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Robust portfolio optimization: a conic programming approach

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Abstract The Markowitz Mean Variance model (MMV) and its variants are widely used for portfolio selection. The mean and covariance matrix used in the model originate from probability distributions that need to be determined empirically. It is well known that these parameters are notoriously difficult to estimate. In addition, the model is very sensitive to these parameter estimates. As a result, the performance and composition of MMV portfolios can vary significantly with the specification of the mean and covariance matrix. In order to address this issue we propose a one-period mean-variance model, where the mean and covariance matrix are only assumed to belong to an exogenously specified uncertainty set. The robust mean-variance portfolio selection problem is then written as a conic program that can be solved efficiently with standard solvers. Both second order cone program (SOCP) and semidefinite program (SDP) formulations are discussed. Using numerical experiments with real data we show that the portfolios generated by the proposed robust mean-variance model can be computed efficiently and are not as sensitive to input errors as the classical MMV's portfolios.

Keywords Mean variance optimization · Robust optimization · Conic programming

1 Introduction

The mean-variance model proposed by Markowitz [8], lies at the core of modern portfolio theory. The model is known to be sensitive to its two main input parameters: the mean and covariance of the returns. The sensitivity of the Markowitz Mean Variance model (MMV) has been investigated and found to be significant in Broadie [2], as well as in Kuhn et al. [6]. In order to address the sensitivity of the model to its

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input parameters it is natural to formulate the original model as a robust optimization problem. The portfolios obtained using robust optimization remain reliable, possibly at the cost of performance, even when the estimates of the input parameters are incorrect. The robust optimization approach has been investigated by Rustem et al. [10] for discrete sets of rival risk and return scenarios, and by Rustem and Howe [11] using a range, or box, uncertainty set for continuous return scenarios. Tütüncü and Koenig in [14] assume componentwise uncertainty sets over the mean return vector and the covariance matrix. They formulate robust portfolio selection as a saddle-point problem that involves semidefinite constraints. Goldfarb and Iyengar [5] develop a robust factor model for the asset returns and cast robust portfolio selection as a SOCP problem that can be solved efficiently. Additionally, El Ghaoui et al. [4] propose a robust approach to the portfolio selection problem using worst-case Value-at-Risk (VaR).

One shortcoming of the work cited above is that the authors consider separate uncertainty sets for the expected return vector $\hat{x}$ and for the covariance matrix Γ. As a result, the probability measure with the minimized mean is different from the probability measure with the maximized second moment. In other words, they assume that the uncertainty sets of expected returns and the covariance matrix are independent. In general, this is not the case. In order to address this problem we propose a modeling framework to capture the uncertainty over model parameters in a consistent manner. Having two uncertainty sets may lead to a worse than worst-case situation and potentially over-conservative results. In practice, the return distribution is not Gaussian and this is a further motivation for adopting a robust approach. Rustem and Howe [11] as well as Ceria and Stubbs [3] propose a robust portfolio model with an uncertainty region over the expected returns. They assume that the covariance matrix of the returns are known exactly, or that the uncertainty over the covariance matrix is specified in terms of a finite number of discrete scenarios (Rustem and Howe [11]).

We propose a robust portfolio optimization and selection model using a conic programming approach that can be applied to the mean-variance framework in general. We introduce uncertainty sets for the mean and the second moment of returns. The uncertainty sets in the literature cited above are either polytopes or ellipsoids. In this paper, we consider an ellipsoidal uncertainty set on the second moment matrix of returns and componentwise bounds on the mean vector of returns. What differentiates this work from others is that we do not have an uncertainty set over the covariance matrix directly, but two uncertainty sets over the mean vector and the second moment matrix of returns simultaneously under a single probability measure. A salient feature of this distinction is that the mean and the covariance of the returns are defined in a consistent manner. Furthermore, we do not assume any specific distributions over returns, but we do assume that the uncertainty sets of the first two moments of the returns are known.

2 Robust portfolio optimization

In this section we progressively construct the proposed robust portfolio selection model. After we introduce some notation in Sect. 2.1, we will then consider the robust portfolio optimization model with an uncertainty region over the expected returns only (Sect. 2.2). This is essentially the same model as in Ceria and Stubbs [3], but serves as a convenient starting point for the model proposed in this paper. In Sect. 2.3.1, we introduce a robust portfolio selection model that considers an uncertainty region over the second moment matrix of returns, assuming the expected returns are known. In Sect. 2.3.2, we extend this model by introducing separate uncertainty sets over the expected returns and the second moment matrix of returns. Numerical results are presented in Sect. 3.

2.1 Notation and conventions

We use $x \in \mathbb{R}^n$ to represent one period returns. For a symmetric matrix A, we use the notation $A \succeq 0$ in order to denote that A is positive semidefinite. We use,

$$S = \{x \in \mathbb{R}^n | \theta(x) = x^T P_i x + 2q_i x + r_i \geq 0,\ i = 1, \ldots, m\},$$

to denote the support of the probability distribution of returns. For some probability measure μ,

$$\int_S x\,d\mu = \hat{x} = \begin{bmatrix} \hat{x}_1 & \ldots & \hat{x}_n \end{bmatrix}^T \qquad \int_S xx^T d\mu = \Sigma,$$

denote the *first moment* (mean) and *second moment* of returns, respectively. The *covariance matrix* is given by

$$\Gamma = \Sigma - \hat{x}\hat{x}^T, \quad \Gamma \succeq 0.$$

The *full second moment matrix* is given by

$$\begin{aligned} \hat{\Sigma} &= \int_S \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}^T d\mu \\ &= \begin{bmatrix} \Sigma & \hat{x} \\ \hat{x}^T & 1 \end{bmatrix}. \end{aligned} \tag{1}$$

We assume that $\hat{\Sigma} \succ 0$. It follows from the Schur complement that $\Gamma = \Sigma - \hat{x}\hat{x}^T \succ 0$. It can be seen that the constraint that defines the set S can also be described by,

$$\begin{aligned} M_0(\theta) &= \int_S \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} d\mu \\ &= \left\langle \hat{\Sigma}, \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \right\rangle \\ &= \langle \Sigma, P_i \rangle + 2q_i^T \hat{x} + r_i \geq 0, \quad i = 1, \ldots, m. \end{aligned}$$

Here $\langle A, B\rangle = \mathbf{Tr}(AB)$ denotes the standard scalar product for the symmetric matrices A, B. $M_0(\theta)$, a notation borrowed from Lasserre [7], represents the set S in terms of the full second moment matrix. Note that $M_0(\theta)$ is a special case of the so-called localizing matrix described in Lasserre [7]. Following Putinar [9] the positive semi-definiteness of the moment and localizing matrices are necessary and sufficient conditions for the elements of $\hat{\Sigma}$ to be the moments of some measure μ supported on S. This result will be directly applied in this paper and we refer interested readers to Lasserre [7] and Putinar [9]. In addition, we introduce the *quadratic matrix variable* denoted by,

$$\bar{X} = \begin{bmatrix} \hat{x}\hat{x}^T & \hat{x} \\ \hat{x}^T & 1 \end{bmatrix}. \tag{2}$$

The term "quadratic matrix variable" reflects the fact that the matrix $\bar{X}$ has entries that are quadratic in $\hat{x}$. We use the term *portfolio quadratic matrix variable* to refer to the following,

$$\hat{W} = \begin{bmatrix} ww^T & w \\ w^T & 1 \end{bmatrix}. \tag{3}$$

Here $w \in W$ are the weights of a portfolio.

Remark 1 In SDP models the matrix variable is linear in its entries. It can be seen from the positive semidefinite constraint below,

$$\begin{bmatrix} X & x \\ x & 1 \end{bmatrix} \succeq 0, \tag{4}$$

that $X - xx^T \succeq 0$. Therefore, $\bar{X}$ and $\hat{W}$ are in general not valid SDP underlying variables. However, according to an interesting result given in Boyd and Vandenberghe [1], the quadratic programming problem defined below,

$$\begin{aligned} \min_{x}\ & \langle A_0, X\rangle + 2b_0^T x + c_0 \\ \text{s.t.}\quad & \langle A_1, X\rangle + 2b_1^T x + c_1 \le 0 \\ & X = xx^T, \end{aligned}$$

can be formulated as the following SDP problem:

$$\begin{aligned} \min_{X,x}\ & \langle A_0, X\rangle + 2b_0^T x + c_0 \\ \text{s.t.}\quad & \langle A_1, X\rangle + 2b_1^T x + c_1 \le 0 \\ & \begin{bmatrix} X & x \\ x & 1 \end{bmatrix} \succeq 0. \end{aligned}$$

Here both A_i, $i = 0, 1$ are symmetric matrices, $b_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$. The result follows from the fact that both problems have the same dual. This result is valid when matrices A_i are positive semidefinite or negative semidefinite. The proof employs the so-called S-procedure and is discussed in Boyd and Vandenberghe [1]. In this paper we will only encounter the convex case, which means $A_i \succeq 0$. We assume that Slater's constraint qualification is satisfied therefore strong duality holds.

2.2 Robustness to first moment estimation errors

We consider the similarities between the MMV model and the approach of Ceria and Stubbs [3] that aims to establish the robustness of the portfolio to uncertainties in the expected returns. The MMV model can be formulated as the following quadratic programming problem (QP).

$$\max_{w \in \mathbb{R}^n} \hat{x}^T w - \lambda w^T \Gamma w$$
$$\text{s.t.} \quad w \in \bar{W},$$

where $\hat{x}$ is the vector of estimated expected returns, the matrix Γ is an estimate of the covariance of the returns, $\bar{W}$ represents a convex set of feasible portfolios (e.g. $\bar{W} = \{w \in \mathbb{R}^n \,|\, \sum_{i=1}^n w_i = 1,\, w \geq 0\}$) and $\lambda \geq 0$ is the relative importance associated with risk, as represented by the portfolio variance. We note that $\lambda \geq 0$ can also be interpreted as the Lagrange multiplier associated with the portfolio risk constraint in $\max_{w \in \mathbb{R}^n} \{\hat{x}^T w \mid w^T \Gamma w \leq \upsilon;\ w \in \bar{W}\}$ for some given value of υ. Ceria and Stubbs [3] consider the worst case scenario on the expected returns by assuming that the vector of true expected returns $\hat{x}_t$ is normally distributed and lies in the confidence region represented by the ellipsoid,

$$(\hat{x}_e - \hat{x}_t)^T \Gamma_e^{-1} (\hat{x}_e - \hat{x}_t) \leq k^2, \tag{5}$$

generated by estimated expected returns $\hat{x}_e$ and a covariance matrix Γ_e of the estimates of expected returns with probability η. Where $k^2 = \chi_n^2(1-\eta)$ and χ_n^2 is the inverse cumulative distribution function of the chi-squared distribution with n degrees of freedom. For a fixed portfolio $(\hat{w})$ the worst case returns are given by,

$$\begin{aligned} &\max_{\hat{x}_e - \hat{x}_t} (\hat{x}_e - \hat{x}_t)^T \hat{w} \\ &\text{s.t.} \quad (\hat{x}_e - \hat{x}_t)^T \Gamma_e^{-1} (\hat{x}_e - \hat{x}_t) \leq k^2. \end{aligned} \tag{6}$$

By constructing the Lagrangian of (6), it is straightforward to obtain $\hat{x}_t^T \hat{w} = \hat{x}_e^T \hat{w} - k\|\Gamma_e^{1/2} \hat{w}\|$. The problem now becomes the robust portfolio selection problem given below.

$$\begin{aligned} &\max_{w \in \mathbb{R}^n} \hat{x}_e^T w - k\|\Gamma_e^{1/2} w\| \\ &\text{s.t.} \quad w^T \Gamma w \leq \gamma^2 \\ &\qquad w \in \bar{W}. \end{aligned} \tag{7}$$

Thus, (7) is the traditional maximum return formulation, augmented with the term $k\|\hat{\Gamma}_e^{1/2} \hat{w}\|$ that reduces the effect of estimation error on the optimal portfolio. This problem can be straightforwardly written as a second order cone programming

(SOCP) problem:

$$\begin{aligned} \max_{w\in\mathbb{R}^n}\ & \hat{x}_e^T w - kt \\ \text{s.t.}\quad & w \in \bar{W}, \\ & \gamma \geq \|\Gamma^{1/2} w\|, \\ & t \geq \|\Gamma_e^{1/2} w\|, \end{aligned} \tag{8}$$

where γ is a standard deviation target. An efficient frontier, similar to that associated with the MMV model, can be constructed by varying γ. We note that Γ is the covariance matrix of returns and Γ_e is the covariance matrix of the estimated expected returns. The latter is related to error arising from the process of estimating $\hat{x}_t$. This model, discussed by Ceria and Stubbs [3], only considers an uncertainty region around the mean vector of returns, and assumes perfect knowledge of the covariance matrix. The main contribution of this paper is concerned with the relaxation of this assumption.

2.3 Robustness to first and second moment estimation errors

The minimum variance formulation of the portfolio selection problem is given below.

$$\begin{aligned} \min_{w\in\mathbb{R}^n}\ & w^T \Gamma w \\ \text{s.t.}\quad & w \in \bar{W}, \\ & w^T \hat{x} \geq R, \end{aligned}$$

where R is the lower limit on the target expected return. Goldfarb and Iyengar [5], Tütüncü and Koenig [14] consider the best portfolio in view of the worst-case return and covariance matrix. We extend these models to provide a covariance definition that is consistent with the worst-case mean computed. In addition to consistency, invoking explicitly the functional dependence of the covariance matrix on the first and second moments provides the correct representation, with the corresponding structural constraint, that is expected to lead to a less conservative robust policy. Consider the following "min-max" robust portfolio problem,

$$\begin{aligned} \min_{w\in\mathbb{R}^n} \max_{\Sigma\in\Sigma_U, \hat{x}\in\hat{x}_U}\ & w^T \Gamma w = w^T(\Sigma - \hat{x}\hat{x}^T) w \\ \text{s.t.}\quad & w \in \bar{W}, \\ & \min_{\hat{x}\in\hat{x}_U} w^T \hat{x} \geq R. \end{aligned} \tag{9}$$

Here the covariance matrix is written as $\Gamma = \Sigma - \hat{x}\hat{x}^T$, Σ is the second moment matrix of returns. Σ_U and $\hat{x}_U$ are the uncertain regions of Σ and $\hat{x}$, respectively. Note that existing solution algorithms can be applied to the model in (9). However, such a direct approach will be based on the assumption that the distribution of the first moments is independent from the distribution of the second moments. This assumption may not be satisfied in practice. In the next section we introduce a framework that does not require this assumption and treats the relationship between first and second moments in a consistent manner.

2.3.1 Robustness to second moment estimation errors

In this subsection we will assume that the expected returns ($\hat{x}$) are known and fixed. Using this assumption we formulate the portfolio optimization problem with worst-case variance as an SDP problem. It follows that when the expected returns ($\hat{x}$) are fixed, the worst-case variance dependents only on the second moment Σ. For a given portfolio w the worst-case can be obtained by solving the problem below.

$$\begin{aligned} \sup_{\mu} & \left\langle ww^T, \int_S xx^T d\mu \right\rangle - \left\langle w, \int_S x d\mu \right\rangle^2 \\ \text{s.t.} \quad & \int_S x d\mu = \hat{x}, \\ & \int_S d\mu = 1, \\ & \mu(x) \geq 0. \end{aligned} \tag{10}$$

The model above is an infinite dimensional optimization problem, and cannot be solved directly. However, it can also be viewed as a variant of the moment problem. Below we use the results from [7] to reformulate the model as a convex SDP problem.

Theorem 2.1 *Suppose that S is a compact, convex semialgebraic set with a nonempty interior, then problem* (10) *is equivalent to the following problem*:

$$\max_{\hat{\Sigma}} \langle \hat{W}, \hat{\Sigma} \rangle - \langle \hat{W}, \bar{X} \rangle$$

$$\text{s.t.} \quad \hat{\Sigma} \succeq 0, \tag{11a}$$

$$\langle A_i, \hat{\Sigma} \rangle = \hat{x}_i, \quad i = 1, \dots, n+1, \tag{11b}$$

$$\langle B_j, \hat{\Sigma} \rangle \geq 0, \quad j = 1, \dots, m. \tag{11c}$$

Here $\hat{\Sigma}$, $\bar{X}$ *and* $\hat{W}$ *are defined by* (1), (2) *and* (3) *respectively.*

The proof follows from Theorem 3.7 in [7]. The constraints (11a) and (11c) are enforcing positive semi-definiteness of the moment and localizing matrices. The matrices A and B in (11) are selected to satisfy the constraints of the problem. The matrices A_i are used to define the first moment constraints (i.e. the first constraint in(10)). For example,

$$A_1 = \begin{bmatrix} 0 & \dots & 0.5 \\ \vdots & \ddots & \vdots \\ 0.5 & \dots & 0 \end{bmatrix}$$

ensures that $\hat{x}$ in the moment matrix $\hat{\Sigma}$ is equal to the specified $\hat{x}_1$. The matrix B_1 is used to define the support S of the return distribution. For example, if S is given by,

$$\{x \in \mathbb{R}^n \mid x^T x \leq 1\}.$$

Then B_1 needs to be specified as,

$$\begin{bmatrix} -\mathbf{I} & 0 \\ 0 & 1 \end{bmatrix},$$

where $\mathbf{I}$ is the n-dimensional identity matrix. The constraints (11c) for $j = 2, \dots, m$ are optional and are used to define constraints on the second moments of the return distribution. For concreteness we give an example on how these matrices are specified for the case when we only have two assets. For this two dimensional example the full second moment matrix is defined by,

$$\hat{\Sigma} = \begin{bmatrix} \hat{x}_1^2 & \hat{x}_1\hat{x}_2 & \hat{x}_1 \\ \hat{x}_2\hat{x}_1 & \hat{x}_2^2 & \hat{x}_2 \\ \hat{x}_1 & \hat{x}_2 & 1 \end{bmatrix}.$$

If we are given an upper bound, say σ_1 on the second moment of asset x_1 we define

$$B_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix},$$

so that $\langle B_2, \hat{\Sigma} \rangle \geq 0$ enforces the constraint $\hat{x}_1^2 \leq \sigma_1$. The same procedure is used to define constraints on the mixed second moments of the returns. For example in order to implement the constraint $\hat{x}_1\hat{x}_2 \leq \sigma_{1,2}$, we define

$$B_3 = \begin{bmatrix} 0 & -0.5 & 0 \\ -0.5 & 0 & 0 \\ 0 & 0 & \sigma_{1,2} \end{bmatrix},$$

so that $\langle B_3, \hat{\Sigma} \rangle \geq 0$ ensures the constraint is satisfied. With this formulation we can define the entries of matrix B_i to satisfy any linear constraints imposed on the moments of the returns.

It follows from Theorem 2.1 that the robust portfolio selection problem in (9) can be re-formulated as follows.

$$\begin{aligned}
\min_{\hat{W}} \max_{\hat{\Sigma}} \ & \langle \hat{W}, \hat{\Sigma} \rangle - \langle \hat{W}, \bar{X} \rangle \\
\text{s.t.} \quad & \hat{\Sigma} \succeq 0, \\
& \langle A_i, \hat{\Sigma} \rangle = \hat{x}_i, \quad i = 1, \ldots, n+1, \\
& \langle B_j, \hat{\Sigma} \rangle \geq 0, \quad j = 1, \ldots, m, \\
& \langle C_p, \hat{W} \rangle \in \bar{W}, \quad p = 1, 2 \ldots, \\
& \hat{W} \succeq 0, \\
& \langle D, \hat{W} \rangle \geq R.
\end{aligned} \tag{12}$$

Where C_p are selected so that the constraints on the weights of the portfolio are satisfied. For example, by choosing

$$C_1 = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} -\mathbf{I} & 0 \\ 0 & 1 \end{bmatrix},$$

we can compute portfolios that lie in the set,

$$\bar{W} = \left\{ w \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} w_i = 1, \; w \geq 0, \; w^T w \leq 1 \right\}.$$

Note that for every positive semidefinite matrices $\hat{\Sigma} - \bar{X}$ the problem (12) is a convex quadratic programming problem with respect to the portfolio variable w. As noted in Remark 1, the linear matrix variable

$$\begin{bmatrix} W & w \\ w & 1 \end{bmatrix},$$

is equivalent to the quadratic matrix variable $\hat{W}$. Therefore (12) is a standard SDP problem. The last constraint in (12) is the performance constraint and it is given by,

$$D = \begin{bmatrix} 0 & \frac{1}{2}\hat{x} \\ \frac{1}{2}\hat{x} & 0 \end{bmatrix},$$

and R is the required return of the portfolio. The result below will help us formulate the robust optimization problem as a conic programming problem that can be solved by standard solvers.

Lemma 2.1 *Suppose that strong duality holds for the max part of* (12), *the "min-max" problem* (12) *can then be expressed by the following "min" SDP problem*:

$$\begin{aligned}
\min_{y,\hat{W}} \ & -\hat{x}^T y - \langle \hat{W}, \bar{X} \rangle \\
\text{s.t.} \quad & -A^T y - B^T \tau - \hat{W} = \Lambda, \\
& \Lambda \succeq 0, \\
& \tau \geq 0, \quad \tau \in \mathbb{R}^m, \\
& \langle C, \hat{W} \rangle \in \bar{W}, \\
& \langle D, \hat{W} \rangle \geq R.
\end{aligned} \tag{13}$$

Proof The proof of this lemma is straightforward and so we omit the details. The basic idea is to write the Lagrangian of the inner maximization problem in (12). We then consider the dual of the inner problem, thus transforming the inner maximization problem to a minimization problem. It follows from strong duality that this transformation is valid (see e.g. [1]). □

The implication of this lemma is that we can solve the robust portfolio selection problem (9) by a single "min" SDP problem instead of a "min-max" problem (12). This means that no special solver needs to be designed for the robust portfolio optimization problem. Without the last performance constraint, (13) yields the optimal portfolio assuming the worst-case variance of returns. If one wants to find out the worst-case mean and variance corresponding the optimal portfolio of (11) (which is also the "max" part of (12)), then (12) needs to be solved for the given portfolio w. Based on the results of this section we next discuss the case where uncertain regions are introduced over *both the expected returns and the second moment matrix of returns*.

2.3.2 The general case

In this subsection we will relax the assumption that the returns are fixed. We will remove the first equality constraint in (10), and replace it with the weaker assumption that the returns belong to an exogenously defined set $[\hat{x}_l, \hat{x}_u] \in \mathbb{R}^n$. Here $\hat{x}_u, \hat{x}_l$ are given component-wise upper and lower bounds of the mean vector, respectively. We will follow the approach developed in Sect. 2.3.1 and we will provide two equivalent robust formulations. The first is based on solving the robust portfolio optimization problem with semi-infinite programming. The second approach is based on the reformulation method introduced in the preceding section and relies on strong duality to solve the robust problem using standard cone programming solvers.

We start our analysis of the "max" function by assuming that vector of portfolio weights (w) is given. With the weights fixed, the value of the "max" function is given by the objective function value of the following problem:

$$
\begin{aligned}
\max_{\hat{\Sigma},\bar{X}} \ & \langle \hat{W}, \hat{\Sigma}\rangle - \langle \hat{W}, \bar{X}\rangle \\
\text{s.t.} \quad & \langle A_i, \hat{\Sigma}\rangle \in [\hat{x}_{il}, \hat{x}_{iu}], \quad i = 1, \dots, n, \\
& \langle A_i, \bar{X}\rangle \in [\hat{x}_{il}, \hat{x}_{iu}], \quad i = 1, \dots, n, \\
& \langle A_i, (\hat{\Sigma} - \bar{X})\rangle = 0, \quad i = 1, \dots, n, \\
& \langle B_j, \hat{\Sigma}\rangle \geq 0, \quad j = 1, \dots, m, \\
& \langle C, \hat{\Sigma}\rangle = 1, \\
& \langle C, \bar{X}\rangle = 1, \\
& \hat{\Sigma} \succeq 0, \\
& \bar{X} \succeq 0, \\
& \hat{\Sigma} - \bar{X} \succeq 0.
\end{aligned}
\tag{14}
$$

Analogous to Theorem 2.1, $\hat{\Sigma} \succeq 0$ and $\langle B_j, \hat{\Sigma}\rangle \geq 0$, $j = 1, \dots, m$ are used to guarantee $\hat{\Sigma}$ to be the full second moment matrix of some probability measure supported on S. The coefficients of the C matrix ensure that the measure under consideration is a probability measure. Hence, $C_{nn} = 1$ and zero everywhere else. The matrices A_i, $i = 1, \dots, n$, are chosen so that $\langle A_i, \hat{\Sigma}\rangle = \hat{x}_i$. The constraints $\langle A_i, (\hat{\Sigma} - \bar{X})\rangle = 0$, $i = 1, \dots, n$, ensure that the mean vectors in the full second moment matrix and the quadratic matrix variable are the same. In this worst-case variance formulation we have a convex QP problem with respect to $\hat{x}$ constrained by $\hat{\Sigma} - \bar{X} \succeq 0$. As was noted in Remark 1 the linear matrix variable,

$$
\begin{bmatrix} X & \hat{x} \\ \hat{x} & 1 \end{bmatrix},
$$

is equivalent to the quadratic variable matrix $\bar{X}$. Therefore (14) can be solved as standard SDP problem.

Using the formulation of the "max" function in (14), the worst-case variance of returns given the uncertainty set for returns, is given by the "min-max" problem below.

$$
\min_{w} \max_{\hat{\Sigma},\bar{X}} q(w, \bar{X}, \hat{\Sigma}) = \langle \hat{W}, \hat{\Sigma} - \bar{X}\rangle, \tag{15}
$$

with the constraints in (14) and the portfolio constraints of (12). In the sequel, we use two different approaches to solve the "min-max" problem. Firstly, following the same procedure introduced in the previous section, we convert the "min-max" problem to a convex "min" SDP problem. The second approach relies on semi-infinite programming (Zakovic and Rustem [15]). We use the second approach as an alternative to verify the validity of the first approach.

By using a similar arguments as in Lemma 2.1, we can reformulate the original model (15) to the model given below.

$$
\begin{aligned}
\min_{\alpha_1,\alpha_2,\alpha_3,\tau,y_u,y_l,z_u,z_l,\Lambda,\hat{W}} \quad & \alpha_1 + \alpha_2 - \hat{x}_l^T(y_l + z_l) + \hat{x}_u^T(y_u + z_u) \\
\text{s.t.} \quad & \alpha_1 C + \alpha_3 A - \tau B - y_l A + y_u A - \Lambda - \hat{W} = \Lambda_1, \\
& \alpha_2 C - \alpha_3 A - z_l A + z_u A + \Lambda + \hat{W} = \Lambda_2, \\
& \langle D, \hat{W} \rangle \in \bar{W}, \\
& \tau \geq 0, \\
& y_u, y_l, z_u, z_l \geq 0, \\
& \Lambda, \Lambda_1, \Lambda_2 \succeq 0, \\
& \langle E, \hat{W} \rangle \geq R.
\end{aligned} \tag{16}
$$

The matrices A, B and C are the same as in (12). The matrix E enforces the performance constraint. We can therefore obtain the optimal portfolio in view of the worst-case mean and variance of returns by solving the single convex "min" SDP problem (16).

It is well known that the "min-max" problem (15) is equivalent to the following semi-infinite programming problem:

$$
\min_{w} \max_{\hat{\Sigma},\bar{X}} q(w, \bar{X}, \hat{\Sigma}) = \begin{bmatrix} w^T & 0 \end{bmatrix} (\hat{\Sigma} - \bar{X}) \begin{bmatrix} w \\ 0 \end{bmatrix}.
$$

The model above is a convex quadratic programming problem with respect to w and a linear SDP problem with respect to $(\hat{\Sigma}, \bar{X})$. By employing the semi-infinite programming algorithm introduced by Zakovic and Rustem [15], we can compute the optimal portfolio and its corresponding worst-case mean and variance through the iterative procedure described below.

{Semi-infinite Programming Algorithm}
Set $A = \{(\hat{\Sigma}_0, \bar{X}_0)\}$
while $U > L$ **do**
  Compute lower bound $L = \min_{w \in W} \max_{\hat{\Sigma},\bar{X}} q(w, \bar{X}, \hat{\Sigma})$ globally
  with $w^* = \arg\min_{w \in W} \max_{(\hat{\Sigma},\bar{X}) \in A} q(w, \bar{X}, \hat{\Sigma})$
  Compute upper bound $U = \max_{(\hat{\Sigma},\bar{X}) \in B} q(w^*, \bar{X}, \hat{\Sigma})$
  with $(\hat{\Sigma}^*, \bar{X}^*) = \arg\max_{(\hat{\Sigma},\bar{X}) \in B} q(w^*, \bar{X}, \hat{\Sigma})$
  $A = A \cup \{(\hat{\Sigma}^*, \bar{X}^*)\}$
end while
Optimal values $(w^*, \hat{\Sigma}^*, \bar{X}^*)$ are attained when $U == L$

Note that the first "min-max" problem in the while loop of the algorithm above can be cast as the following SOCP problem.

$$\begin{aligned} &\min_{\hat{w}} \max_{\hat{\Gamma}\in\bigcup_{i=0,\dots,p}(\hat{\Sigma}_i,\bar{X}_i)} \hat{w}^T\hat{\Gamma}_i\hat{w} = \|\hat{\Gamma}_i^{1/2}\hat{w}\|^2 \\ &\text{s.t.} \quad w \in \bar{W}. \end{aligned}$$

The problem above is equivalent to,

$$\begin{aligned} &\min_{\hat{w}} t \\ &\text{s.t.} \quad \|\hat{\Gamma}_i^{1/2}\hat{w}\| \le t, \quad i = 0,\dots,p, \\ &\qquad w \in \bar{W}, \end{aligned}$$

where $w \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\hat{w} \equiv [w \quad 0]^T$. In each iteration, the lower bound $L = t^2$ is computed and compared with the upper bound U generated by the second maximization problem. The second maximization problem in the algorithm is a linear SDP problem that can be solved efficiently. After each iteration, if $U > L$ then the set A is enlarged by the new $(\hat{\Sigma}^*, \bar{X}^*)$ generated by the second maximization problem. The additional constraint will cause the objective function value of the SOCP problem to increase in the next iteration. The iterations terminate when $U = L$. Our numerical experiments show that almost same results (within $1E-4$) can be attained by solving the robust portfolio selection problem via these two approaches. Comparing these two approaches, the single SDP approach is easier to implement and faster than the semi-infinite approach. The reason is that the SDP approach only computes a single linear SDP problem whereas the semi-infinite programming approach needs many iterations (each iteration contains an SCOP problem and a linear SDP problem) to reach the optimal solution. Note that this optimal solution provides not only the optimal portfolio but also the corresponding worst-case moments. The SDP approach however, requires another step to obtain the corresponding worst-case moments. The additional step consists of solving (14) given the optimal portfolio determined above. The semi-infinite formulation is an alternative way to solve the robust portfolio selection problem and also provides a way to empirically test the validity of the SDP model. We can use the following algorithm to generate a discrete approximation to the robust efficient frontier:

Robust Efficient Frontier Algorithm:

1. Solve problem (13) or (16) without the performance constraint in order to compute optimal portfolio w_{min}.
2. Solve problem (11) or (14) to attain the worst-case mean and variance $\hat{x}_{wc}$, Γ_{wc}.
3. Set $R_{min} = \hat{x}_{wc}^T w_{min}$, $R_{max} = \hat{x}_{wc}^T w_{max}$ and $\Delta = R_{max} - R_{min}$.
4. Choose N, the number of desired points on the efficient frontier. For $R \in \{R_{min} + \frac{\Delta}{N-1}, R_{min} + 2\frac{\Delta}{N-1}, \dots, R_{min} + (N-1)\frac{\Delta}{N-1}\}$ solve problem (16) with constraints $\hat{x}_{wc}^T w = R$.

In this algorithm, w_{min} corresponds to the risk averse portfolio with respect to the worst-case risk measure

$$\min_{w} \max_{\hat{\Sigma}, \bar{X}} \quad q(w, \bar{X}, \hat{\Sigma}),$$

without a performance requirement as given in (16). This essentially corresponds to the worst-case portfolio performance. On the other hand w_{max}, represents the best portfolio return with respect to the worst-case $\hat{x}$ with no consideration of risk. This is just the problem $\max_w \min_{\hat{x}} \hat{x}^T w$ with constraints only on $\hat{x}$ and w.

3 Numerical results

In this section we illustrate how the proposed framework can be used to solve practical robust portfolio optimization models. For the SOCP and SDP problems we used SeDuMi version 1.1 developed by Sturm [12] and SDPT3 by Toh et al. [13]. The computational experiments were performed on a Pentium IV 3.2 GHZ PC with 1 G RAM.

The first numerical experiment illustrates the effect of estimation errors in expected returns and the improvement in robustness attained by the proposed approach. We take Broadie's [2] true and estimated data as inputs. The estimated, true and actual efficient frontiers are then generated. The estimated frontier is calculated by using the estimated mean vector, the estimated covariance matrix of returns and the portfolios computed by the mean-variance model. The true efficient frontier is calculated by employing the true mean vector, the true covariance matrix of returns and the portfolios computed by the mean-variance model. The actual frontier is calculated by utilizing the true mean vector, the true covariance matrix of returns and the portfolios computed by the mean-variance model but using the estimated inputs. We then solve the model in (8) with the estimated expected returns and the true expected returns to calculate the estimated robust efficient frontier and the actual robust efficient frontier, respectively. Suppose that the true expected returns falls in the confidence region with probability of 95%. This assumption implies $k = 1.0703$ in (7). The resulting efficient frontiers are shown in Fig. 1. It can be seen that the robust portfolios perform better than MMV's portfolios in terms of sensitivity to the input data. The numerical results also show that the robust portfolios are more diversified than MMV's portfolios.

We next report on numerical experiments that illustrate the effect of uncertainty on the covariance matrix of returns. The robust efficient frontiers are also presented. When the expected returns are specified, we optimize the portfolios assuming the worst-case variance, as in model (13). For simplicity, suppose that $S := \{x \in \mathbb{R}^5 | x^T x \leq 0.1\}$ and the expected returns and the covariance matrix of returns are the same as in Broadie [2]. An efficient frontier for the MMV model and the worst-case variances with the given portfolios using (11) are computed using the robust efficient frontier algorithm and the formulation in (13). The optimal portfolios are obtained assuming the worst-case variances. The result is shown in Fig. 2.

When the expected returns are only given by componentwise upper and lower bounds, we implement the two models proposed previously to solve the robust portfolio selection problem. Using the semi-infinite approach we can obtain the worst-case expected returns, the covariance matrix of returns and the corresponding optimal

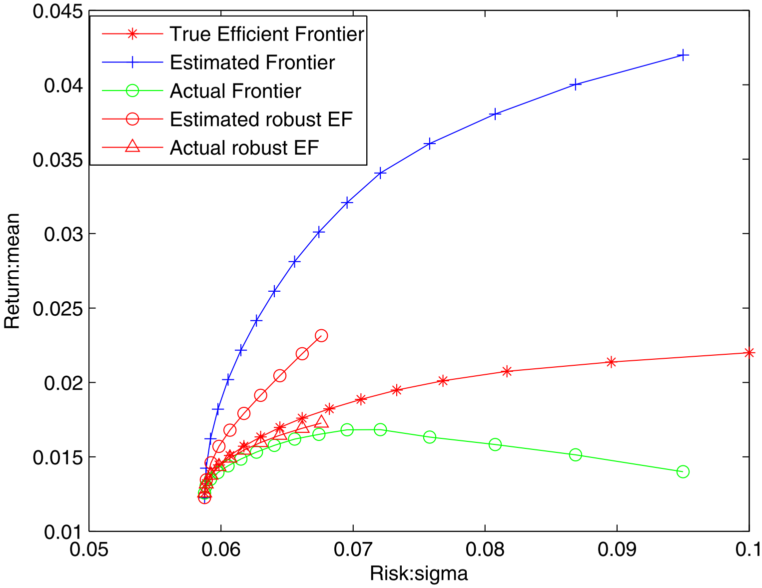

Fig. 1 Efficient frontiers

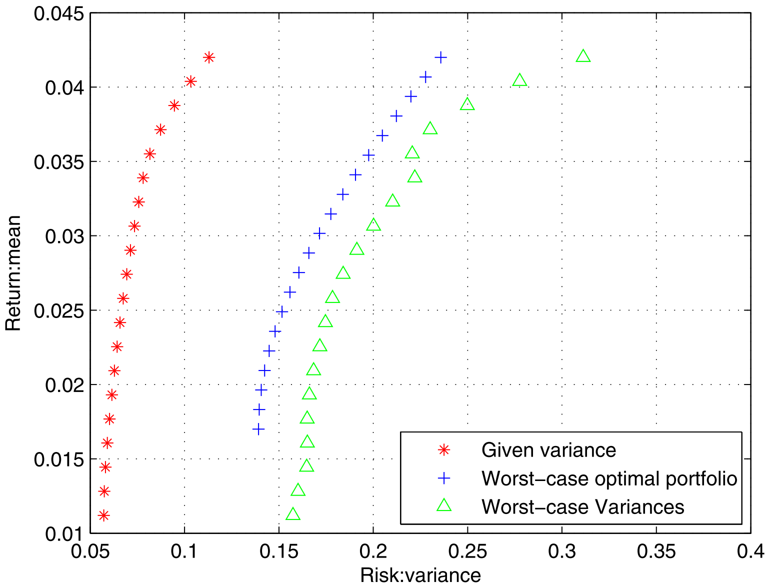

Fig. 2 Efficient frontier, worst-case variance, and worst-case optimal portfolio

portfolio simultaneously. However, it is slower than using the linear SDP model ((14) and (16)). By choosing upper bounds and lower bounds of the expected returns, the semi-infinite algorithm needs about 90 iterations to converge to the optimal value. On

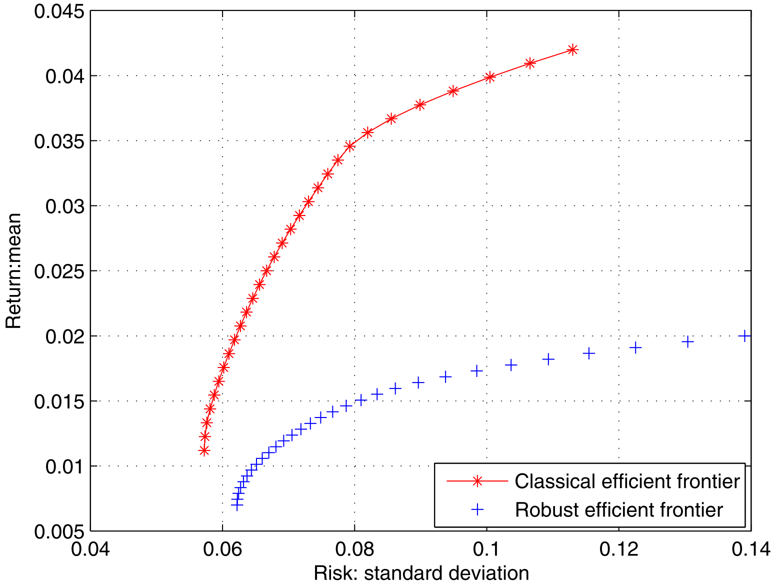

Fig. 3 Robust efficient frontier

the other hand, we can solve a single convex SDP problem (16) to gain the optimal portfolio assuming the worst-case variance. To this end, the classical efficient frontier and the robust efficient frontier generated by the aforementioned algorithm are shown in Fig. 3. Note that Fig. 3 only demonstrates how different the robust efficient frontier can be compared with the classical efficient frontier. The performance of the robust portfolio is largely dependent on the settings of the uncertainty regions.

3.1 Backtesting

We arbitrarily chose 5 stocks (BT, BP, Barclays, Bay system, HSBO) from the FTSE100 as our portfolio. We employed the robust portfolio optimization and selection model proposed in this paper to compare the returns with the index FTSE100 from 3rd January 2005 to 22th January 2008. We took 10 weeks as a historical window from which we approximately determine the full second moment matrices. We used the highest and the lowest values in the full second moment matrices as the upper and lower bound values. According to (16), a robust portfolio can be obtained by assuming that in the future week the mean and the second moment matrix of returns would not transgress from the bounds provided. In addition, we take three points from the frontiers to run the backtesting: (1) the minimum risk level (i.e., (16) without performance requirements), (2) half of the maximum risk level (model (16) with performance requirement $R = 50\% R_{max}$), and (3) maximum risk level (model (16) with performance requirement $R = R_{max}$). The historical window keeps moving which means that we are always looking back for the latest 10 weeks in order to construct a portfolio for the next week. In this backtesting experiment, we also include the Markowitz mean-variance portfolio, the worst case portfolio proposed by Tütüncü

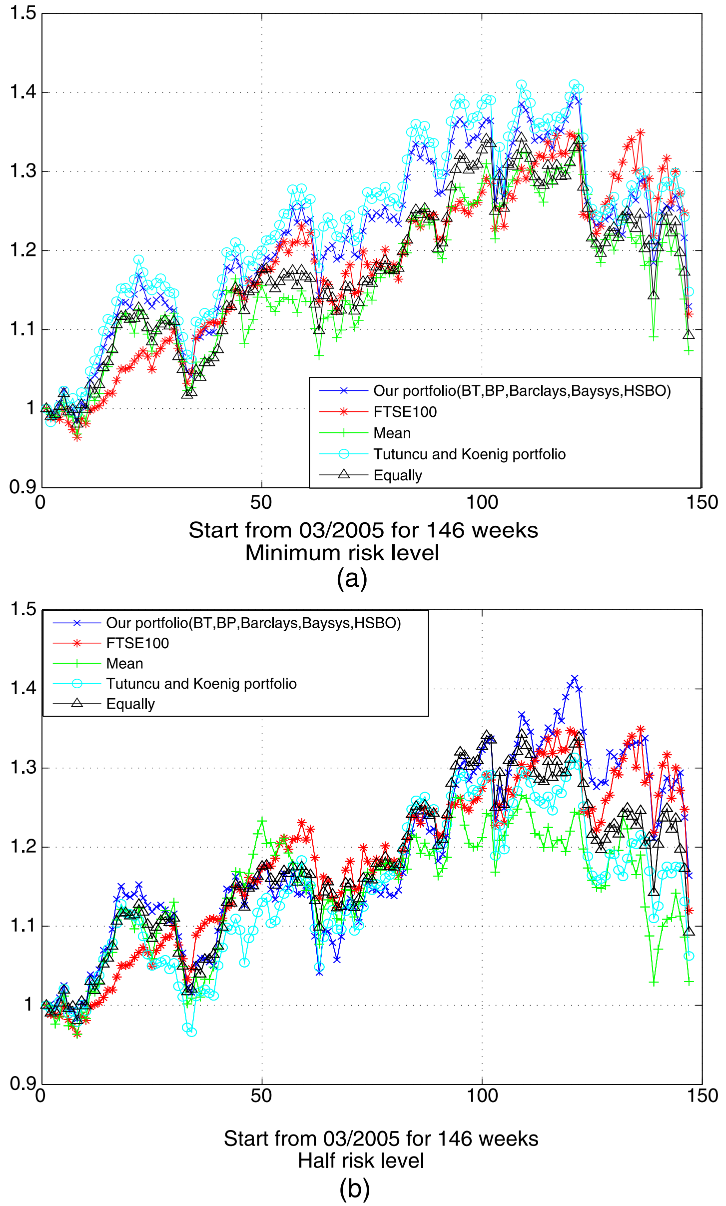

Fig. 4 Backtesting results

and Koenig [14] and the equally weighted portfolio. For the mean-variance portfolio, we calculate the mean and variance of our historic window as inputs of the Markowitz model. In the case of Tütüncü and Koenig's portfolio, the mean vector always takes the lower bound values. Equally weighted portfolio can be obtained when S is described by a sphere. The results are shown in Fig. 4. It can be seen that the portfolio

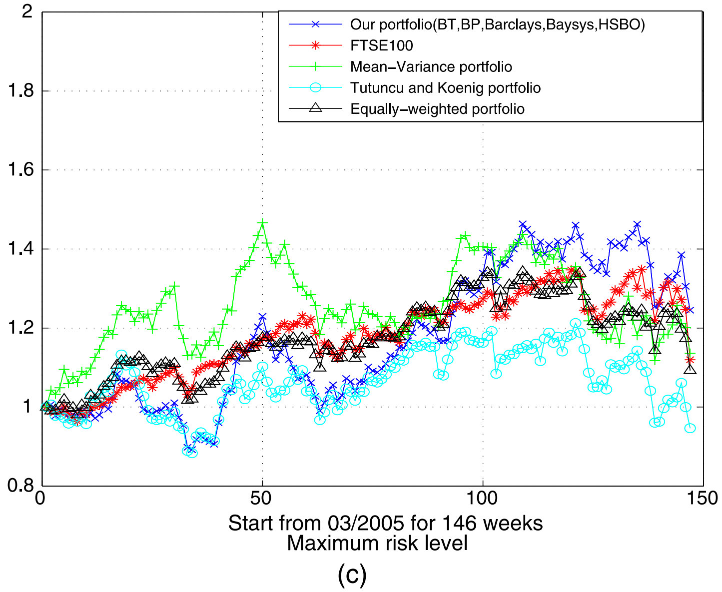

Fig. 4 (*Continued*)

calculated by our model performs well, in terms of expected returns, compared with the existing methods.

4 Conclusion

We have discussed the formulation of the one time period robust portfolio selection model as a conic programming problem. Uncertain regions were introduced to both the expected returns and the second moment matrix of returns. The resulting robust portfolio selection problems were formulated and solved as conic programs using public domain solvers. It is shown that the robust portfolios perform more reliably than MMV portfolios as robust portfolios are less sensitive to input errors.

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