic types in the standard table is listed as 'Power, not Immunity, Counter-power, not Counter-immunity' which corresponds to the conjunction of one-agent act positions (K_1) for a and (K_1) for b and thus the symbolic expression:

$$\mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \mathrm{PE}_b F \wedge \mathrm{PE}_b \neg F$$

The 15th atomic type in the table, to pick just one other example, is listed as 'Liberty, not Power, Immunity, Counter-power, Counter-immunity'. This corresponds to the conjunction of one-agent act positions (K_6) for a and (K_2) for b and thus the symbolic expression:

$$\mathrm{O}\neg\mathrm{E}_a F \wedge \mathrm{PE}_a \neg F \wedge \mathrm{P}\neg\mathrm{E}_a \neg F \wedge \mathrm{O}\neg\mathrm{E}_b F \wedge \mathrm{O}\neg\mathrm{E}_b \neg F$$

The complete listing and numbering used by the previous authors together with the corresponding symbolic expressions in each case can be found in [Herrestad, 1996, Appendix].

Each of Kanger's 26 atomic types can be expressed as a conjuction of two of the 6 single-agent types (K_1)–(K_6) by virtue of equation (8) (and Theorem 4.1 below).

Kanger gives a complete and exhaustive analysis of all the logically possible atomic types. In general, all maxi-conjunctions of the form $[\![\pm\, \Phi]\!]$ have this property of exhaustiveness. Moreover, all (consistent) boolean compounds of expressions in Φ are logically equivalent to a (non-empty) disjunction of elements from $[\![\pm\, \Phi]\!]$. As observed by Makinson [1986], the maxi-conjunctions can be given an algebraic interpretation (as atoms of a Boolean algebra). For certain logics (those of type *EMCP*, though not for weaker ones), they give the constituents of a distributive normal form in the underlying modal logics. (They are not quite yet a normal form: for that we would need to consider not just the sentences of Φ but also all of their subsentences.)

The value of Makinson's suggestion, besides the conciseness of the notation, is that the characterisation of positions in terms of maxi-conjunctions emphasises their character rather than the specific procedures by which they happen to be generated. There are many different ways of generating the same set of maxi-conjunctions. The following elementary property of maxi-conjunctions is particularly useful, and is the basis for a whole family of such procedures.

Theorem 4.1 *For any choice scheme $\Phi = \Phi_1 \cup \Phi_2$ (Φ_1 and Φ_2 not necessarily distinct):*

1. $[\![\Phi_1]\!] \cdot [\![\Phi_2]\!] \subseteq [\![\Phi]\!]$
2. $[\![\pm\, \Phi]\!] = [\![\pm\, \Phi_1]\!] \cdot [\![\pm\, \Phi_2]\!]$

Proof. Straightforward. See [Sergot, 2001]. ■

Computationally: to generate the set of maxi-conjunctions $[\![\pm\Phi]\!]$, decompose the scheme (or set of sentences) Φ into smaller, not necessarily disjoint, subsets Φ_1 and Φ_2 (there are many different strategies for this step); (recursively) compute the sets of maxi-conjunctions $[\![\pm\, \Phi_1]\!]$ and $[\![\pm\, \Phi_2]\!]$, possibly in parallel; form all conjunctions of expressions from these sets of maxi-conjunctions; discard those conjunctions that are logically inconsistent. The steps, especially the last two steps, may be co-routined for efficiency. It is straightforward to code any such procedure as a computer program, requiring only an implementation of the inconsistency check for the generated conjunctions. Although this is not difficult—it is only fragments of the underlying modal logics that are required—it is not particularly useful either. In Section 6 we show how a little additional manipulation eliminates the need for theorem-proving techniques altogether, at least for the most common types of modal logic.

As an example, the method used to generate classes of normative positions in [Jones and Sergot, 1993] (and in [Jones and Parent, 2008]) is a special case of Theorem 4.1. For illustration, in [Jones and Sergot, 1993] the generation of what are there called the 'normative fact positions'

(11) $[\![\pm\, \mathrm{O} \pm F]\!]$

proceeds as follows. Form two tautologies $\mathrm{O}F \vee \neg\mathrm{O}F$ and $\mathrm{O}\neg F \vee \neg\mathrm{O}\neg F$. Their conjunction is another tautology. Re-write it as a disjunction of conjunctions by picking one disjunct from each in all combinations, to obtain $(\mathrm{O}F \wedge \mathrm{O}\neg F) \vee (\mathrm{O}F \wedge \neg\mathrm{O}\neg F) \vee (\neg\mathrm{O}F \wedge \mathrm{O}\neg F) \vee (\neg\mathrm{O}F \wedge \neg\mathrm{O}\neg F)$. The first disjunct of this expression is logically inconsistent and so can be deleted; the others can be simplified. That procedure can be presented as a special case of Theorem 4.1 as follows:

$$\begin{aligned}\llbracket \pm\,\mathrm{O} \pm F \rrbracket &= \llbracket \pm\,\mathrm{O}F \rrbracket \cdot \llbracket \pm\,\mathrm{O}\neg F \rrbracket && \text{(by Theorem 4.1)} \\ &= \begin{pmatrix} \mathrm{O}F \\ \neg\mathrm{O}F \end{pmatrix} \cdot \begin{pmatrix} \mathrm{O}\neg F \\ \neg\mathrm{O}\neg F \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{O}F \\ \mathrm{O}\neg F \\ \mathrm{P}F \wedge \mathrm{P}\neg F \end{pmatrix} && \text{(with logical redundancies removed)}\end{aligned}$$

Equation (8) expressing Kanger's two-agent atomic types as conjunctions of one-agent types is also a special case of Theorem 4.1. This follows immediately from:

$$\pm\,\mathrm{O} \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm F \;=\; \pm\,\mathrm{O} \pm \mathrm{E}_a \pm F \;\cup\; \pm\,\mathrm{O} \pm \mathrm{E}_a \pm F$$

There will be other examples presently.

Lars Lindahl [1977] presents a refinement and further development of Kanger's analysis. The second part of his book deals also with aspects of 'change' of normative positions. That part of Lindahl's account will not be pursued here.

Lindahl constructs his analysis on the following set of normative one-agent act positions:

(12) $\Big\llbracket \pm\,\mathrm{P} \llbracket \pm\,\mathrm{E}_a \pm F \rrbracket \Big\rrbracket$

where now there is a maxi-conjunction expression within the scope of the P operator. In words, (12) is the set of maxi-conjunction expressions of the form $\pm\,\mathrm{P}A$, where each A is itself a maxi-conjunction of sentences of the form $\pm\,\mathrm{E}_a \pm F$. The iterated bracket notation is again from [Makinson, 1986].

There are three *act positions* in the set

(13) $\llbracket \pm\,\mathrm{E}_a \pm F \rrbracket$

K_1	is logically equivalent to	$(T_1 \vee T_3)$
K_2	… … …	T_6
K_3	… … …	T_5
K_4	… … …	T_2
K_5	… … …	T_7
K_6	… … …	T_4

Table 1. Normative one-agent act positions

They are:

(A_1) $E_a F$

(A_2) $E_a \neg F$

(A_3) $\neg E_a F \wedge \neg E_a \neg F$

The third of these (A_3) is the 'passivity' of agent a with respect to state of affairs F, which following Lindahl we also write using the abbrevation $\mathrm{Pass}_a F$.

There are $2^3 - 1 = 7$ expressions in the set (12). They are, numbered as in [Lindahl, 1977] and with logical redundancies removed:

(T_1) $PE_a F \wedge PE_a \neg F \wedge P\mathrm{Pass}_a F$

(T_2) $PE_a F \wedge O\neg E_a \neg F \wedge P\mathrm{Pass}_a F$

(T_3) $PE_a F \wedge PE_a \neg F \wedge \neg P\mathrm{Pass}_a F$

(T_4) $O\neg E_a F \wedge PE_a \neg F \wedge P\mathrm{Pass}_a F$

(T_5) $OE_a F$

(T_6) $O\mathrm{Pass}_a F$

(T_7) $OE_a \neg F$

(T_2) and (T_4) can be written equivalently as:

(T'_2) $PE_a F \wedge \neg PE_a \neg F \wedge P\mathrm{Pass}_a F$

(T'_4) $\neg PE_a F \wedge PE_a \neg F \wedge P\mathrm{Pass}_a F$

Lindahl's construction gives a finer-grained analysis than Kanger's. For the one-agent types, five of the six in Kanger's (9) are logically equivalent to five of the seven in Lindahl's (12), as summarized in Table 1.

On Lindahl's analysis, therefore, Kanger's type (K_1) can be decomposed:

(K_1) $PE_a F \wedge PE_a \neg F$

is logically equivalent to a disjunction of two of Lindahl's types, viz.

$(\mathrm{T}_1)\quad \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \mathrm{PPass}_a F$

$(\mathrm{T}_3)\quad \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \neg\mathrm{PPass}_a F$

For an example of (T_3), consider a judge (a) who is permitted to see to it that the prisoner is imprisoned (F) and permitted to see to it that the prisoner is not imprisoned ($\neg F$); but a is not permitted to do neither of these: $\neg\mathrm{PPass}_a F$.

In place of Kanger's two-agent types (8), Lindahl has the following set of positions:

(14) $$\left[\!\!\left[\pm \mathrm{P} [\![\pm \mathrm{E}_a \pm F]\!] \right]\!\!\right] \cdot \left[\!\!\left[\pm \mathrm{P} [\![\pm \mathrm{E}_b \pm F]\!] \right]\!\!\right]$$

There are $7 \times 7 = 49$ conjunctions to consider, of which 35 are internally consistent. These are Lindahl's 'individualistic' normative two-agent act positions. The significance of 'individualistic' will be explained in a moment. Lindahl's construction again gives a finer-grained analysis than Kanger's: some of Kanger's 26 two-agent 'atomic types' (7) are logically equivalent to disjunctions of Lindahl's corresponding 35 types (14). We omit the details: the next section presents a general result and a computational method to perform this kind of calculation.

Notice that, since P is the dual of O, Kanger's one-agent positions (9) may be written equivalently as $[\![\pm \mathrm{P} \pm \mathrm{E}_a \pm F]\!]$. The expression within the maxi-conjunction brackets may be seen in two ways: either as a scheme of *four* (not mutually exclusive) act positions $\pm \mathrm{E}_a \pm F$ prefixed by $\pm \mathrm{P}$, or as *two* mutually exclusive act positions $\mathrm{E}_a \pm F$ prefixed by $\pm \mathrm{P} \pm$. What is obtained by combining the second view, $\pm \mathrm{P} \pm$, with the three mutually exclusive act positions $[\![\pm \mathrm{E}_a \pm F]\!]$ used by Lindahl? In other words, consider the following:

(15) $$\left[\!\!\left[\pm \mathrm{P} \pm [\![\pm \mathrm{E}_a \pm F]\!] \right]\!\!\right] = \left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!] \right]\!\!\right]$$

(The equality here is because P and O are duals.) This is the construction used in Jones and Sergot's account of normative positions [1992; 1993]. It turns out that for the logics employed by Kanger and Lindahl the positions in set (15) are exactly the same seven as those in Lindahl's simpler form (12). By Theorem 4.1 the following holds irrespective of the logic of O:

(16) $$\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!] \right]\!\!\right] = \left[\!\!\left[\pm \mathrm{P} [\![\pm \mathrm{E}_a \pm F]\!] \right]\!\!\right] \cdot \left[\!\!\left[\pm \mathrm{O} [\![\pm \mathrm{E}_a \pm F]\!] \right]\!\!\right]$$

But when the logic of O is of type *EMCP* (or stronger), then also (as shown later in Section 6, Theorem 6.1):

$$\text{(17)}\quad \left[\!\left[\pm\,\mathrm{O}\pm[\![\pm\,\mathrm{E}_a\pm F]\!]\right]\!\right] = \left[\!\left[\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_a\pm F]\!]\right]\!\right]$$

For weaker logics the equality (17) does not hold. In that case the Jones-Sergot form (15) gives a more refined analysis than Lindahl's (12).

There is another important respect in which Lindahl extends Kanger's analysis of two-agent 'atomic types'. In [Lindahl, 1977, Ch.5] the account is extended to what are called 'collectivistic two-agent types', to cover the case where, for instance, there is an obligation on two agents which does not apply to either of them individually:

$$\mathrm{O}(\mathrm{E}_a F \vee \mathrm{E}_b F)\ \wedge\ \neg\mathrm{O}\mathrm{E}_a F\ \wedge\ \neg\mathrm{O}\mathrm{E}_b F$$

Lindahl is there addressing the *co-ordination* of a and b's actions, which introduces distinctions that cannot be expressed by conjunctions of the 'individualistic' types (14). The reason is simply that, in the logics employed, P does not distribute over conjunction (nor O over disjunction): $(\mathrm{P}A \wedge \mathrm{P}B) \rightarrow \mathrm{P}(A \wedge B)$ is *not* a theorem for arbitrary A and B. For instance, $\mathrm{PE}_a F\ \wedge\ \mathrm{PE}_b F$ is consistent with both $\mathrm{P}(\mathrm{E}_a F \wedge \mathrm{E}_b F)$ and $\neg\mathrm{P}(\mathrm{E}_a F \wedge \mathrm{E}_b F)$.

Lindahl's 'collectivistic' two-agent positions are obtained by the following construction:

$$\text{(18)}\quad \left[\!\left[\pm\,\mathrm{P}\,[\![\pm\begin{pmatrix}\mathrm{E}_a\\ \mathrm{E}_b\end{pmatrix}\pm F]\!]\right]\!\right] = \left[\!\left[\pm\,\mathrm{P}\,\left([\![\pm\,\mathrm{E}_a\pm F]\!]\cdot[\![\pm\,\mathrm{E}_b\pm F]\!]\right)\right]\!\right]$$

In the *EMCP*-equivalent Jones-Sergot form these positions are:

$$\text{(19)}\quad \left[\!\left[\pm\,\mathrm{O}\pm[\![\pm\begin{pmatrix}\mathrm{E}_a\\ \mathrm{E}_b\end{pmatrix}\pm F]\!]\right]\!\right] = \left[\!\left[\pm\,\mathrm{O}\pm\left([\![\pm\,\mathrm{E}_a\pm F]\!]\cdot[\![\pm\,\mathrm{E}_b\pm F]\!]\right)\right]\!\right]$$

For the logics employed by Kanger and Lindahl, there are $2^7 - 1 = 127$ 'collectivistic normative two-agent act positions' in the sets (18) and (19). Each collectivistic type implies one of the 'individualistic' types (14); each of the 'individualistic' types is logically equivalent to a disjunction of one or more of the collectivistic types. This can be seen by reference to the table compiled by [Lindahl, 1977, p180], or, as shown in later sections, from a general property of maxi-conjunctions which holds when the logic of O is of type *EMCP*.

[Sergot, 2001] presents a generalised theory of normative positions that builds upon Makinson's maxi-conjunction characterisation. It is summarised in the next two sections, and addresses the following questions in particular:

(1) How can the account be generalised to the case of n agents? This is a possibility mentioned by Lindahl but not developed by him, presumably because of the size and number of the symbolic expressions to be manipulated.

(2) How can the account be generalised to deal with related states of affairs, in the same kind of way that the 'collectivistic' positions generalise the 'individualistic'? Consider two neighbours, a and b. Let F represent that there is a fence at the front of their adjoining properties, and G that there is a fence at the back of their properties. Suppose both neighbours are permitted to see to it that there is a fence at the front, $\mathrm{PE}_a F \wedge \mathrm{PE}_b F$, and permitted to see to it that there is a fence at the back, $\mathrm{PE}_a G \wedge \mathrm{PE}_b G$. We might nevertheless want to distinguish between the case represented by $\mathrm{P}(\mathrm{E}_a F \wedge \mathrm{E}_a G) \wedge \mathrm{P}(\mathrm{E}_b F \wedge \mathrm{E}_b G)$ and the case represented by $\neg\mathrm{P}(\mathrm{E}_a F \wedge \mathrm{E}_a G) \wedge \neg\mathrm{P}(\mathrm{E}_b F \wedge \mathrm{E}_b G)$. It is conceivable that there could be other constraints, such as that represented by $\mathrm{O}(\mathrm{E}_a F \leftrightarrow \mathrm{E}_a G)$, i.e. $\neg\mathrm{P}(\mathrm{E}_a F \wedge \neg\mathrm{E}_a G) \wedge \neg\mathrm{P}(\mathrm{E}_a G \wedge \neg\mathrm{E}_a F)$. These distinctions cannot be expressed in the Kanger-Lindahl framework.

(3) To what extent can these various constructions be generalised to other, weaker logics than those employed by Kanger and Lindahl? Which features of the theory are properties of the specific logics employed, and which of maxi-conjunctions in general?

(4) Lindahl's construction yields a finer-grained analysis than Kanger's. Is there similarly a finer-grained analysis than Lindahl's? Is there a finest analysis?

The last question can be answered as follows. For one agent a and one state of affairs F, Lindahl bases his analysis on the set of three act positions $[\![\pm\,\mathrm{E}_a \pm F]\!]$. But a finer analysis can be obtained by taking instead the act positions from the following scheme:

(20) $[\![\pm\,\mathrm{E}_a \pm F]\!]\cdot[\![\pm\,F]\!]$

We might call these 'cumulative fact/act positions'. There are four such positions:

(A_1)	$\mathrm{E}_a F$	
(A_2)	$\mathrm{E}_a \neg F$	
(A_{3a})	$F \wedge \neg\mathrm{E}_a F$	(which is equivalent to $\mathrm{Pass}_a F \wedge F$)
(A_{3b})	$\neg F \wedge \neg\mathrm{E}_a \neg F$	(which is equivalent to $\mathrm{Pass}_a F \wedge \neg F$)

Lindahl's 'passive' act position (A_3) does not distinguish between (A_{3a}) and (A_{3b}).

$$
\begin{array}{ll}
\mathrm{T}_1 & \left\{ \begin{array}{l} \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \\ \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \neg \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \\ \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \neg \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \end{array} \right. \\
\mathrm{T}_2 & \left\{ \begin{array}{l} \mathrm{PE}_a F \wedge \neg \mathrm{PE}_a \neg F \wedge \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \\ \mathrm{PE}_a F \wedge \neg \mathrm{PE}_a \neg F \wedge \neg \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \\ \mathrm{PE}_a F \wedge \neg \mathrm{PE}_a \neg F \wedge \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \neg \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \end{array} \right. \\
\mathrm{T}_3 & \left\{ \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \neg \mathrm{PPass}_a F \right. \\
\mathrm{T}_4 & \left\{ \begin{array}{l} \neg \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \\ \neg \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \neg \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \\ \neg \mathrm{PE}_a F \wedge \mathrm{PE}_a \neg F \wedge \mathrm{P}(F \wedge \neg \mathrm{E}_a F) \wedge \neg \mathrm{P}(\neg F \wedge \neg \mathrm{E}_a \neg F) \end{array} \right. \\
\mathrm{T}_5 & \left\{ \mathrm{OE}_a F \right. \\
\mathrm{T}_6 & \left\{ \begin{array}{l} \mathrm{OPass}_a F \wedge \mathrm{O} F \\ \mathrm{OPass}_a F \wedge \mathrm{O} \neg F \\ \mathrm{OPass}_a F \wedge \mathrm{P} F \wedge \mathrm{P} \neg F \end{array} \right. \\
\mathrm{T}_7 & \left\{ \mathrm{OE}_a \neg F \right.
\end{array}
$$

Table 2. Normative one-agent cumulative fact/act positions

The corresponding single-agent 'normative act positions' are:

(21) $\left[\!\left[\pm \mathrm{O} \pm \left[\!\left[\pm \mathrm{E}_a \pm F \right]\!\right] \cdot \left[\!\left[\pm F \right]\!\right] \right]\!\right]$

There are $2^4 - 1 = 15$ conjunctions in the set (21), as compared with the seven (T_1)–(T_7) constructed in Lindahl's analysis. They are listed in Table 2. Three are identical to Lindahl's (T_3), (T_5) and (T_7); the other four of Lindahl's types are each logically equivalent to a disjunction of three conjunctions from (21). Just as Lindahl is able to give examples to illustrate the ambiguity in Kanger's type (K_1), so it is easy to find examples to illustrate the ambiguities in Lindahl's types (T_1), (T_2), (T_4), (T_6). Consider (T_1) for example, and suppose that a neighbour a is permitted to see to it that there is a fence (F), permitted to see to it that there is no fence, and permitted to remain passive with respect to there being a fence. It may be, however, that if there is a fence then a must see to it, in other words that $\mathrm{O}(F \to \mathrm{E}_a F)$, equivalently $\neg \mathrm{P}(F \wedge \mathrm{E}_a F)$, is true. That possibility is covered by the second of the (T_1) refinements in Table 2 but not by the other two.

For two-agent positions, the corresponding expressions for 'individualistic' and 'collectivistic' positions are, respectively:

$$(22)\quad \left[\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm F]\!]\right]\!\right] \cdot \left[\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_b \pm F]\!] \cdot [\![\pm F]\!]\right]\!\right]$$

$$(23)\quad \left[\!\left[\pm\mathrm{O}\pm[\![\pm\begin{pmatrix}\mathrm{E}_a\\ \mathrm{E}_b\end{pmatrix}\pm F]\!]\cdot[\![\pm F]\!]\right]\!\right] = \left[\!\left[\pm\mathrm{O}\pm[\![\pm\mathrm{E}_a\pm F]\!]\cdot[\![\pm\mathrm{E}_b\pm F]\!]\cdot[\![\pm F]\!]\right]\!\right]$$

When the logic of O is of type *EMCP* or stronger, constructions (21) for one agent and (23) for any pair of agents are—effectively—the finest-grained set of normative positions that can be constructed for a given state of affairs, respectively. The next section explains what is meant by 'finest-grained'.

The account can be generalised, to any (finite) number of agents $\{a, b, \ldots\}$ not just two, and any (finite) number of separate states of affairs $\{F, G, \ldots\}$ not just one. Consider for instance the following construction:

$$(24)\quad \left[\!\left[\pm \mathrm{O} \pm [\![\pm \begin{pmatrix}\mathrm{E}_a\\ \mathrm{E}_b\\ \vdots\end{pmatrix} \pm \begin{pmatrix}F\\ G\\ \vdots\end{pmatrix}]\!] \cdot [\![\pm \begin{pmatrix}F\\ G\\ \vdots\end{pmatrix}]\!]\right]\!\right]$$

There are still more complex classes of normative positions if we allow also iterations of the action modalities. We will give some examples in Section 7 below.

5 Partitions

Lindahl's construction yields a finer-grained analysis than Kanger's. But Kanger's analysis is also exhaustive, in the sense that his 'atomic types' are logically consistent, mutually exclusive, and their disjunction is a tautology. Kanger's analysis and Lindahl's analysis are both exhaustive, but Lindahl's is finer than Kanger's. We now formalise these notions.

We begin by defining a syntactic version of the standard notion of a *partition* of a set whereby a set is partitioned into non-empty disjoint subsets. All definitions are given with respect to some underlying logic Λ. Since Λ is usually obvious from context we write $\vdash A$ for $A \in \Lambda$. The only assumption we make in this section is that Λ includes classical propositional logic, i.e. contains all tautologies *PL* and is closed under modus ponens.

Definition 5.1 *Let* $\mathbf{P} = \{P_1,\ P_2, \ldots\}$ *be a set of sentences and* Q *a sentence of the language of* Λ*. Then* $\mathbf{P} = \{P_1,\ P_2, \ldots\}$ *is a* Λ-partition *of* Q *iff it satisfies the following conditions:*

1. *every element* P_i *of* $\mathbf{P}$ *is logically consistent:* $\nvdash \neg P_i$*;*

2. *every element* P_i *of* $\mathbf{P}$ *logically implies* Q*:* $\vdash P_i \to Q$*;*

3. *distinct elements of* $\mathbf{P}$ *are mutually exclusive:* $\vdash \neg(P_i \wedge P_j) \quad (i \neq j)$*;*

4. *the set* $\mathbf{P}$ *'exhausts'* Q*:* $\vdash Q \to \bigvee_{P \in \mathbf{P}} P$*.*

Conditions (2) and (4) together are: $\vdash Q \leftrightarrow \bigvee_{P \in \mathbf{P}} P$*.*

When Q *is a tautology we shall say that* $\mathbf{P}$ *is a* complete Λ*-partition, or simply a* Λ*-partition. Where context permits we omit the* Λ*-prefix and simply say 'partition'. In what follows partitions will be finite sets.*

Example 5.2 *All of the following (the terminology is from [Jones and Sergot, 1993]) are (complete) partitions:*

- *fact positions:* $[\![\pm F]\!] = \{F,\ \neg F\}$*;*
- *Lindahl's one-agent act positions:*

$$[\![\pm \mathrm{E}_a \pm F]\!] = \{\mathrm{E}_a F,\ \mathrm{E}_a \neg F,\ \mathrm{Pass}_a F\};$$

- *normative fact positions:* $[\![\pm \mathrm{O} \pm F]\!] = \{\mathrm{O}F,\ \mathrm{O}\neg F,\ \mathrm{P}F \wedge \mathrm{P}\neg F\}$*;*
- *Lindahl's normative one-agent act positions* (T_1)*–*(T_7)*:*

$$\Big[\!\!\Big[\pm \mathrm{P}\,[\![\pm \mathrm{E}_a \pm F]\!]\Big]\!\!\Big]$$

In general, any maxi-conjunction of the form $[\![\pm\Phi]\!]$ *is a (complete) partition. In contrast:*

- *The act positions used by Kanger,* $\pm \mathrm{E}_a \pm F$*, are not mutually exclusive, whereas* $\mathrm{E}_a \pm F = \{\mathrm{E}_a F,\ \mathrm{E}_a \neg F\}$ *are mutually exclusive but do not form a complete partition.*

Naturally, if $\{P_1, \ldots, P_n\}$ *is a set of consistent, mutually exclusive sentences, then* $\{P_1, \ldots, P_n\}$ *is a partition of* $P_1 \vee \ldots \vee P_n$*.*

Λ-partitions are just syntactic analogues of the standard notion of a partition of a set. The two are easily related. For any model $\mathcal{M}$ of Λ, let $\|Q\|^{\mathcal{M}}$ denote the 'truth set' of Q, i.e. the set of possible worlds of $\mathcal{M}$ at which Q is true. The exact structure of $\mathcal{M}$ does not matter. Then the set of sentences $\mathbf{P} = \{P_1,\ P_2, \ldots\}$ is a Λ-partition of Q when, for all models $\mathcal{M}$ of Λ, the sets $\|P_1\|^{\mathcal{M}}, \|P_2\|^{\mathcal{M}}, \ldots$ partition the set $\|Q\|^{\mathcal{M}}$.

In view of this observation, it would be possible to eliminate the need for Definition 5.1 altogether and use instead the set-theoretic language indicated above, identifying each sentence with the set of all maximal consistent sets that contain it, and taking the notion of partition in its ordinary set-theoretic sense. We will stick to the syntactic version of Definition 5.1, however, because its application is more immediate in the present context. Furthermore, given a set of sentences, it is still necessary to check whether they constitute a partition, and for this purpose Definition 5.1 is more useful. We record in this section a number of properties of (syntactic) partitions that will be used later. All of them are easy to check, either directly from Definition 5.1 or by translating first to the set-theoretic analogue.

Proposition 5.3 *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be partitions of some sentence* R*. Then the set of conjunctions* $\mathbf{P} \cdot \mathbf{Q}$ *is non-empty and is also a partition of* R*.*

In the above, $\mathbf{P} \cdot \mathbf{Q}$ must be non-empty, else R is logically inconsistent and $\mathbf{P}$ and $\mathbf{Q}$ could not be partitions. We now define some relations between partitions.

Definition 5.4 *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be partitions of some sentence* R*.* $\mathbf{P}$ *and* $\mathbf{Q}$ *are* equivalent *(*$\mathbf{P} \equiv \mathbf{Q}$*) iff their elements are pairwise logically equivalent, i.e. iff there is a bijection* $f\colon \mathbf{P} \to \mathbf{Q}$ *such that* $\vdash P \leftrightarrow f(P)$ *for all elements* P *of* $\mathbf{P}$*.*

Definition 5.5 *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be partitions of some sentence* R*.* $\mathbf{P}$ *is a* refinement *of* $\mathbf{Q}$ *(*$\mathbf{P} \geq \mathbf{Q}$*) iff every element of* $\mathbf{P}$ *logically implies some element of* $\mathbf{Q}$*:*

$$\mathbf{P} \geq \mathbf{Q} \quad \textit{iff} \quad \forall P \in \mathbf{P}\ \exists Q \in \mathbf{Q} \ \ \textit{such that} \ \vdash P \to Q.$$

When $\mathbf{P} \geq \mathbf{Q}$ *we shall also say that partition* $\mathbf{P}$ refines *partition* $\mathbf{Q}$*.*

Proposition 5.6 *Let* $\mathbf{P}$*,* $\mathbf{Q}$*,* $\mathbf{R}$ *be partitions of some sentence* S*.*

1. $\mathbf{P} \equiv \mathbf{Q}$ *iff* $\mathbf{P} \geq \mathbf{Q}$ *and* $\mathbf{Q} \geq \mathbf{P}$*;*
2. $\mathbf{P} \cdot \mathbf{Q} \geq \mathbf{P}$ *and* $\mathbf{P} \cdot \mathbf{Q} \geq \mathbf{Q}$*;*
3. $\mathbf{P} \cdot \mathbf{Q} \equiv \mathbf{P}$ *iff* $\mathbf{P} \geq \mathbf{Q}$*;*
4. *Moreover, the conjunction operator* $\cdot$ *is the 'meet' operator (glb) for partitions: if* $\mathbf{R} \geq \mathbf{P}$ *and* $\mathbf{R} \geq \mathbf{Q}$ *then* $\mathbf{R} \geq \mathbf{P} \cdot \mathbf{Q}$*.*

Example 5.7

- *Here is an instance of a general property to be established in a moment:*

$$[\![\pm P]\!] \cdot [\![\pm Q]\!] \ \geq \ [\![\pm P]\!]$$

- *[Lindahl, 1977, p100] provides a table comparing his atomic (one-agent) types with those of Kanger, reproduced as Table 1 above. From the table it is clear that Lindahl's types (which are a (complete) partition) are a refinement of Kanger's:*

$$\left[\!\left[\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right] \;\geq\; [\![\pm\,\mathrm{O} \pm \mathrm{E}_a \pm F]\!]$$

In later sections we shall be able to establish this relationship without having to compute these sets explicitly. It holds when the logic of O *is of type EMCP. See Example 5.9 and Theorems 6.1 and 6.3 below.*

- *The procedure used in [Jones and Sergot, 1993] constructs a set of maxi-conjunctions that is a refinement of Lindahl's normative one-agent act positions:*

$$\left[\!\left[\pm\,\mathrm{O} \pm [\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right] \;\geq\; \left[\!\left[\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right]$$

This is just a corollary of Theorem 4.1 and does not depend on the logic of O*. See Example 5.9 below. When the logic of* O *is of type EMCP we have also*

$$\left[\!\left[\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right] \;\geq\; \left[\!\left[\pm\,\mathrm{O} \pm [\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right]$$

i.e., an equivalence. See Theorem 6.1.

- *Lindahl's 'collectivistic' two-agent types are a refinement of the 'individualistic' types:*

$$\left[\!\left[\pm\,\mathrm{P}\,[\![\pm \begin{pmatrix}\mathrm{E}_a \\ \mathrm{E}_b\end{pmatrix} \pm F]\!] \right]\!\right] \;\geq\; \left[\!\left[\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right] \cdot \left[\!\left[\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_b \pm F]\!] \right]\!\right]$$

This can be seen by examination of the table compiled by [Lindahl, 1977, p180] but again it can be established, without evaluating the two expressions in full, by means of general properties of maxi-conjunctions. It holds when the logic of O *is of type EMCP. See Theorems 6.1 and 6.3 below.*

- *Normative positions based on cumulative fact/act postions* (21) *are a refinement of Lindahl's normative one-agent act positions:*

$$\begin{aligned}\left[\!\left[\pm\,\mathrm{O} \pm [\![\pm\,\mathrm{E}_a \pm F]\!] \cdot [\![\pm F]\!] \right]\!\right] \;&\geq\; \left[\!\left[\pm\,\mathrm{O} \pm [\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right] \\ &\geq\; \left[\!\left[\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_a \pm F]\!] \right]\!\right]\end{aligned}$$

This can be seen by inspection of Table 2 above. It holds because $[\![\pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm F]\!] \geq [\![\pm \mathrm{E}_a \pm F]\!]$. *In general when* O *is of type EMCP,* $\mathbf{A} \geq \mathbf{B}$ *implies* $[\![\pm \mathrm{O} \pm \mathbf{A}]\!] \geq [\![\pm \mathrm{O} \pm \mathbf{B}]\!]$. *See Theorem 6.3 below.*

- *There is a similar relationshp between the corresponding two agent 'collectivistic' positions:*

$$\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \begin{pmatrix}\mathrm{E}_a \\ \mathrm{E}_b\end{pmatrix} \pm F]\!] \cdot [\![\pm F]\!]\right]\!\!\right] \geq \left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \begin{pmatrix}\mathrm{E}_a \\ \mathrm{E}_b\end{pmatrix} \pm F]\!]\right]\!\!\right]$$

The following property is very useful. It follows from Theorem 4.1 and Proposition 5.6, part (2).

Proposition 5.8 *For sets of sentences* $\Phi_1 \subseteq \Phi_2$*:* $[\![\pm \Phi_2]\!] \geq [\![\pm \Phi_1]\!]$.

Example 5.9 *Since* P *is the dual of* O, $\pm\mathrm{P}\,[\![\pm\mathrm{E}_a \pm F]\!] \subseteq \pm\mathrm{O}\pm[\![\pm\mathrm{E}_a \pm F]\!]$, *and hence*

$$\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!]\right]\!\!\right] \geq \left[\!\!\left[\pm \mathrm{P}\,[\![\pm \mathrm{E}_a \pm F]\!]\right]\!\!\right]$$

as observed in Example 5.7 above. Similarly, $\mathrm{E}_a \pm F \subseteq [\![\pm \mathrm{E}_a \pm F]\!]$ *so*

$$\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!]\right]\!\!\right] \geq [\![\pm \mathrm{O} \pm \mathrm{E}_a \pm F]\!]$$

Definition 5.10 *For* $\mathbf{P}$ *a set of sentences and* Q *any expression:*

$$\mathbf{P}/Q \;=_{\mathrm{def}}\; \{P \in \mathbf{P} \mid P \wedge Q \textit{ consistent}\}.$$

For example: suppose that in the analysis of some scenario or set of regulations, it is determined that $\mathrm{OE}_a F$ is true. The library example of Section 3 is of this form. Then

$$\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_b \pm F]\!]\right]\!\!\right] \Big/ \mathrm{OE}_a F$$

represents the (Jones-Sergot) normative one-agent act positions consistent with $\mathrm{OE}_a F$. The 'collectivistic' two-agent act positions consistent with $\mathrm{OE}_a F$ are given by the expression:

$$\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \begin{pmatrix}\mathrm{E}_a \\ \mathrm{E}_b\end{pmatrix} \pm F]\!]\right]\!\!\right] \Bigg/ \mathrm{OE}_a F$$

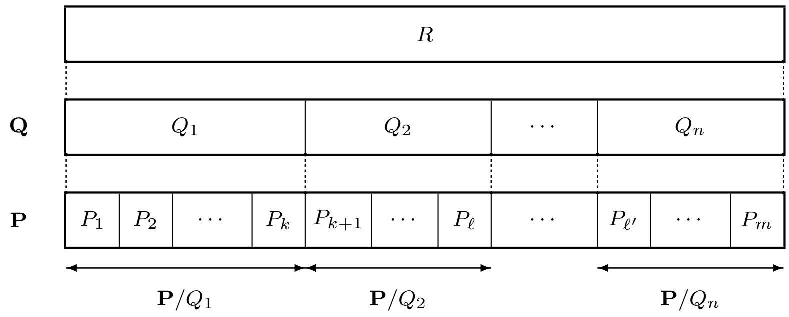

Figure 2. Partitions $\mathbf{P}$ and $\mathbf{Q}$ of R with $\mathbf{P} \geq \mathbf{Q}$

We can say much more about the structure of partitions $\mathbf{P}$ and $\mathbf{Q}$ in the case that $\mathbf{P}$ is a refinement of $\mathbf{Q}$. When $\mathbf{P} \geq \mathbf{Q}$ and Q is an element of $\mathbf{Q}$ then $\mathbf{P}/Q$ is also the set of elements of $\mathbf{P}$ that logically imply Q. Indeed, when $\mathbf{P} \geq \mathbf{Q}$ and Q is an element of $\mathbf{Q}$ then $\mathbf{P}/Q$ is a Λ-partition of Q. And further: the set $\mathbf{P}$ itself is partitioned (standard set notion) into the collection of disjoint subsets $\mathbf{P}/Q_i$ where the Q_i are the elements of $\mathbf{Q}$. The relationships are summarised in Figure 2. (The rectangles can be seen as Venn diagrams of the corresponding truth sets, moved apart to show the structure of the two partitions.)

We are now in a position to summarise the relationship between Kanger's (one-agent) 'atomic types', Lindahl's more refined version, the more complicated construction used in [Jones and Sergot, 1993], and the maxi-conjunctions identified at the end of Section 4 as a further refinement still. We include for completeness the set of 'normative fact positions' $[\![\pm \mathrm{O} \pm F]\!]$. The Kanger and Lindahl forms are not refinements of this last one. They have a weaker relationship which we term an *elaboration.*

Definition 5.11 *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be partitions of some sentence* R. $\mathbf{P}$ *is an* elaboration *of* $\mathbf{Q}$ *(*$\mathbf{P} \succeq \mathbf{Q}$*) iff for every* $Q \in \mathbf{Q}$ *there is a* $P \in \mathbf{P}$ *such that* $\vdash P \to Q$.

Example 5.12 *Consider the 'one-agent act positions' used by Lindahl:*

$$[\![\pm \mathrm{E}_a \pm F]\!] \;=\; \{\mathrm{E}_a F,\ \mathrm{E}_a \neg F,\ \mathrm{Pass}_a F\}$$

Since E_a *is a 'success' operator,* $[\![\pm \mathrm{E}_a \pm F]\!]$ *is an elaboration of* $[\![\pm F]\!]$. *But* $[\![\pm \mathrm{E}_a \pm F]\!]$ *is not a* refinement *of* $[\![\pm F]\!]$ *because* $\mathrm{Pass}_a F = \neg \mathrm{E}_a F \wedge \neg \mathrm{E}_a \neg F$ *does not imply any element of* $[\![\pm F]\!]$.

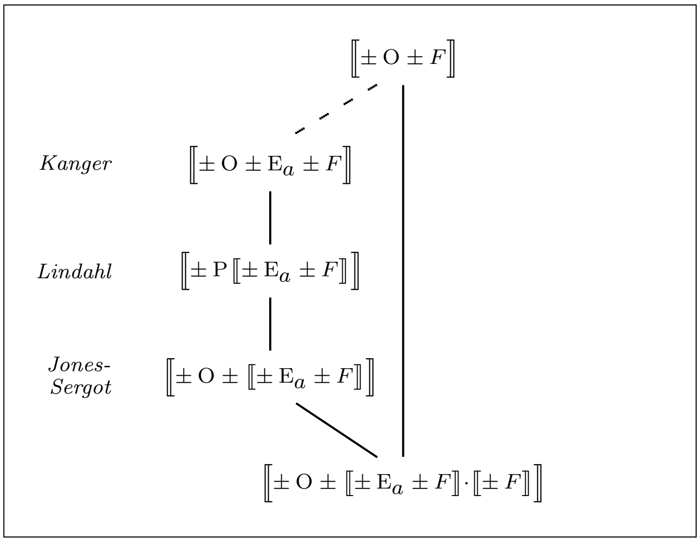

Figure 3. Normative one-agent act positions

It is possible to establish various relationships between refinements, elaborations and equivalences of partitions, but we shall not do so here. The relationships between the various forms of one-agent positions are summarised in Figure 3. The broken line represents an elaboration. The solid lines are refinements. The partitions at the bottom of the diagram are refinements (elaborations) of those higher up.

The relationships between Lindahl's individualistic and collectivistic normative positions are summarised in Figure 4.

Finally, the following properties are useful for performing (hand) computations.

Proposition 5.13 *Let* $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ *be partitions of some sentence* S *such that* $\mathbf{P} \geq \mathbf{R}$. *Then for any* $R \in \mathbf{R}$: $\mathbf{P} \cdot \mathbf{Q}/R \;=\; (\mathbf{P}/R) \cdot (\mathbf{Q}/R)$.

As a special case, for any choice schemes (or sets of sentences) Φ_1 *and* Φ_2, *and any sentence* $A \in (\pm\,\Phi_1 \cup \pm\,\Phi_2)$:

$$([\![\pm\,\Phi_1]\!] \cdot [\![\pm\,\Phi_2]\!])/A \;=\; ([\![\pm\,\Phi_1]\!]/A) \cdot ([\![\pm\,\Phi_2]\!]/A).$$

We will refer to these properties when looking at some small examples later.

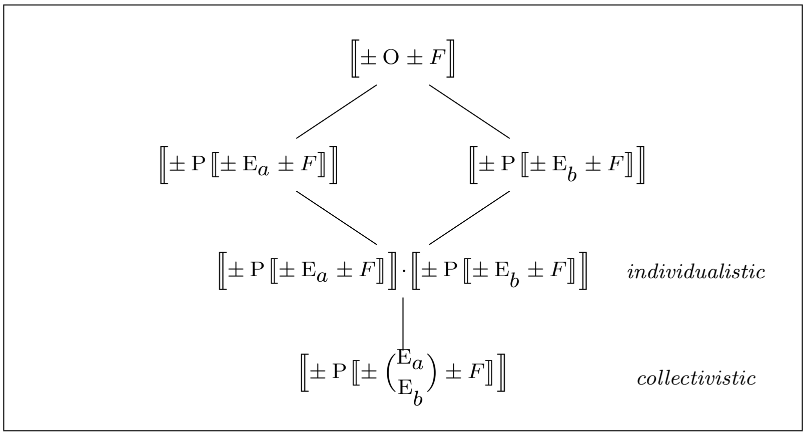

Figure 4. Lindahl's individualistic and collectivistic positions

6 Normative positions

There are two main questions to consider:

(a) Given logic Λ and scheme (set of sentences) Φ, what is the set of maxi-conjunctions $[\![\pm\,\Phi]\!]$?

(b) For given logic Λ, which schemes (sets of sentences) Φ yield the most meaningful, or useful, sets of maxi-conjunctions $[\![\pm\,\Phi]\!]$?

6.1 Maxi-conjunctions for logics of type *EMCP*

We begin by looking at a special case of question (a), focussing on maxi-conjunctions of the form:

(25) $[\![\pm\,\mathrm{O}\pm\mathbf{A}]\!] = [\![\pm\,\mathrm{P}\pm\mathbf{A}]\!]$ ($\mathbf{A}$ a complete partition)

The equality is because O and P are duals.

We assume only that $\mathbf{A}$ is a complete partition. We shall not take into account the structure of sentences in $\mathbf{A}$ and the possibility of rules and axiom schemas in Λ that would allow reductions of certain iterated modalities. In this article we restrict attention to the logics employed by Kanger and Lindahl: type *EMCP* for the logic of O and type *ET* for the action modalities E_x. Elsewhere [Sergot, 1996] we set out the structure of maxi-conjunctions of the form (25) for a range of logics from type *EP* to type *EMCP*, and beyond.

For O of type *EMCP* and $\mathbf{A}$ a complete partition, the maxi-conjunctions in $[\![\pm\,\mathrm{O}\pm\mathbf{A}]\!]$ have a particularly simple form.

Theorem 6.1 *Let $\mathbf{A} = \{A_1, \ldots, A_n\}$ be a complete partition. When the logic of* O *is of type EMCP the set of maxi-conjunctions:*

$$[\![\pm \mathrm{O} \pm \mathbf{A}]\!] = [\![\pm \mathrm{P} \pm \mathbf{A}]\!]$$

is equivalent (Definition 5.4) to the set of conjunctions of the form

$$(26) \quad \pm \mathrm{P}A_1 \wedge \ldots \wedge \mathrm{P}A_j \wedge \ldots \wedge \pm \mathrm{P}A_n$$

that is, conjunctions such that, for each $A_i \in \mathbf{A}$, there is a conjunct of the form $\mathrm{P}A_i$ or $\neg\mathrm{P}A_i$, and at least one conjunct is of the form $\mathrm{P}A_j$.

We write $\pi\mathbf{A}$ to stand for any conjunction of the form (26)*. $\pi^+\mathbf{A}$ is the set of the permissible A_i in $\pi\mathbf{A}$, i.e.*

$$\pi^+\mathbf{A} \;=_{\text{def}}\; \{A_i \in \mathbf{A} \mid \pi\mathbf{A} \vdash \mathrm{P}A_i\}$$

$\pi^-\mathbf{A}$ is the set of the 'prohibited' A_i, i.e.

$$\pi^-\mathbf{A} \;=_{\text{def}}\; \{A_i \in \mathbf{A} \mid \pi\mathbf{A} \vdash \neg\mathrm{P}A_i\} \;=\; \mathbf{A} - \pi^+\mathbf{A}.$$

Proof. See [Sergot, 2001]. In outline: every conjunction $\pi\mathbf{A}$ of the form (26) is consistent, and maximal for expressions falling under the scheme $\pm\,\mathrm{P}\mathbf{A}$. The conjunction $\neg\mathrm{P}A_1 \wedge \ldots \wedge \neg\mathrm{P}A_n$, where there is no conjunct of the form $\mathrm{P}A_j$, is inconsistent. The remaining expressions to consider are those falling under the scheme $\pm\,\mathrm{P}\neg\mathbf{A}$, i.e. those of the form $\pm\,\mathrm{P}\neg A_j$, $A_j \in \mathbf{A}$. It can be readily checked that every such expression is either inconsistent with or implied by every conjunction of form (26). ∎

Corollary 6.2 *When the logic of* O *is of type EMCP, and* $\mathbf{A}$ *is a complete partition:*

$$[\![\pm \mathrm{O} \pm \mathbf{A}]\!] \;\equiv\; [\![\pm \mathrm{P}\mathbf{A}]\!].$$

The corollary generalises the remarks in Section 4 on the equivalence, when O is of type *EMCP*, between Lindahl's form for normative one-agent and two-agent act positions, (12) and (18) respectively, and the forms (15) and (19) employed in [Jones and Sergot, 1993] for the same purpose.

Notice that in order to specify any element $\pi\mathbf{A}$ of $[\![\pm \mathrm{O} \pm \mathbf{A}]\!]$ it is sufficient to specify the permissible elements $\pi^+\mathbf{A}$. For O of type *EMCP* and $\mathbf{A}$ a complete partition, $[\![\pm \mathrm{O} \pm \mathbf{A}]\!]$ can thus be represented by the set of non-empty subsets of $\mathbf{A}$. [Talja, 1980] takes a special case of this observation as the starting point for an algebraic treatment of the [Lindahl, 1977] account of 'change' of normative positions. Notice also that when $\pi^+\mathbf{A}$ is a singleton, and O is of type *EMCP* (or stronger), the conjunction $\pi\mathbf{A}$ can be written equivalently in a simpler form: when $\pi^+\mathbf{A} = \{A_j\}$, $\pi\mathbf{A}$ is logically equivalent to $\mathrm{O}\,A_j$.

For example, Lindahl's normative one-agent act positions (12) are given by the expression $[\![\pm\,\mathrm{P}\,[\![\pm\,\mathrm{E}_a \pm F]\!]\,]\!]$. There are three act positions in the partition $[\![\pm\,\mathrm{E}_a \pm F]\!]$, viz. $\{\,\mathrm{E}_a F,\ \mathrm{E}_a\neg F,\ \mathrm{Pass}_a F\,\}$. There are $2^3 - 1 = 7$ non-empty subsets of $[\![\pm\,\mathrm{E}_a \pm F]\!]$, and hence 7 elements in (12). They were listed earlier using Lindahl's numbering (T_1)–(T_7). The application of Theorem 6.1 is clearer when (T_2) and (T_4) are re-written in the logically equivalent forms (T'_2) and (T'_4).

As one more example, Jones and Parent [2008] study what they call *normative-informational* positions as a contribution to the investigation of such rights as the right to silence, the right to know and the right to conceal information.

Let $\mathrm{I}_j A$ represent that 'agent j is informed/told that A'. Let $\mathrm{O}_k A$ represent that 'it is obligatory for agent k that A'. P_k is the dual. The logic of each O_k is a normal logic of type *KD*, which is type *EMCP* together with a rule of necessitation $A/\mathrm{O}_k A$. As observed earlier, the rule of necessitation plays no role in the generation of normative positions for logics of this type.

The Jones-Parent normative-informational positions are given by the expression:

(27) $[\![\pm\,\mathrm{O}_k \pm [\![\pm\,\mathrm{I}_j \pm A]\!]\,]\!]$

The logic of I_j is taken to be a classical logic of type K. There are four *informational positions* in the set $[\![\pm\,\mathrm{I}_j \pm A]\!]$:

(I_1) $\mathrm{I}_j A \wedge \neg\mathrm{I}_j\neg A$

(I_2) $\mathrm{I}_j\neg A \wedge \neg\mathrm{I}_j A$

(I_3) $\neg\mathrm{I}_j A \wedge \neg\mathrm{I}_j\neg A$

(I_4) $\mathrm{I}_j A \wedge \mathrm{I}_j\neg A$

(I_1) and (I_2) are called the *straight truth/straight lie* positions, depending on whether A is or is not the case. (I_3) represents the *silence* position. (I_4) represents the *conflicting information* position.

There are $2^4 - 1 = 15$ non-empty subsets of $[\![\pm\mathrm{I}_j \pm A]\!]$ and so 15 normative-informational positions of type (27). They are symmetric in A and $\neg A$. Jones and Parent re-write some of them in more readable equivalent form but our purpose here is merely to illustrate the application of Theorem 6.1.

Note that the difference between a 'straight truth' and a 'straight lie' is the difference between $A \wedge \mathrm{I}_j A \wedge \neg\mathrm{I}_j\neg A$ on the one hand and $\neg A \wedge \mathrm{I}_j A \wedge$

$\neg \mathrm{I}_j \neg A$ (or $A \wedge \mathrm{I}_j \neg A \wedge \neg \mathrm{I}_j A$) on the other. Suppose then we consider the following more refined class of normative-informational positions:

(28) $[\![\pm\, \mathrm{O}_k \pm [\![\pm\, \mathrm{I}_j \pm A]\!]{\cdot}[\![\pm A]\!]]\!]$

There are 8 informational positions in $[\![\pm\, \mathrm{I}_j \pm A]\!]{\cdot}[\![\pm A]\!]$ and so $2^8 - 1 = 255$ normative-informational positions of type (28), symmetric in A and $\neg A$.

6.2 Refinement structures

As will be established presently, when O is of type *EMCP*, then $\mathbf{A} \geq \mathbf{B}$ implies $[\![\pm\, \mathrm{O} \pm \mathbf{A}]\!] \geq [\![\pm\, \mathrm{O} \pm \mathbf{B}]\!]$. There is much more that can be said about the structure of such maxi-conjunctions, however.

We now summarise the structure of conjunctions of the form

$$[\![\pm\, \mathrm{O} \pm \mathbf{A}]\!] / \pi\mathbf{B} \qquad (\mathbf{A} \geq \mathbf{B})$$

The question is also of considerable practical significance. (It is the basis of the automated inference methods presented in [Sergot, 2001].)

Suppose $B_j \in \pi^+\mathbf{B}$, i.e. $\pi\mathbf{B}$ is an element of $[\![\pm\, \mathrm{O} \pm \mathbf{B}]\!]$ containing a conjunct $\mathrm{P}B_j$. Since $\mathbf{A} \geq \mathbf{B}$ there is some set of elements $\mathbf{A}/B_j = \{A_1^j, \ldots, A_{m_j}^j\}$ such that $\vdash B_j \leftrightarrow (A_1^j \vee \ldots \vee A_{m_j}^j)$. By O.RE, $\vdash \mathrm{P}B_j \leftrightarrow \mathrm{P}(A_1^j \vee \ldots \vee A_{m_j}^j)$, and when O is of type *EMCP*, then also $\vdash \mathrm{P}B_j \leftrightarrow (\mathrm{P}A_1^j \vee \ldots \vee \mathrm{P}A_{m_j}^j)$. It follows that every element $\pi\mathbf{A}$ of $[\![\pm\, \mathrm{O} \pm \mathbf{A}]\!] / \pi\mathbf{B}$ must have at least one conjunct $\mathrm{P}A_i^j$, i.e. every $\pi^+\mathbf{A}$ contains at least one element of $\mathbf{A}/B_j$.

Conversely, suppose $B_j \in \pi^-\mathbf{B}$. Then, since $\vdash A_i^j \to B_j$ for every A_i^j in $\mathbf{A}/B_j$, it follows when O is of type *EMCP* that $\vdash \neg\mathrm{P}B_j \to \neg\mathrm{P}A_i^j$.

Theorem 6.3 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be complete partitions such that* $\mathbf{A} \geq \mathbf{B}$. *Suppose* $\pi\mathbf{B}$ *is an element of* $[\![\pm\, \mathrm{O} \pm \mathbf{B}]\!]$; $\pi\mathbf{B}$ *is logically equivalent to a conjunction of the form:*

$$\neg\mathrm{P}B_1 \wedge \ldots \wedge \neg\mathrm{P}B_k \wedge \mathrm{P}B_{k+1} \wedge \ldots \wedge \mathrm{P}B_n \qquad (k \geq 1)$$

i.e. $\pi^-\mathbf{B} = \{B_1, \ldots, B_k\}$ *and* $\pi^+\mathbf{B} = \{B_{k+1}, \ldots, B_n\}$. *When* O *is of type EMCP, every element of* $[\![\pm\, \mathrm{O} \pm \mathbf{A}]\!] / \pi\mathbf{B}$ *is logically equivalent to a conjunction of the form:*

$$\neg\mathrm{P}B_1 \wedge \ldots \wedge \neg\mathrm{P}B_k \wedge \pi(\mathbf{A}/B_{k+1}) \wedge \ldots \wedge \pi(\mathbf{A}/B_n).$$

Proof. In the previous discussion. ■

It follows that when O is of type *EMCP*, $\mathbf{A} \geq \mathbf{B}$ implies $[\![\pm \mathrm{O} \pm \mathbf{A}]\!] \geq [\![\pm \mathrm{O} \pm \mathbf{B}]\!]$.

Example 6.4 *Suppose we are given the truth of* $\mathrm{O}F$ *(*F *represents, let us suppose, that there is a fence between two adjoining properties) and we wish to investigate what this implies about obligations of some agent* a*. We wish to determine the normative positions of form* (21) *that are consistent with* $\mathrm{O}F$*, i.e.*

(29) $\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm F]\!]\right]\!\!\right] \Big/ \mathrm{O}F$

Proceed as follows. $\mathrm{O}F$ *can be written equivalently as* $\mathrm{P}F \wedge \neg\mathrm{P}\neg F$*. All conjunctions* (29) *will thus be equivalent to conjunctions* $\neg\mathrm{P}\neg F \wedge C$ *where* C *is a conjunction of the form* $\pi([\![\pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm F]\!]/F)$*. Consider now* $[\![\pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm F]\!]/F$*. By Proposition 5.13 this is* $\{F \wedge \mathrm{E}_a F,\ F \wedge \neg\mathrm{E}_a F\} \equiv \{\mathrm{E}_a F,\ F \wedge \neg\mathrm{E}_a F\}$*. There are three non-empty subsets of this set, and so, by Theorem 6.3, three normative positions in set* (29)*. They are (equivalent to):*

$$\begin{pmatrix} \neg\mathrm{P}\neg F \wedge \mathrm{PE}_a F \wedge \neg\mathrm{P}(F \wedge \neg\mathrm{E}_a F) \\ \neg\mathrm{P}\neg F \wedge \neg\mathrm{PE}_a F \wedge \mathrm{P}(F \wedge \neg\mathrm{E}_a F) \\ \neg\mathrm{P}\neg F \wedge \mathrm{PE}_a F \wedge \mathrm{P}(F \wedge \neg\mathrm{E}_a F) \end{pmatrix} \equiv \begin{pmatrix} \mathrm{OE}_a F \\ \mathrm{O}F \wedge \neg\mathrm{PE}_a F \\ \mathrm{O}F \wedge \mathrm{PE}_a F \wedge \mathrm{P}\neg\mathrm{E}_a F \end{pmatrix}$$

In similar fashion we may calculate which of the 'collectivistic' normative positions of form (23) *for two agents* a *and* b *are consistent with, say* $\mathrm{OE}_a F$*:*

$$\left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm F]\!] \cdot [\![\pm F]\!]\right]\!\!\right] \Big/ \mathrm{OE}_a F \ = \\ \left[\!\!\left[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm \mathrm{E}_b \pm F]\!] \cdot [\![\pm F]\!]\right]\!\!\right] \Big/ \mathrm{OE}_a F$$

These positions will be (equivalent to) conjunctions of the form $\mathrm{OE}_a F \wedge C$*: to determine* C *we need to consider*

$$\begin{aligned} &[\![\pm \mathrm{E}_a \pm F]\!] \cdot [\![\pm \mathrm{E}_b \pm F]\!] \cdot [\![\pm F]\!] \Big/ \mathrm{E}_a F \\ &\qquad = ([\![\pm \mathrm{E}_a \pm F]\!]/\mathrm{E}_a F) \cdot ([\![\pm \mathrm{E}_b \pm F]\!]/\mathrm{E}_a F) \\ &\qquad = \{\mathrm{E}_a F \wedge \mathrm{E}_b F,\ \mathrm{E}_a F \wedge \neg\mathrm{E}_b F\} \end{aligned}$$

There are three non-empty subsets, and so again three normative positions of the form we seek. They are (equivalent to):

$$\begin{pmatrix} \mathrm{OE}_a F \wedge \mathrm{PE}_b F \wedge \neg\mathrm{P}\neg\mathrm{E}_b F \\ \mathrm{OE}_a F \wedge \neg\mathrm{PE}_b F \wedge \mathrm{P}\neg\mathrm{E}_b F \\ \mathrm{OE}_a F \wedge \mathrm{PE}_b F \wedge \mathrm{P}\neg\mathrm{E}_b F \end{pmatrix} \equiv \begin{pmatrix} \mathrm{OE}_a F \wedge \mathrm{OE}_b F \\ \mathrm{OE}_a F \wedge \mathrm{O}\neg\mathrm{E}_b F \\ \mathrm{OE}_a F \wedge \mathrm{PE}_b F \wedge \mathrm{P}\neg\mathrm{E}_b F \end{pmatrix}$$

The procedure illustrated in the previous example is quite mechanical, and is readily automated. It is the basis of the automated inference methods presented in [Sergot, 2001].

The example also illustrates an important advantage of basing the generation of normative positions on cumulative fact/act positions of the form $[\![\pm\, \mathrm{E}_a \pm F]\!] \cdot [\![\pm\, F]\!]$ in preference to the simpler act positions $[\![\pm\, \mathrm{E}_a \pm F]\!]$ employed by Lindahl. Not only is the resulting analysis more precise, but Lindahl's act positions are not a refinement of $[\![\pm\, F]\!]$ and so the computational methods just described cannot be exploited, except in a messy and rather indirect way.

For O of type *EMCP*, $[\![\pm \mathrm{O} \pm \mathbf{A}]\!]$, and hence $[\![\pm \mathrm{P}\mathbf{A}]\!]$, is the most refined set of normative positions that can be constructed from a partition $\mathbf{A}$. It is essentially the basis of a disjunctive normal form for the fragment of the logic consisting of sentences falling under the schemes $\pm\, \mathrm{O} \pm \mathbf{A}$ and $\mathbf{A}$ and their subsentences [Sergot, 2001].

What of the act positions? Which act positions $\mathbf{A}$ yield the most refined set of normative positions $[\![\pm\, \mathrm{O} \pm \mathbf{A}]\!]$? Here the answer is more complicated because it depends on the specific properties of the action modalities employed besides E.RE and E.T. Full discussion of the possibilities is far beyond the scope of this article. For practical purposes it seems reasonable to restrict attention to act expressions containing propositional atoms or their negations within the scope of an action operator. That rules out of consideration act expressions such as $\mathrm{E}_a(p \wedge q)$, $\mathrm{E}_a(p \wedge \neg q)$, $\mathrm{E}_a(p \vee q)$, and so on. In principle there is nothing problematic about allowing these more general forms of act expressions; in practice, it is not clear that the added level of precision is worth the extra trouble.

So, as a practical compromise, for a (finite) set of agents $\mathcal{A}g = \{a, b, \ldots\}$ and a (finite) set of propositional atoms $\mathcal{P}rops = \{p, q, \ldots\}$ it seems reasonable to focus on act positions of the following form:

$$(30) \quad \left\langle \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \\ \vdots \end{pmatrix} \pm \begin{pmatrix} p \\ q \\ \vdots \end{pmatrix} \right\rangle =_{\text{def}} \left[\!\!\left[\pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \\ \vdots \end{pmatrix} \pm \begin{pmatrix} p \\ q \\ \vdots \end{pmatrix} \right]\!\!\right] \cdot \left[\!\!\left[\pm \begin{pmatrix} p \\ q \\ \vdots \end{pmatrix} \right]\!\!\right]$$

This is the form of act expression supported by the automated analysis program described in [Sergot, 2001].

The number of positions in $[\![\pm\, \mathrm{O} \pm \mathbf{A}]\!]$ when O is of type *EMCP* is $2^{|\mathbf{A}|} - 1$. When $\mathbf{A}$ is of the form (30) and there are m agents in $\mathcal{A}g$ and n propositional variables in $\mathcal{P}rops$, the number of act positions is $2^{(m+1)n}$. The number of normative positions is then $2^{2^{(m+1)n}} - 1$. Although it is easy to write a computer program to generate all these expressions, that is a *very* large number of positions to examine even when m and n are small. It can nevertheless be practical to examine positions of this complex form because the analysis can be broken down into simple stages using the refinement results outlined in this section.

7 Example

The previous sections presented an extended and generalised version of the Kanger-Lindahl theory of normative positions. This framework is an important but still incomplete component of a full formal theory of duties, rights and other complex normative relations. We comment on some of the missing ingredients in Section 8 below.

[Sergot, 2001] describes how the procedures described in previous sections can be implemented in a computer program that is intended to facilitate application of the theory to the analysis of practical examples, either for the purpose of interpretation and disambiguation of legal texts, rules, and regulations, or in the design and specification of a new set of norms. A typical example is the case discussed in [Jones and Sergot, 1992; Jones and Sergot, 1993] concerning access 'rights' to sensitive medical information in a hospital database [Ting, 1990]. The problem here is to clarify and expand an incomplete and very imprecise statement of requirements into a precise specification at some desired level of detail.

In order to conduct such an analysis, the general strategy is to pick some scheme $[\![\pm \mathrm{O} \pm \mathbf{A}]\!]$ which represents the problem under consideration at the appropriate level of detail. The objective of the analysis is to identify which position in this *target partition* holds in the (real or hypothetical) circumstances under consideration. In practice, there will often be points of detail on which we will be unable or unwilling to decide. In that case the result of the analysis will be a disjunction of positions.

As suggested in previous sections such an analysis can be conducted by a process of progressive refinement. At each stage the analysis completed so far is used to constrain the choice of possible positions at the next level of detail. Given a target partition $[\![\pm \mathrm{O} \pm \mathbf{A}]\!]$, find a sequence of refinements $\mathbf{A}_0 \leq \mathbf{A}_1 \leq \ldots \leq \mathbf{A}_N \leq \mathbf{A}$ and proceed as follows. First determine which position $\pi_0\mathbf{A}_0$ of $[\![\pm \mathrm{O} \pm \mathbf{A}_0]\!]$ holds in the given circumstance. Then consider the candidate positions at the next level of detail: determine position $\pi_1\mathbf{A}_1$ from the candidate set $[\![\pm \mathrm{O} \pm \mathbf{A}_1]\!] / \pi_0\mathbf{A}_0$. Now consider $[\![\pm \mathrm{O} \pm \mathbf{A}_2]\!] / \pi_1\mathbf{A}_1$, and so on, until left with the task of identifying a position from the target partition, which will be an element of $[\![\pm \mathrm{O} \pm \mathbf{A}]\!] / \pi_N\mathbf{A}_N$. As described in the previous section, the calculation of the candidate positions at each individual step is simple (especially when O is of type *EMCP*) and quite mechanical.

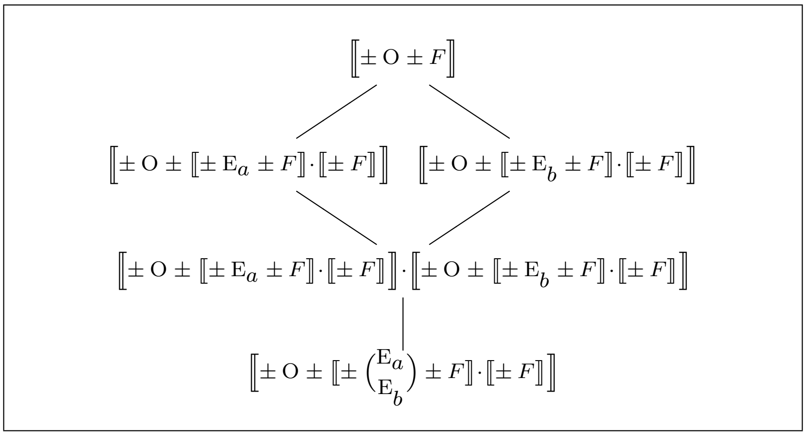

Figure 5. Positions for two agents a and b and one state of affairs F

In practice the procedure is more complicated because usually it will not be a *sequence* of refinements that has to be considered but a more elaborate structure. Figure 5 shows the refinement structure for the case of two agents a and b and one state of affairs F. Figure 6 shows the structure for the case of one agent a and $\mathcal{P}rops = \{F,\ G\}$. In each case, the analysis would begin with the partitions at the top of the figure and work its way down to the more refined partitions shown lower down.

We present here a small example of how this can work. The example is for illustration only; longer accounts with detailed transcripts from the automated system and supplementary comments are provided in [Sergot, 2001] and in [Sergot and Richards, 2000].

The example is a modified version of Ronald Lee's [1988] car park example discussed briefly in Section 3. It concerns the specification of which categories of staff are permitted and not permitted to park in a car park. We will use it to make a number of different points to Lee's. We choose it because it is familiar and requires no further explanation. In Lee's example, administrators are permitted to park in the car park. We will ignore other categories of staff here.

Consider the following scenario:

> a is an administrator, permitted to park in the car park. a has two cars, *car*-a_1 and *car*-a_2. b is a disgraced administrator, banned from the car park. b has one car, *car*-b. c is a passer-by. g is the gatekeeper, charged with controlling access to the car park and ensuring the rules are obeyed.

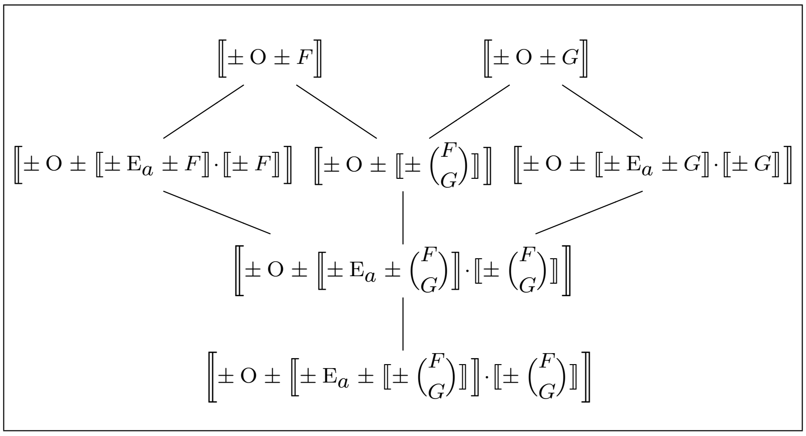

Figure 6. Positions for one agent a and two states of affairs F and G

We will not attempt to cover every feature of the example. In particular the representation of what it means to say that the gatekeeper g is responsible for ensuring that the rules of the car park are obeyed raises a number of difficult points which are outside the scope of this article.

Let $p(a_1)$, $p(a_2)$, $p(b)$ represent that cars *car-*a_1, *car-*a_2, *car-b* are parked in the car park, respectively. We take it that the following at least is implicit and obvious from the scenario description as given above: that it is not permitted that *car-b* is parked in the car park, $\neg \mathrm{P} p(b)$; that it is permitted but not obligatory that *car-*a_1 is parked in the car park, $\mathrm{P} p(a_1) \wedge \mathrm{P} \neg p(a_1)$; and that it is permitted but not obligatory that *car-*a_2 is parked in the car park, $\mathrm{P} p(a_2) \wedge \mathrm{P} \neg p(a_2)$.

What else holds according to the rules of the car park (as we imagine them to be from the scenario and previous experience of typical car parks)? In order to investigate the possibilities in a systematic fashion, and to identify any points requiring further clarification, the task is to pick out one or, in the case of some residual uncertainty, several of the positions from the following *target partition*:

$$(31) \quad \left[\!\!\left[\pm \mathrm{O} \pm \left\langle \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \\ \mathrm{E}_c \\ \mathrm{E}_g \end{pmatrix} \pm \begin{pmatrix} p(a_1) \\ p(a_2) \\ p(b) \end{pmatrix} \right\rangle \right]\!\!\right]$$

We want to restrict attention to those positions in the target partition that are consistent with the initial assertions and thus to compute

$$(32)\quad \left[\!\left[\pm\, \mathrm{O} \pm \left\langle \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \\ \mathrm{E}_c \\ \mathrm{E}_g \end{pmatrix} \pm \begin{pmatrix} p(a_1) \\ p(a_2) \\ p(b) \end{pmatrix} \right\rangle \right]\!\right] \Big/ \neg \mathrm{P} p(b) \;\wedge\; (\mathrm{P} p(a_1) \wedge \mathrm{P}\neg p(a_1)) \;\wedge\; (\mathrm{P} p(a_2) \wedge \mathrm{P}\neg p(a_2))$$

The problem can be simplified by focussing first on, say, the two car owners a and b, and analyzing

$$(33)\quad \left[\!\left[\pm\, \mathrm{O} \pm \left\langle \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm \begin{pmatrix} p(a_1) \\ p(a_2) \\ p(b) \end{pmatrix} \right\rangle \right]\!\right] \Big/ \neg \mathrm{P} p(b) \;\wedge\; (\mathrm{P} p(a_1) \wedge \mathrm{P}\neg p(a_1)) \;\wedge\; (\mathrm{P} p(a_2) \wedge \mathrm{P}\neg p(a_2))$$

This in turn can be simplified to sub-problems

$$(34)\quad \left[\!\left[\pm\, \mathrm{O} \pm \left\langle \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm \begin{pmatrix} p(a_1) \\ p(a_2) \end{pmatrix} \right\rangle \right]\!\right] \Big/ \neg \mathrm{P} p(b) \;\wedge\; (\mathrm{P} p(a_1) \wedge \mathrm{P}\neg p(a_1)) \;\wedge\; (\mathrm{P} p(a_2) \wedge \mathrm{P}\neg p(a_2))$$

and

$$(35)\quad \left[\!\left[\pm\, \mathrm{O} \pm \left\langle \pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm p(b) \right\rangle \right]\!\right] \Big/ \neg \mathrm{P} p(b) \;\wedge\; (\mathrm{P} p(a_1) \wedge \mathrm{P}\neg p(a_1)) \;\wedge\; (\mathrm{P} p(a_2) \wedge \mathrm{P}\neg p(a_2))$$

The automated analysis program described in [Sergot, 2001] provides a graphical interface to help visualize the structure of these sub-problems, and to keep track of the analysis as it proceeds.

Consider (34). Some questions are immediate. Presumably $\mathrm{P}(\neg p(a_1) \wedge \neg p(a_2))$ is true in the car park. But is it the case that $\mathrm{P}(p(a_1) \wedge p(a_2))$? Is it permitted for both of administrator a's cars to be parked at the same time? In a practical setting, this would need to be checked with the car park authorities, or left undetermined if it were not regarded as important. One purpose of the analysis to identify points of detail that may have remained undetected otherwise.

Similarly $\mathrm{PE}_a p(a_1)$ and $\mathrm{PE}_a \neg p(a_1)$ seem straightforward. But what of $\mathrm{P}(p(a_1) \wedge \neg \mathrm{E}_a p(a_1))$ and $\mathrm{P}(\neg p(a_1) \wedge \neg \mathrm{E}_a \neg p(a_1))$, equivalently, $\mathrm{O}(p(a_1) \to$

$\mathrm{E}_a p(a_1)$) and $\mathrm{O}(\neg p(a_1) \to \mathrm{E}_a \neg p(a_1))$? It might be tempting to read the first as saying that if *car*-a_1 is parked then it must have been the administrator a who parked it. But note that expression $\mathrm{E}_a p(a_1)$ does not necessarily signify 'a parks *car*-a_1'; a may bring about $p(a_1)$ in some different way, perhaps even unintentionally. The correct reading of $\mathrm{E}_a p(a_1)$ depends on which version of the logic of action is employed and its semantics. There are many variations. We will make a few further remarks in Section 8 below. And similarly for the question $\mathrm{O}(\neg p(a_1) \to \mathrm{E}_a \neg p(a_1))$.

What of $\mathrm{PE}_b p(a_1)$ and $\mathrm{PE}_b \neg p(a_1)$? Again, we might be tempted to read the first as asking whether the banned administrator b is permitted to park a's car, though again that really depends on how precisely the action modality is to be read. And similarly for the second question. Note that in general $\mathrm{E}_a F$ does not imply $\neg \mathrm{E}_b F$ for other agents $b \neq a$. a and b could act jointly to bring about F, or could even act unintentionally in such a way that each brings about F.

Switching now to the sub-problem (35): $\neg \mathrm{P} p(b)$ implies both $\neg \mathrm{PE}_a p(b)$ and $\neg \mathrm{PE}_b p(b)$ in the logics we are employing. Presumably $\mathrm{PE}_b \neg p(b)$ is true in the car park. But is it the case that $\mathrm{PE}_a \neg p(b)$? Is a permitted to see to it that b's car is not parked? That is far from clear. It will depend on what precisely the act expression $\mathrm{E}_a \neg p(b)$ represents. We will return briefly to some of these points in Section 8.

[Sergot, 2001] and [Sergot and Richards, 2000] present detailed transcripts of a full exploration of partition (33) in the example. Depending on the answers given to earlier questions, about a dozen questions are required to determine a unique position in the partitions (34) and (35); from that about a dozen more pick out a unique position from the partition (33). An exploration of the original target position (31) where there are other agents c and g to consider in addition can be undertaken in similar fashion.

8 Discussion

8.1 Alchourrón-Bulygin's normative systems, and conditional positions

We will comment briefly on the connnection between the mapping out of classes of normative positions and Alchourrón and Bulygin's [1971] formalisation of a normative system. A normative system $\mathcal{N}$ maps a *universe of cases* to *solutions*. The universe of cases is the set of all possible fact combinations that can be constructed from a given set $\mathcal{P}rops$ of propositional variables. In the maxi-conjunction notation, it is $[\![\pm \mathcal{P}rops]\!]$. Where there is one action F for which solutions are specified, a (consistent and complete) normative system $\mathcal{N}$ is a mapping of the form:

(36) $\mathcal{N} : [\![\pm \mathcal{P}rops]\!] \mapsto [\![\pm \mathrm{O} \pm F]\!]$

As observed earlier, when the logic of O is type *EMCP* (or Standard Deontic Logic, type *KD*) $[\![\pm \mathrm{O} \pm F]\!]$ is (with logical redundancies removed) the set of mutually exclusive normative 'fact positions' $\{\mathrm{O}F, \mathrm{O}\neg F, \mathrm{P}F \wedge \mathrm{P}\neg F\}$, or in words, 'obligatory', 'prohibited/forbidden', 'facultative'.

More generally, for a set of propositional variables $\mathcal{P}rops$ and actions $\{F_1, \ldots, F_n\}$, a consistent and complete normative system $\mathcal{N}$ maps the universe of cases to solutions as follows:

(37) $\mathcal{N} : [\![\pm \mathcal{P}rops]\!] \mapsto [\![\pm \mathrm{O} \pm F_1]\!] \cdot \ldots \cdot [\![\pm \mathrm{O} \pm F_n]\!]$

which is

$$\mathcal{N} : [\![\pm \mathcal{P}rops]\!] \mapsto \begin{pmatrix} \mathrm{O}F_1 \\ \mathrm{O}\neg F_1 \\ \mathrm{P}F_1 \wedge \mathrm{P}\neg F_1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} \mathrm{O}F_n \\ \mathrm{O}\neg F_n \\ \mathrm{P}F_n \wedge \mathrm{P}\neg F_n \end{pmatrix}$$

Note that a mapping $\mathcal{N}'$ of this alternative form:

(38) $\mathcal{N}' : [\![\pm \mathcal{P}rops]\!] \mapsto [\![\pm \mathrm{O} \pm \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}]\!]$

defines normative system $\mathcal{N}'$ as a refinement (in the sense used by Alchourrón and Bulygin) of the normative system $\mathcal{N}$: the set of solutions in $\mathcal{N}'$ is a refinement (in the sense of this article) of the set of solutions in $\mathcal{N}$.

Viewed in this way, the solutions in expressions (36)–(38) are classes of normative positions of a rather simple kind, where no agent is specified. More generally then, one could define a normative system as mapping a universe of cases to sets of *normative positions*, of arbitrary degrees of precision, as exemplified by the following possible forms (among many others):

$$\mathcal{N} : [\![\pm \mathcal{P}rops]\!] \mapsto \Big[\!\!\Big[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!]\Big]\!\!\Big]$$

$$\mathcal{N} : [\![\pm \mathcal{P}rops]\!] \mapsto \Big[\!\!\Big[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_a \pm F]\!]\Big]\!\!\Big] \cdot \Big[\!\!\Big[\pm \mathrm{O} \pm [\![\pm \mathrm{E}_b \pm F]\!]\Big]\!\!\Big]$$

$$\mathcal{N} : [\![\pm \mathcal{P}rops]\!] \mapsto \Big[\!\!\Big[\pm \mathrm{O} \pm [\![\pm \begin{pmatrix} \mathrm{E}_a \\ \mathrm{E}_b \end{pmatrix} \pm F]\!]\Big]\!\!\Big]$$

$$\mathcal{N} : [\![\pm \mathcal{P}rops]\!] \mapsto \Big[\!\!\Big[\pm \mathrm{O} \pm [\![\pm \begin{pmatrix} \mathrm{E}_a \\ \vdots \\ \mathrm{E}_b \end{pmatrix} \pm \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}]\!] \cdot [\![\pm \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}]\!]\Big]\!\!\Big]$$

Alchourrón and Bulygin's formalisation can thus be seen as a special case of a much more general account.

Similarly, a rule-based representation language such as that employed in [Lee, 1988] (Example 3.3, Section 3) can be seen as a set of if/then rules whose consequents are agent-free normative positions of a very simple kind. A more general representation language would have if/then rules of the form

if *conditions* then *normative-position*

where *normative-position* is one of some class of normative positions, of arbitrary complexity and precision depending on the needs of the application.

The representation of *conditional* (normative) positions is far from straightforward. It is not just the additional combinatorial complexity that would have to be addressed; there are also strong interactions between conditional structures and deontic logic, and between conditional structures and the treatment of action adopted. For example, unless all actions can be assumed to be instantaneous (an assumption which is made in some of the works cited above) there is a great deal to sort out. If we say that Alice is permitted to park her car if, and only if, it is raining, and if the action of parking takes some significant length of time, do we check that it is raining when she begins to park, or when she completes the job? Do we require it to be raining throughout the entire process? The first of these seems the most natural but that would require quite far-reaching adjustments to the logic of action that has been employed.

8.2 Extended forms of act expression

For certain purposes we might consider extending the initial class of act expressions from which the normative positions are constructed. Some regulations pertain not to *individual* agent positions of the form $\mathrm{E}_x F$, but to what have been termed interpersonal *control* positions, e.g. of type $\mathrm{E}_x\,\mathrm{E}_y F$ or $\mathrm{E}_x \neg\mathrm{E}_y F$. Indeed, the ability to iterate action operators in this way is one of the generally perceived benefits of employing this approach to the treatment of action.

Consider the car park example. The banned administrator b's car may not be parked, $\neg\mathrm{P}\, p(b)$. It follows in the logic that the banned administrator b may not see to it that his car is parked, $\neg\mathrm{P}\,\mathrm{E}_b\, p(b)$. Consider now the responsibilities of the gatekeeper. It seems reasonable to say that the gatekeeper g is permitted to see to it that the banned administrator does not park his car, or more generally that $\mathrm{P}\,\mathrm{E}_g \neg\mathrm{E}_b\, p(b)$ holds according to the rules of the car park. (One might even be tempted to say that there is an *obligation* on the gatekeeper g to see to it that b does not park his car. However, as discussed in the introductory sections, an expression of the form $\mathrm{O}\,\mathrm{E}_g\neg\mathrm{E}_b\,p(b)$ does not represent such an obligation adequately. We will not discuss it further.) One would surely not *insist*, however, that g sees to it that $\mathrm{E}_b\neg p(b)$—surely we would expect that $\mathrm{P}\neg\mathrm{E}_g\,\mathrm{E}_b\neg p(b)$ holds in the car park. Are there any other possibilities?

Ingmar Pörn [1977] has applied similar position-generating techniques to the systematic study of what he called 'control' and 'influence' positions, and in particular to classes of positions of the following forms:

$$(39)\quad \Big[\!\!\Big[\pm\,\mathrm{E}_b \pm\ [\![\pm\,\mathrm{E}_a \pm F]\!]\Big]\!\!\Big]$$

$$(40)\quad \Big[\!\!\Big[\pm\,\mathrm{E}_b \pm\ \mathit{Can} \pm\ [\![\pm\,\mathrm{E}_a \pm F]\!]\Big]\!\!\Big]$$

Here *Can* is a modality for a notion of (practical) possibility.

[Sergot and Richards, 2000] have considered normative positions of the following general form:

$$(41)\quad \Big[\!\!\Big[\pm\,\mathrm{O} \pm [\![\pm \begin{pmatrix}\mathrm{E}_x\\ \vdots\\ \mathrm{E}_y\end{pmatrix} \pm \begin{pmatrix}\mathrm{E}_x\\ \vdots\\ \mathrm{E}_y\end{pmatrix} \pm \begin{pmatrix}F\\ \vdots\\ G\end{pmatrix}]\!]\cdot[\![\pm \begin{pmatrix}F\\ \vdots\\ G\end{pmatrix}]\!]\Big]\!\!\Big]$$

The general principles and methods of construction are exactly as presented in previous sections, though much more complicated in application. For simplicity [Sergot and Richards, 2000] consider in detail only the simpler case of normative positions of the following form:

$$(42)\quad \Big[\!\!\Big[\pm\,\mathrm{O} \pm [\![\pm\,\mathrm{E}_x \pm \mathrm{E}_y F]\!]\cdot[\![\pm\,\mathrm{E}_x \pm \mathrm{E}_y\neg F]\!]\cdot[\![\pm\,\mathrm{E}_y \pm F]\!]\cdot[\![\pm F]\!]\Big]\!\!\Big]$$

The act-expressions are

$$(43)\quad [\![\pm\,\mathrm{E}_x \pm \mathrm{E}_y F]\!]\cdot[\![\pm\,\mathrm{E}_x \pm \mathrm{E}_y\neg F]\!]\cdot[\![\pm\,\mathrm{E}_y \pm F]\!]\cdot[\![\pm F]\!]$$

There are 16 act-expressions in this set, symmetric in F and $\neg F$, and hence $2^{16}-1$ normative positions of type (42).

Note that in some versions of action/agency, notably the 'stit' logics, it is not meaningful to say x 'sees to it' that y 'sees to it' that F for $x \neq y$ (see e.g. the discussion in [Belnap and Perloff, 1988]). In those logics, $\neg\mathrm{E}_x\mathrm{E}_y F$ is a theorem for all $x \neq y$. If a 'stit' version is adopted for E_x, then the list of act positions (43) can be simplified. There are 12 act-expressions in that case, and $2^{12}-1$ corresponding normative positions.

If we look now at the car park and the gatekeeper's control over the banned administrator then we need to consider which of the following act expressions can be permitted given $\mathrm{P}\,(\mathrm{E}_g \neg \mathrm{E}_b\, p(b) \wedge \neg p(b))$ (supposing, as we do, that this is true in the car park):

$$(44)\quad [\![\pm\, \mathrm{E}_g \pm \mathrm{E}_a\, p(b)]\!] \cdot [\![\pm\, \mathrm{E}_g \pm \mathrm{E}_b\, \neg p(b)]\!] \cdot [\![\pm\, \mathrm{E}_b \pm p(b)]\!] \cdot [\![\pm\, p(b)]\!]$$

Applying the methods of the previous sections, we obtain the following set of mutually exclusive act expressions. At least one of them must be permitted, but there may be more than one.

(a) $\mathrm{E}_g\, \mathrm{E}_b \neg p(b) \wedge \mathrm{E}_g \neg \mathrm{E}_b\, p(b)$

(b) $\mathrm{E}_b \neg p(b) \wedge \neg \mathrm{E}_g\, \mathrm{E}_b \neg p(b) \wedge \mathrm{E}_g \neg \mathrm{E}_b\, p(b)$

(c) $\neg p(b) \wedge \mathrm{E}_g \neg \mathrm{E}_b \neg p(b) \wedge \mathrm{E}_g \neg \mathrm{E}_b\, p(b)$

(d) $\neg p(b) \wedge \neg \mathrm{E}_g \neg \mathrm{E}_b \neg p(b) \wedge \mathrm{E}_g \neg \mathrm{E}_b\, p(b)$

In the case of a 'stit' logic for the action modalities, the first of these can be eliminated as it is logically inconsistent, and the second can be simplified by removing the second conjunct.

In each of these expressions b's car is not parked and g's actions are such as to ensure that b does not see to it that b's car is parked. In each case however the interaction between g and b is subtly different. Which of these acts are permitted in the car park (as we imagine it to be)?

It is not easy to give a concise reading to these expressions. A careful reading of each would be quite involved, and more importantly, would again depend critically on what precisely the action modalities are taken to represent. Apart from the huge number of new positions that are created, even with a relatively small number of agents and states of affairs, it is very far from clear whether there is any real value in providing this level of analysis. As the example illustrates, deciphering these complex expressions is far from straightforward. One may be offering a level of precision that is simply unusable in practice.

One of the main difficulties in deciphering the control positions is in interpreting negatives. It is hard to decide what 'x does not see to it that F is not the case' actually means. This is made all the harder because it is unclear what 'not being parked' means exactly: do we mean that the car was never in the car park, or that it was in the car park and was then removed? This can make a big difference. We turn to that next.

It is very easy to imagine a car park in which the gatekeeper g is permitted to prevent a banned car from parking but not permitted to remove a car even if it is illegally parked. With the presently available resources all we can say is that $\mathrm{P}\mathrm{E}_g \neg p(b)$ —the gatekeeper is permitted to see to it that b's car is not parked. Clearly some kind of termporal extension is required.

One possible approach is to follow a suggestion made by von Wright [1968; 1983], Segerberg [1992], and Hilpinen [1997]. We will follow the terminology of Hilpinen's version; the others are essentially the same. There are two components: first, the idea that actions are associated with transitions between states; and second, a distinction between transitions corresponding to the agent's activity and transitions corresponding to the agent's inactivity. The latter are transitions where the agent lets 'nature take its own course'. There are then eight possible modes of agency, and because of the symmetry between F and $\neg F$, four basic forms to consider:

- x brings it about that F ($\neg F$ to F, x active);
- x lets it become the case that F ($\neg F$ to F, x inactive);
- x sustains the case that F (F to F, x active);
- x lets it remain the case that F (F to F, x inactive).

As discussed by Segerberg and Hilpinen there remain a number of fundamental problems to resolve in this account. Moreover, not discussed by those authors, the picture is considerably more complicated when there are the actions of other agents to take into account and not just the effect of nature's taking its course.

To illustrate one possible line of development, Sergot [2008a; 2008b] presents a formalism which combines a logic of action of the 'brings it about' kind with a transition-based treatment of action. Leaving aside the details, an expression $0{:}F$ is true at a transition when F is true at its initial state; $1{:}F$ is true when F is true at the final state of a transition. The distinctions above can then be expressed as follows. The first ('brings it about that') and third ('sustains the case that') are:

(45) $\mathrm{E}_x(0{:}\neg F \wedge 1{:}F)$, equivalently (as it turns out) $0{:}\neg F \wedge \mathrm{E}_x 1{:}F$

(46) $\mathrm{E}_x(0{:}F \wedge 1{:}F)$, equivalently $0{:}F \wedge \mathrm{E}_x 1{:}F$

The second and fourth cases, where x is inactive, can be expressed as follows

(47) $(0{:}\neg F \wedge 1{:}F) \wedge \neg\mathrm{E}_x(0{:}\neg F \wedge 1{:}F)$

(48) $(0{:}F \wedge 1{:}F) \wedge \neg\mathrm{E}_x(0{:}F \wedge 1{:}F)$

These four cases are mutually exclusive.

With these additional resources we are able to distinguish between seeing to it that a car not parked in the car park remains not parked (approximately, preventing a car from entering), and seeing to it that a car which was parked is no longer parked (approximately, removing it). In the (imaginary) car park, the first is permitted for the gatekeeper g, the second is not:

(49) $\mathrm{PE}_g(0{:}\neg p(b) \wedge 1{:}\neg p(b)) \quad \text{and} \quad \neg\mathrm{PE}_g(0{:}p(b) \wedge 1{:}\neg p(b))$

In the logic, these expressions are equivalent to, respectively

(50) $\mathrm{P}(0{:}\neg p(b) \wedge \mathrm{E}_g 1{:}\neg p(b)) \quad \text{and} \quad \neg\mathrm{P}(0{:}p(b) \wedge \mathrm{E}_g 1{:}\neg p(b))$

One could make a case that in the car park we have in mind, $\mathrm{P}(0{:}\neg p(b) \wedge \mathrm{E}_g 1{:}\neg p(b))$ could be strengthened to

(51) $\mathrm{O}(0{:}\neg p(b) \rightarrow \mathrm{E}_g 1{:}\neg p(b))$

These brief examples are offered as suggestions for further lines of development. We will not discuss them further here.

More generally, the various examples in this article are intended in part to illustrate some of the difficulties of employing the 'brings it about' or 'sees to it that' treatment of action in the representation of practical problems. These are very abstract treatments of action. There is often a temptation in particular to read expressions containing E_x with emphasis on the 'end result' feature and insufficient attention to the agency component. Where $p(x)$ stands for 'x's car is parked', for example, it can be tempting to read the expression $\mathrm{E}_x p(x)$ as 'x parks his car', and further, $\mathrm{OE}_x p(x)$ as a representation of an ought-to-do statement that 'x ought to park his car'. But this is not what these expressions say. What they do say depends on the semantics of the action logic adopted. One problem is that in most versions the semantics of the action operators is very abstract indeed, making it very difficult to see how to interpret some expressions in a practical setting.

For example, in the car park it seems intuitively right to say that the banned administrator b is not permitted to park the administrator a's car, or rather, not permitted to see to it that the administrator a's car is parked. But is this correctly represented by $\neg\mathrm{P}\,\mathrm{E}_b\, p(a_1)$? In the logics employed, $\mathrm{P}\,\mathrm{E}_b\, p(a_1)$ is consistent with the following

(52) $\mathrm{P}\,(\mathrm{E}_a\, p(a_1) \wedge \mathrm{E}_b\, p(a_1))$

Are the administrator a and the banned administrator b, perhaps when acting together, permitted to park the administrator a's car? Perhaps they act in such a way that both bring it about that the car is parked (or remains parked). One can imagine circumstances where that would seem to be reasonable, and we could certainly create other similar examples where it would be so. We have $\mathrm{P}\,(\mathrm{E}_a\, p(a_1) \wedge \mathrm{E}_b\, p(a_1))$, and since the logic contains all instances of $\mathrm{P}\,(A \wedge B) \to \mathrm{P}\,B$, we have also:

$$\mathrm{P}\,(\mathrm{E}_a\, p(a_1) \wedge \mathrm{E}_b\, p(a_1)) \to \mathrm{P}\mathrm{E}_b\, p(a_1)$$

It seems that $\mathrm{P}\,\mathrm{E}_b\, p(a_1)$ is likely to be true in the car park after all, if we consider all possible imaginable combinations of actions by a and b.

The erroneous reading of such expressions seems very easy to slip into. For instance, Lindahl uses the example of two adjoining properties, one of which is owned by an agent called John. When discussing the possible normative relations between John and his neighbour in regard to various kinds of acts, including the painting of the neighbour's house white, Lindahl suggests: "... a case in which John is completely unauthorized to influence the situation (since it is no business of his): John may neither bring about nor prevent the main building on his neighbour's property being painted white."[Lindahl, 1977, pp93–94]. In the light of the previous discussion, this is unlikely to be correct. More likely, there are permitted circumstances in which John and his neighbour *between them* act in such a way that they both bring about that the neighbour's property is painted white. It would then follow that John *is* permitted to influence the situation, even though the colour of his neighbour's house is no business of his. The conjunction $\mathrm{E}_a\, p(a_1) \wedge \mathrm{E}_b\, p(a_1)$ may but does not necessarily signify (intentional) joint action by a and b. It could be that both a and b choose independently to see to it that $p(a_1)$. It could be that both bring it about that $p(a_1)$ by chance. It could be that one does it intentionally and the other by chance. Nor does the conjunction represent a composite agent a-and-b-together.

Whether or not these general observations apply for a particular choice of action logic will depend on the details of that choice and on the semantics. The point is that the theory of normative positions makes only minimal assumptions about the properties of the action modalities. For practical applications, it will be necessary to look at some of the detailed choices.

8.3 Limitations

The Kanger-Lindahl theories have several well-documented limitations. Lindahl [1994] himself argues that Kanger's attempted classification of types of *rights* is better seen as a typology of *duties*.

There are two main shortcomings. As a formalisation of the Hohfeldian scheme, the theory of normative positions does not address the feature Hohfeld called '(legal) power'. It has long been understood that 'power' in the sense of (legal) capacity or 'competence' cannot be reduced to permission, and must also be distinguished from the 'can' of practical possibility. An agent can have 'power', to effect a marriage say, without necessarily having the permission nor the practical possibility of exercising that power. The example is from [Makinson, 1986]. Jones and Sergot [1996] argue that 'power' in this Hohfeldian sense is to be understood as a special case of a more general phenomenon, whereby in the context of a given normative system or institution, designated kinds of acts, when performed by designated agents in specific circumstances, *count as* acts that create or modify specific kinds of institutional relations and states of affairs. This switches attention from the formalisation of permission to the formalisation of the *count as* relation more generally.

The second shortcoming of the theory of normative positions, when viewed as a theory of duties and rights or as a formalisation of the Hohfeldian framework, is that it fails to deal with the notion of *counterparty*—the idea that when a party x owes an obligation or duty to party y that such-and-such, or when y has a claim-right against x that such-and-such, then the *counterparty* y has a special relationship in the normative relation between x and y that is not shared by other agents.

There are two main views of how to treat the *counterparty*: as *claimant* or as *beneficiary*. Discussions of the relative merits are sometimes framed as if they were competing accounts for the same notion. It is more helpful to see them not as competitors but as meaningful and distinct notions in their own right. In some cases claimant and beneficiary coincide, in other cases they do not. Both views however present severe challenges to an adequate formal characterisation.

The counterparty as claimant notion is associated with 'power'. Thus a commonly expressed view of what it means to be a counterparty is in terms of a conditional power: 'A relative duty in the law is owed to the party who has the legal power to initiate proceedings to enforce that duty.' [Wellman, 1989]

Makinson [1986] puts it like this:

> The informal account that suggests itself is that x bears an obligation to y that F under the system N of norms iff in the case that F is not true then y has the power under the code N to initiate legal action against x for non-fulfillment of F (or in the case of a moral rather than a legal code, iff in such a case y is 'entitled to complain' of x for non-fulfillment of F). [Makinson, 1986, p423]

There is nevertheless a fundamental difficulty. Generally speaking, a party y has a power to *initiate* legal action against x even when x has no obligation to y, even when the legal action is initiated on what will turn out to be completely unsubstantiated grounds, or perhaps even frivolously. What is missing is the idea that when x does bear an obligation to y, y has the power to initiate legal action with some expectation of *success*. One could not say there is a *guarantee* of success because legal action by its nature is never that certain. But some extra ingredient is essential to eliminate speculative, unsubstantiated or frivolous legal actions. It is very far from clear how one might approach a characterisation of that idea.

Some authors have preferred to take the view that what it means to be a counterparty is to be the *beneficiary* of another's duty or obligation. That notion also remains a serious challenge to formal characterisation. Herrestad and Krogh [1995] for instance, along with others, have proposed adding an index to the obligation operator to designate the beneficiary. Let $\underset{x \to y}{\mathrm{O}} F$ represent that there is directed obligation that F on the bearer x that is for the benefit of the counterparty y. (It is the obligation that is of benefit to y, not necessarily the content F of the obligation itself.) This device allows useful distinctions to be expressed though adding an index in itself obviously does not provide any insight into what the beneficiary is.

One simple suggestion, which nevertheless shows much promise, has been made by Lars Lindahl [1994] as a variation of the Andersonian reduction.

Where x and y are (names of) agents, let propositional constants $W(x, y)$ be read as x 'is wronged by' y. Let $\underset{x \to y}{\mathrm{O}} F$ represent that x is the bearer of a directed obligation (relative duty) to y that F, or on Lindahl's suggested reading, that 'y has a *right-proper* versus x to the effect that F'. Define $\underset{x \to y}{\mathrm{O}}$ in terms of $W(y, x)$ as follows:

(53) $\underset{x \to y}{\mathrm{O}} F \ =_{\mathrm{def}} \ \Box(\neg F \to W(y, x))$

In words, x owes an obligation to y that F (y has a right-proper versus x that F) when, if it is not the case that F, then y is wronged by x. Lindahl takes $\Box$ to be a normal (alethic) modality of type KT; one could consider other options.

Let S be the Andersonian propositional constant representing that a violation or Something Bad has occurred. It is natural to add the axiom schema:

(54) $W(x, y) \to S$ (for all x and y)

The usual Andersonian reduction

(55) $\mathrm{O}F \leftrightarrow \Box(\neg F \to S)$

(56) $\neg\Box S$

then makes the logic of each $\underset{x \to y}{\mathrm{O}}$ Standard Deontic Logic (a normal logic of type *KD*). We also get, for all x, y, w and z:

(57) $\mathrm{O}\neg W(x,y)$

(58) $\underset{x \to y}{\mathrm{O}} F \to \neg \underset{z \to w}{\mathrm{O}} \neg F$

(59) $\underset{x \to y}{\mathrm{O}} F \to \mathrm{O} F$

The idea is simple but it can be refined in several interesting respects. For instance, as Lindahl points out, one can make a case for the following additional schema:

(60) $\Box(\mathrm{E}_x W(x,y) \to W(y,x))$

If x himself sees to it that x is wronged by y, then y is wronged by x.

With the addition of some rather simple general properties of E_x and $\Box$, which we omit here in the interests of space, it is possible to derive the following:

(61) $\underset{x \to y}{\mathrm{O}} \mathrm{E}_x F \to \underset{y \to x}{\mathrm{O}} \neg \mathrm{E}_y \neg F$

If x owes a duty to y to see to it that F then y owes a duty to x not to see to it that $\neg F$. This seems entirely plausible. One can investigate several variations along these lines.

9 Conclusion

We have presented an account of the theory of normative positions, as originally developed by Kanger and Lindahl, and in the generalised and extended form developed in [Sergot, 2001] building on David Makinson's maxi-conjunction characterisation. The methods for mapping out and investigating classes of 'positions' are quite general and are independent of the choice of specific deontic and action logics, though specific results can be obtained for the special case where the underlying logics are those employed by Kanger and Lindahl. The deontic logic component is (a very slightly weakened version of) Standard Deontic Logic. The action logic component makes minimal assumptions: the action logic could be strengthened and refined in many ways.

A secondary aim of this article has been to illustrate the inherent complexity of normative concepts such as duty, right, authorisation, responsibility, commitment, which are encountered not just in legal discourse, but in any description of regulated and organised agent interaction. The theory of normative positions as presented here is an important but limited component of a formal treatment of this complex network of concepts. It is already clear even from this limited theory that there is no point in searching for some, possibly large but nevertheless identifiable, set of basic types—'lowest common denominators' in Hohfeld's words—in terms of which all normative relations between any (two) agents could be articulated. The representation of such relations can be taken to arbitrary levels of detail and complexity. There are nevertheless grounds to believe that a more comprehensive formal account could be developed, together with the automated support tools necessary for its practical use.

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