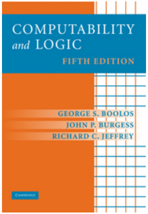
Turing Machines and Separation Logic
(Work in Progress)


Jian Xu Xingyuan Zhang
PLA University of Science and Technology

Christian Urban
King's College London

Why Turing Machines?

- we wanted to formalise computability theory
- at the beginning, it was just a student project



Computability and Logic (5th. ed)
Boolos, Burgess and Jeffrey

- found an inconsistency in the definition of halting computations (Chap. 3 vs Chap. 8)

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- we wanted to formalise computability theory
  - TMs are a fantastic model of low-level code
  - completely unstructured

  - good testbed for verification techniques
  - Can we verify a program with 38 Mio instructions?
  - we can delay implementing a concrete machine model (for OS/low-level code verification)
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  - completely unstructured Spaghetti Code
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Some Previous Works

(but not interested in low-level code)

- Norrish formalised computability theory in HOL starting from the lambda-calculus
  - for technical reasons we could not follow his work
  - some proofs use TMs (Wang tilings)
- Asperti and Ricciotti formalised TMs in Matita
  - no undecidability result $\Rightarrow$ interest in complexity
  - their UTM operates on a different alphabet than the TMs it simulates

"In particular, the fact that the universal machine operates with a different alphabet with respect to the machines it simulates is annoying." (Asperti and Ricciotti)

The Big Picture

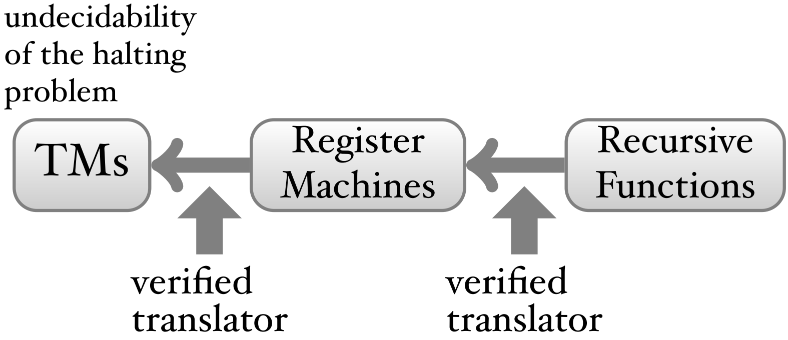
undecidability
of the halting
problem

TMs

Register Machines

Recursive Functions

The Big Picture



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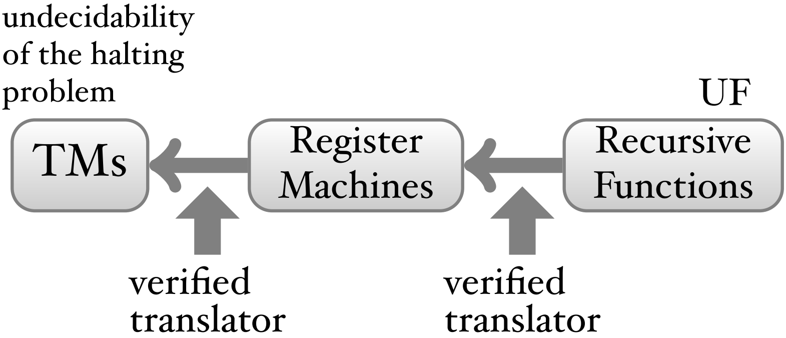
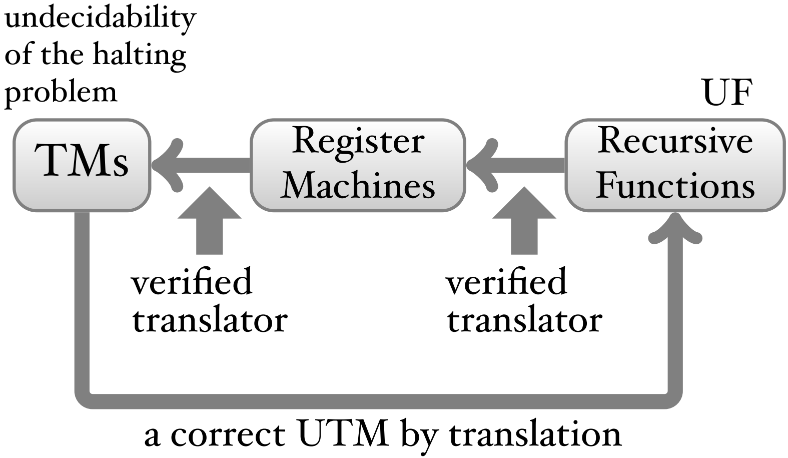
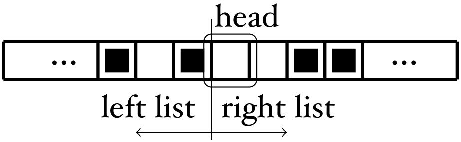


The Big Picture



Turing Machines

- tapes are lists and contain *0*s or *1*s only



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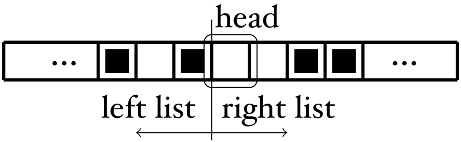

- *steps* function:

  What does the TM calculate after it has executed n steps?

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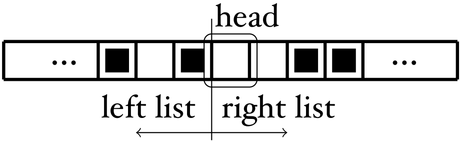

- *steps* function:

  What does the TM calculate after it has executed n steps?

- designate the *0*-state as "halting state" and remain there forever, i.e. have a *Nop*-action

Register Machines

- programs are lists of instructions

I	$::=$	*Goto L*	jump to instruction L
	$\mid$	*Inc R*	increment register R by one
	$\mid$	*Dec R L*	if the content of R is non-zero, then decrement it by one otherwise jump to instruction L

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Spaghetti Code!

Recursive Functions

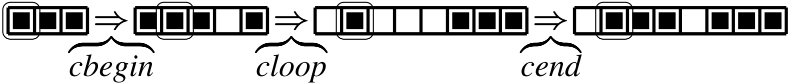
$$
\begin{array}{llll}
rec & ::= & Z & \text{zero-function} \\
 & \mid & S & \text{successor-function} \\
 & \mid & Id^n_m & \text{projection} \\
 & \mid & Cn^n\, f\, gs & \text{composition} \\
 & \mid & Pr^n\, f\, g & \text{primitive recursion} \\
 & \mid & Mn^n\, f & \text{minimisation}
\end{array}
$$

- $eval :: rec \Rightarrow nat\ list \Rightarrow nat$
  can be defined by recursion (HOL has *Least*)
- you define
  - addition, multiplication, logical operations, quantifiers...
  - coding of numbers (Cantor encoding), UF

Copy Turing Machine

- TM that copies a number on the input tape

$$copy \stackrel{def}{=} cbegin\ ;\ cloop\ ;\ cend$$



$cbegin \stackrel{def}{=}$
$[(W_0, 0), (R, 2), (R, 3),$
$(R, 2), (W_1, 3), (L, 4),$
$(L, 4), (L, 0)]$

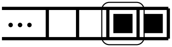
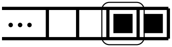
$cloop \stackrel{def}{=}$
$[(R, 0), (R, 2), (R, 3),$
$(W_0, 2), (R, 3), (R, 4),$
$(W_1, 5), (R, 4), (L, 6),$
$(L, 5), (L, 6), (L, 1)]$

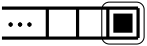
$cend \stackrel{def}{=}$
$[(L, 0), (R, 2), (W_1, 3),$
$(L, 4), (R, 2), (R, 2),$
$(L, 5), (W_0, 4), (R, 0),$
$(L, 5)]$

Dither Machine

- TM that is the identity with *1* and loops with *0*

start tape

halting case:  $\Rightarrow$ 

non-halting case:  $\Rightarrow$ loops

$$dither \stackrel{def}{=} [(W_0, 1), (R, 2), (L, 1), (L, 0)]$$

Hoare Logic for TMs

- Hoare-triples

$\{P\}\ p\ \{Q\} \stackrel{def}{=}$

$\forall\, tp.$
if $P\ tp$ holds then
$\exists\, n.$ such that
$is\_final\ (steps\ (1,\ tp)\ p\ n)\ \wedge$
$Q\ holds\_for\ (steps\ (1,\ tp)\ p\ n)$

Hoare Logic for TMs

- Hoare-triples and Hoare-pairs:

$\{P\}\ p\ \{Q\} \stackrel{def}{=}$
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$Q\ holds\_for\ (steps\ (1,\ tp)\ p\ n)$

$\{P\}\ p \uparrow \stackrel{def}{=}$
$\forall\, tp.$
if $P\ tp$ holds then
$\forall\, n.\ \neg\ is\_final\ (steps\ (1,\ tp)\ p\ n)$

Some Derived Rules

$$\frac{P' \mapsto P \qquad \{P\}\, p\, \{Q\} \qquad Q \mapsto Q'}{\{P'\}\, p\, \{Q'\}}$$

$$\frac{\{P\}\, p_1\, \{Q\} \qquad \{Q\}\, p_2\, \{R\}}{\{P\}\, p_1 \,;\, p_2\, \{R\}} \qquad \frac{\{P\}\, p_1\, \{Q\} \qquad \{Q\}\, p_2 \uparrow}{\{P\}\, p_1 \,;\, p_2 \uparrow}$$

Undecidability

$$contra \stackrel{def}{=} copy \; ; \; H \; ; \; dither$$

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$$contra \stackrel{def}{=} copy\ ;\ H\ ;\ dither$$

- Suppose H decides whether *contra* called with the code of *contra* halts, then

$$P_1 \stackrel{def}{=} \lambda tp.\ tp = ([],\ \langle code\ contra \rangle)$$
$$P_2 \stackrel{def}{=} \lambda tp.\ tp = ([0],\ \langle (code\ contra,\ code\ contra) \rangle)$$
$$P_3 \stackrel{def}{=} \lambda tp.\ \exists\, k.\ tp = (0^k,\ \langle 0 \rangle)$$

$$\dfrac{\dfrac{\{P_1\}\ copy\ \{P_2\} \qquad \{P_2\}\ H\ \{P_3\}}{\{P_1\}\ copy\ ;\ H\ \{P_3\}} \qquad \{P_3\}\ dither \uparrow}{\{P_1\}\ contra \uparrow}$$

Undecidability

$$contra \overset{def}{=} copy \,;\, H \,;\, dither$$

- Suppose H decides wether *contra* called with the code of *contra* does not halt, then

$$Q_1 \overset{def}{=} \lambda tp.\ tp = ([\,],\ \langle code\ contra\rangle)$$
$$Q_2 \overset{def}{=} \lambda tp.\ tp = ([0],\ \langle (code\ contra,\ code\ contra)\rangle)$$
$$Q_3 \overset{def}{=} \lambda tp.\ \exists\, k.\ tp = (0^k,\ \langle 1\rangle)$$

$$\cfrac{\cfrac{\{Q_1\}\ copy\ \{Q_2\} \qquad \{Q_2\}\ H\ \{Q_3\}}{\{Q_1\}\ copy\ ;\ H\ \{Q_3\}} \qquad \{Q_3\}\ dither\ \{Q_3\}}{\{Q_1\}\ contra\ \{Q_3\}}$$

Hoare Reasoning

- reasoning is quite demanding, e.g. the invariants of the copy-machine:

$$I_1\ n\ (l, r) \stackrel{def}{=} (l, r) = ([], 1^n) \qquad \text{(starting state)}$$

$$I_2\ n\ (l, r) \stackrel{def}{=} \exists\, i\, j.\ 0 < i \land i + j = n \land (l, r) = (1^i, 1^j)$$

$$I_3\ n\ (l, r) \stackrel{def}{=} 0 < n \land (l, tl\ r) = (0{::}1^n, [])$$

$$I_4\ n\ (l, r) \stackrel{def}{=} 0 < n \land (l, r) = (1^n, [0, 1]) \lor (l, r) = (1^{n-1}, [1, 0, 1])$$

$$\begin{aligned} I_0\ n\ (l, r) \stackrel{def}{=}\ & 1 < n \land (l, r) = (1^{n-2}, [1, 1, 0, 1]) \lor \qquad \text{(halting state)} \\ & n = 1 \land (l, r) = ([], [0, 1, 0, 1]) \end{aligned}$$

$$\begin{aligned} J_1\ n\ (l, r) \stackrel{def}{=}\ & \exists\, i\, j.\ i + j + 1 = n \land (l, r) = (1^i, 1{::}1{::}0^j @ 1^j) \land 0 < j \lor \\ & 0 < n \land (l, r) = ([], 0{::}1{::}0^n @ 1^n) \qquad \text{(starting state)} \end{aligned}$$

$$J_0\ n\ (l, r) \stackrel{def}{=} 0 < n \land (l, r) = ([0], 1{::}0^n @ 1^n) \qquad \text{(halting state)}$$

$$K_1\ n\ (l, r) \stackrel{def}{=} 0 < n \land (l, r) = ([0], 1{::}0^n @ 1^n) \qquad \text{(starting state)}$$

$$K_0\ n\ (l, r) \stackrel{def}{=} 0 < n \land (l, r) = ([0], 1^n @ 0{::}1^n) \qquad \text{(halting state)}$$

Midway Conclusion

- feels awfully like reasoning about machine code
- compositional constructions / reasoning is not at all frictionless
- sizes

	sizes:
UF	*140843* constructors
URM	*2* Mio instructions
UTM	*38* Mio states

*old version: URM (*12* Mio) UTM (*112* Mio)

The Trouble With Hoare-Triples

- Whenever we wanted to prove

$$\{P\}\ p\ \{Q\}$$

(1) we had to find invariants for each state (not easy)

(2) we had to find a termination order proving that p terminates (not easy either)

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very little opportunity for automation

Inspiration from other Works

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Stealing

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- Jensen, Benton, Kennedy (**2013**), *High-Level Separation Logic for Low-Level Code*
- Myreen (**2008**), *Formal Verification of Machine-Code Programs*, PhD thesis
- Klein, Kolanski, Boyton (**2012**), *Mechanised Separation Algebra*

Better Composability

- an idea from Jensen, Benton & Kennedy who looked at X86 assembler programs and macros

- assembler for TMs:

$$
\begin{array}{l}
\mathit{move\_one\_left} \stackrel{\mathit{def}}{=} \\
\qquad \Lambda\ \mathit{exit}. \\
\qquad\quad \mathit{Inst}\ (L, \mathit{exit})\ (L, \mathit{exit})\ ; \\
\qquad\quad \mathit{Label}\ \mathit{exit}
\end{array}
$$

$\Rightarrow$ represent "state" labels as functions (with bound variables $\Rightarrow$ locality)

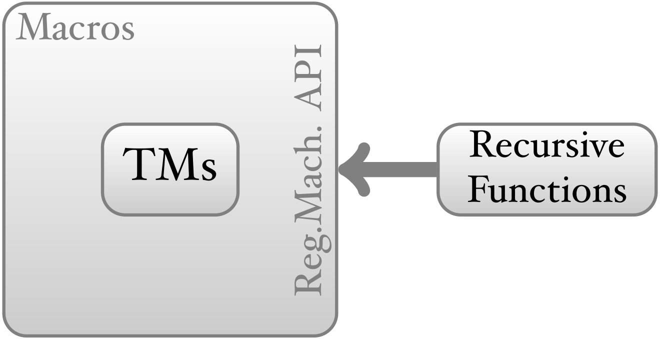
Better Composability

$$\begin{array}{l} move\_left\_until\_zero \stackrel{def}{=} \\ \qquad \Lambda\ start\ exit. \\ \qquad\quad Label\ start\ ; \\ \qquad\quad if\_zero\ exit\ ; \\ \qquad\quad move\_left\ ; \\ \qquad\quad jmp\ start\ ; \\ \qquad\quad Label\ exit \end{array}$$

$$if\_zero\ e \stackrel{def}{=} \Lambda\ exit.\ Inst\ (W_0, e), (W_1, exit); Label\ exit$$

$$jmp\ e \stackrel{def}{=} Inst\ (W_0, e), (W_1, e)$$

An RM-API with TMs



- Suppose the first four registers of an RM contain 1,2,0 and 3, then the encoding is

Inc a

$$
\begin{aligned}
\mathit{Inc}\ a \stackrel{\mathit{def}}{=}\ & \\
& \mathit{locate}\ a\ ;\\
& \mathit{right\_until\_zero}\ ;\\
& \mathit{move\_right}\ ;\\
& \mathit{shift\_right}\ ;\\
& \mathit{move\_left}\ ;\\
& \mathit{left\_until\_double\_zero}\ ;\\
& \mathit{write\_one}\ ;\\
& \mathit{left\_until\_double\_zero}\ ;\\
& \mathit{move\_right}\ ;\\
& \mathit{move\_right}
\end{aligned}
$$

Separation Algebra

- use some infrastructure introduced by Klein et al in Isabelle/HOL
- and an idea by Myreen

$$\{\!|p|\!\}\ c\ \{\!|q|\!\}$$

p, c, q will be assertions in a separation logic

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e.g. $\{\!|st\ i \star hd\ n \star ones\ u\ v \star zero\ (v + 1)|\!\}$

Separation Triples

$$\{\!|p|\!\}\ c\ \{\!|q|\!\} \stackrel{def}{=}$$
$$\forall\ cf\ r.$$
$$(p \star c \star r)\ cf \text{ implies}$$
$$\exists\ k.\ (q \star c \star r)\ (steps\ k\ cf)$$

$$\{\!|\ st\ i \star hd\ v \star zero\ u \star ones\ (u + 1)\ v\ |\!\}$$
$$i{:}[left\_until\_zero]{:}j$$
$$\{\!|\ st\ j \star hd\ u \star zero\ u \star ones\ (u + 1)\ v\ |\!\}$$

Inductions over *ones*

- What most simplifies the work is that we can do inductions over the "input" (inductively defined assertions)

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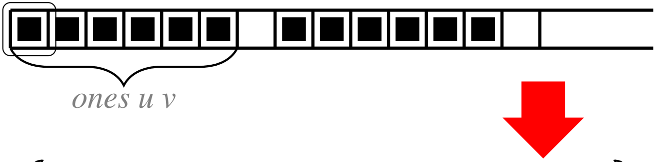
- Suppose *right_until_zero*:



$$\{\!|\ st\ i \star hd\ u \star zero\ (v + 1) \star ones\ u\ v\ |\!\}$$
$$i{:}[right\_until\_zero]{:}j$$
$$\{\!|\ st\ j \star hd\ (v + 1) \star zero\ (v + 1) \star ones\ u\ v\ |\!\}$$

Automation

- we introduced some tactics for handling sequential programs

$$\{\!|p|\!\}\ i{:}[c_1\ ;\ \ldots\ ;\ c_n]{:}j\ \{\!|q|\!\}$$

- for loops we often only have to do inductions on the length of the input (e.g. how many *0*s/*1*s are on the tape)
- no termination measures are needed

Register Machines

- We could also use Jensen's et al work to give a more appropriate view on register machines

$$\{\!|p|\!\}\ i{:}[rm\_c]{:}j\ \{\!|q|\!\}$$

- Rule for *Inc*

$$\begin{array}{ll} RM. & \{\!|\ pc\ i \star m\ a\ v\ |\!\} \\ & i{:}[\ Inc\ a\ ]{:}j \\ & \{\!|\ pc\ j \star m\ a\ (Suc\ v)|\!\} \end{array}$$

- Rules for *Dec*

$$\begin{array}{ll} RM. & \{\!|(pc\ i \star m\ a\ (Suc\ v))|\!\} \\ & i\ {:}[\ Dec\ a\ e\ ]{:}\ j \\ & \{\!|pc\ j \star m\ a\ v|\!\} \end{array} \qquad \begin{array}{ll} RM. & \{\!|\ pc\ i \star m\ a\ 0\ |\!\} \\ & i{:}[\ Dec\ a\ e\ ]{:}j \\ & \{\!|\ pc\ e \star m\ a\ 0\ |\!\} \end{array}$$

Conclusion

- What started out as a student project, turned out to be much more fun than first thought.
- Where can you claim that you proved the correctness of a *38* Mio instruction program? (ca. *7000* is the soa)
- We learned a lot about current verification technology for low-level code (we had no infrastructure: CPU model).
- The existing literature on TMs & RMs leave out quite a bit of the story (not to mention contains bugs).