Internet Auction Processes and Mechanisms

by

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Abstract

The nature of E-commerce over the Internet has seen significant changes over the years. Instead of companies selling items to consumers, consumers are increasingly selling items to fellow consumers on a global-scale, and Internet auctions have been the mechanism of choice in achieving this. In fact, auctioning allows the departure from the fixed price model, which some regard as too rigid to be able to respond swiftly to varying supply and demand fluctuations and changes, and the Internet plays a pivotal role in catalysing the widespread acceptance of such a variable pricing model on a global scale.

Internet auctions exhibit characteristics which are often not shared by conventional auctions, e.g. auctions of fixed duration which encourage sniping (bidders submit their bids moments before the close of an auction thereby preventing other bidders from submitting counter-bids), the acceptance of multiple bids in a single auction, and a maximum threshold whereby the auction will terminate at that price point. Internet auctions have significantly greater scope incorporating algorithms of increased complexity than conventional auction procedures. In this thesis, the characteristics and properties of different Internet auction algorithms are modelled mathematically based on a series of operational assumptions which characterise the arrival rate of bids, as well as the distribution from which the private values of buyers are sampled. From this, closed-form expressions of several key performance metrics are determined, including the average selling price in a given auction, as well as the average auction duration. In cases where a seller may be selling a commodity and auctions repeat themselves with the same items for sale multiple times, the income per unit time may also be quantified. Simulation experiments have been performed and analysed in the context of the mathematical models, and reasonable agreements are observed.

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Chapter 1

Introduction

1.1 Motivation and Significance

The prevalence of the Internet has ushered in the near-frictionless dissemination of data. In recent times, information flow is increasingly being monetised and distributed. Two trends can be observed: (i) the rise of goods and services purchased over the Internet, as evident by a 24.9% increase in Internet sales by businesses in the United Kingdom in 2009 alone (Office for National Statistics, 2010), and (ii) a marked adoption of user-generated content, which sees half of the top ten internationally visited sites in 2010 as being related to user-generated content (*Facebook*, *YouTube*, *Wikipedia*, *Blogger*, and *Twitter*, in order of site visits); notably five years ago there were none (Alexa, 2010). The crossing-over of these two trends has resulted in the augmentation of the E-commerce market by consumer-to-consumer (C2C) exchanges of good and services, in many cases complementing, and in the others supplanting, the pre-existing models of business-to-business (B2B) and business-to-consumer (B2C) exchanges. Instead of companies selling items to consumers, consumers are now selling items among themselves, with a common mechanism of achieving this being the auction. In fact, auctioning allows for a departure from the fixed price model, which some regard as too rigid to be able to respond rapidly to supply and demand fluctuations and changes. The pervasiveness and ubiquity of the Internet has played a pivotal role in catalysing the widespread acceptance of such a variable pricing model.

The use of auctions as a means of resource allocation has existed since antiquity, though becoming widespread only since the 1600's. In fact, the first use of an auction was recorded by the first-known historian and, "Father of History", Herodotus, who described the auctioning of brides, from the fairest to the least fair, in Illyria (modern day Bosnia and Herzegovina), to men who would become their husbands (Herodotus, 2008). There were several known rules surrounding this mechanism: (i) the process had to take place in a central location where all bids were observed, (ii) all girls of marriageable age were to adhere to this custom and were not permitted to find a husband outside this mechanism, and (iii) a man who bought a girl at the auction had to give security in order to ensure that he would, in fact, make her his wife. This historic observation highlights a few nuances of the auction mechanism—namely, the transparency of bids, regularisation of the objects put up for auction, and the contractual obligation between buyer and seller—from which parallels with contemporary auctions can be drawn.

In the eighteenth century, various auction houses sprang up in order to provide the service of selling assets that were generally considered illiquid, such as fine art, rare books and antiques. Among the auction houses founded in the 1700's, some are still in existence today. These include *Sotheby's* (1744), *Christie's* (1766) and *Bonhams* (1793). The largest of these is *Christie's*, with its 2009 sales totalling £2.1 billion, while the most expensive painting sold—*Jackson Pollock's No.5, 1948*—was auctioned by *Sotheby's* in 2006 at a 2011 inflation-adjusted price of £93.8 million. In more recent times, however, the public may be familiar with the concept of auctions through the proliferation of *eBay*, an online auction website, although perhaps the unconventional route that *Google* took in 2004 in allocating its initial public offering via a Dutch auction had also given significant exposure to the auction mechanism. Corporations often employ auctions in commodity allocation and these include, but are not limited to, auctions of carbon credits, government bonds and sections of the electromagnetic spectrum.

One way to view an auction is to regard it as the determination of bidders' valuations by the seller with the hindrance of concealed information from bidders (Cowell, 2006). The value of the object being sold (or lot) can either be the same for everyone and bids will vary according to the accuracy of the information a bidder holds, or each bidder will have his own private valuation that is unaffected by the valuations of those around him, whether known to him or not.

Five types of auctions are commonly addressed in literature:

1. The *English auction* involves public announcements of gradually increasing bids until a single bidder remains, who pays for the lot at the price of the last bid. This is also known as an open ascending price auction and the process is shown in Figure 1-1.

2. The *Dutch auction* or *reverse auction* is the reverse of this and public announcements of gradually decreasing bids are made until a single bidder remains, who agrees to the exchange of services or goods at the price of the last bid. This is also known as an open descending price auction and the process is shown in Figure 1-2.

3. In the *sealed-bid first-price auction*, all bids are submitted in private and the winner will pay the price that he had bid.

4. In the *sealed-bid second-price auction* or *Vickrey auction*, all bids are submitted in private and the winner will pay the price that the "runner-up" had bid, i.e. the next highest price. The process is shown in Figure 1-3.

Further to this, there are other independent properties that can be incorporated when designing an auction (Parsons et al., 2011). These other properties are listed below and a taxonomy of auctions is shown in Figure 1-4.

- **Combinatorial:** auctions that are combinatorial see multiple heterogeneous goods auctioned together.

- **Dimensionality:** in a singularly-dimensional auction, the bid is completely defined by the price of the lot, whereas in a many-dimensional auction, the bid

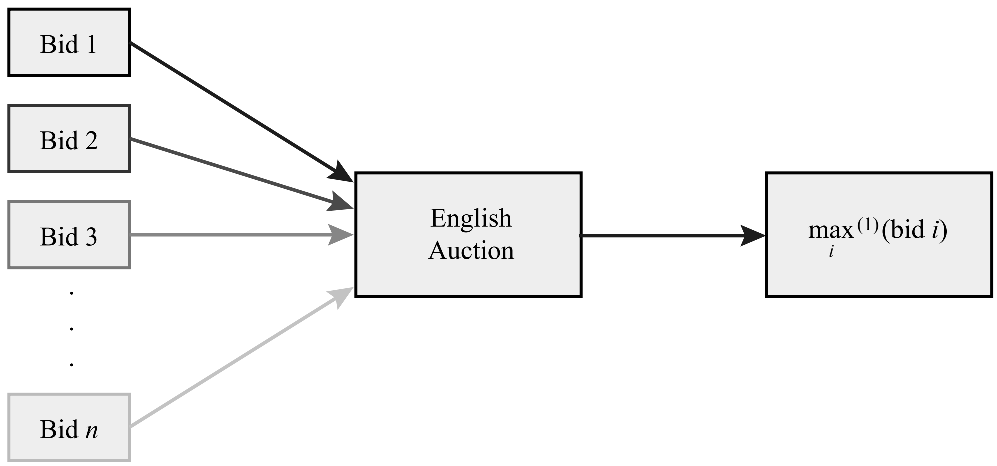

Figure 1-1: English auction process.

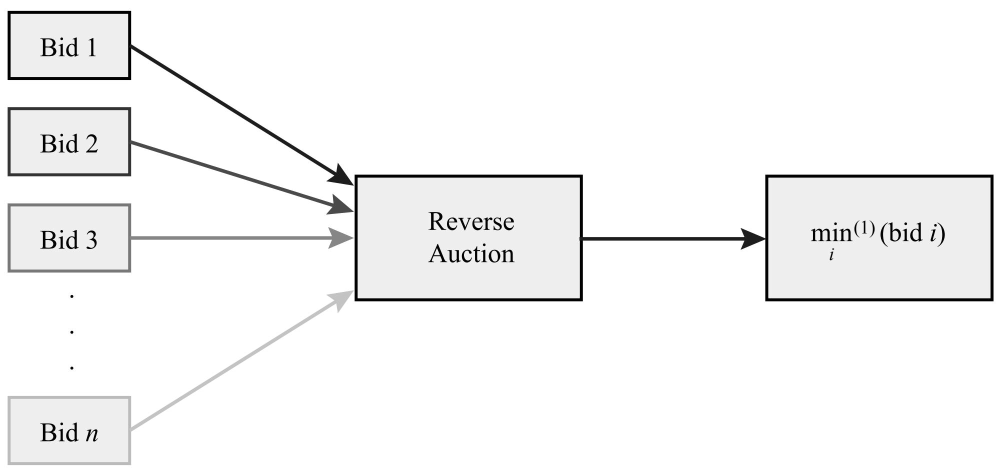

Figure 1-2: Reverse auction process.

may be a function of other attributes such as the timely delivery of the lot or the length and amount of the insurance contract taken out on that lot.

- **Sidedness:** in a one-sided auction, bidders are either all sellers or all buyers. In a two-sided auction, bids are submitted by both buyers and sellers and these are matched by the auctioneer.

In fact, in auctions without time restrictions, the English and the sealed-bid second-price auctions have been shown to be equivalent, while the Dutch and the

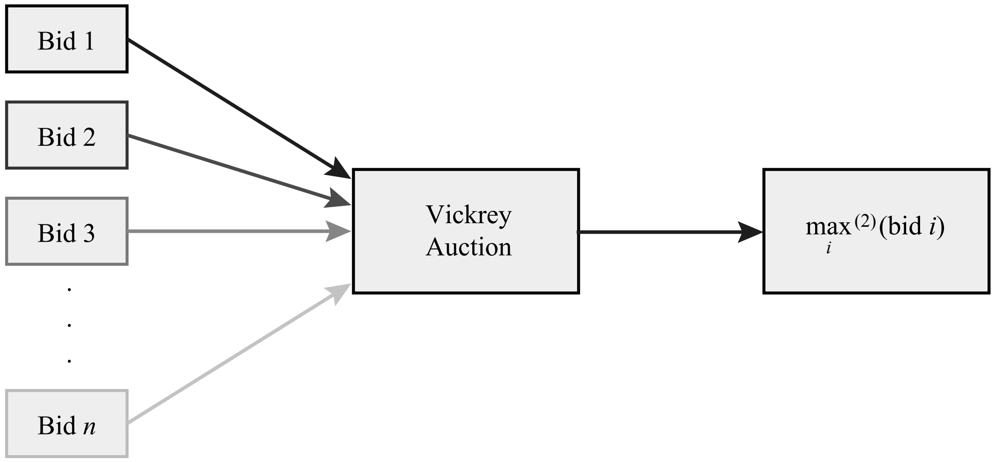

Figure 1-3: Vickrey auction process.

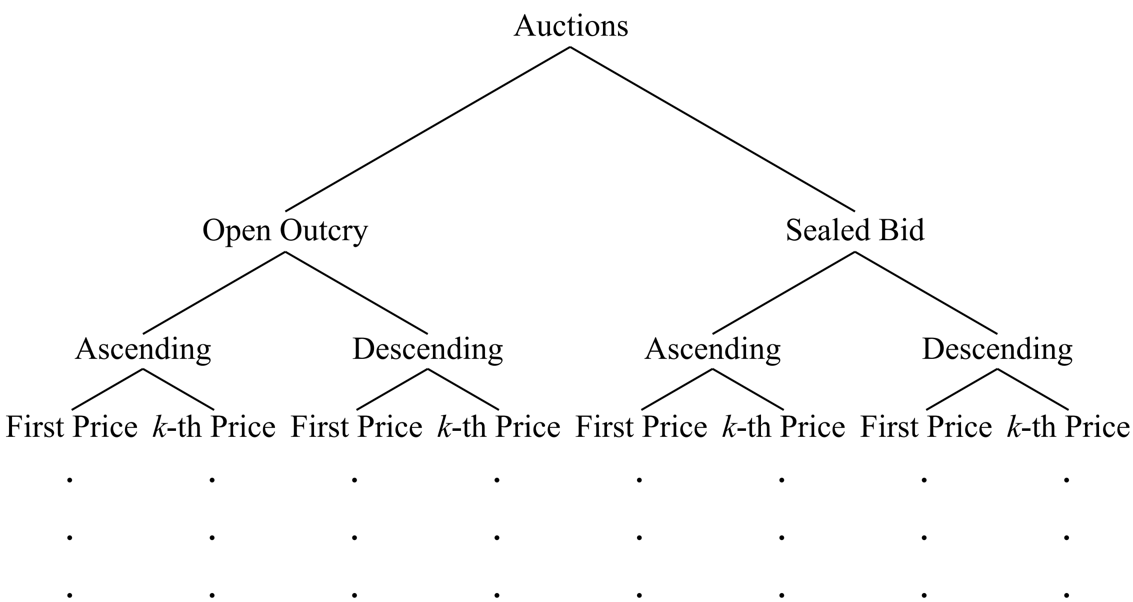

Figure 1-4: A taxonomy of auctions.

sealed-bid first-price auctions have also been shown to be equivalent. Furthermore, this is generalised by the Revenue Equivalence Theorem (shown in its entirety in Theorem 1), which states that the seller will obtain the same revenue for all auctions where: (i) the bidder with the highest bid always wins, (ii) the bidder with the lowest bid expects zero surplus, (iii) all bidders are risk neutral, and (iv) the private values of all bidders are drawn independently from the same distribution (Myerson, 1981). By allowing sellers to equate the revenue generating power of different types of auctions that may otherwise seem incomparable, the Revenue Equivalence Theorem assists sellers in auction mechanism design. It is equally interesting to observe cases where the Revenue Equivalence Theorem does not hold, e.g. if the private values of bidders have been shown to be related and not independently drawn from the same distribution, then it can be shown that the English auction yields a higher level of revenue than the sealed-bid first-price auction (Milgrom and Weber, 1982).

Theorem 1 *The seller's expected utility from a feasible auction mechanism is completely determined by the probability function p and the numbers $U_i(p, x, a_i)$ for all i.*

That is, once we know who gets the object in each possible situation (as specified by p) and how much expected utility each bidder would get if his value estimate were at its lowest possible level a_i, then the seller's expected utility from the auction does not depend on the payment function x. Thus, for example, the seller must get the same expected utility from any two auction mechanisms which have the properties that (1) the object always goes to the bidder with the highest value estimate above t_o and (2) every bidder would expect zero utility of his value estimate were at its lowest possible level. If the bidders are symmetric and all $e_i = 0$ and $a_i = 0$, then the Dutch auctions and progressive auctions studied in (Vickrey, 1961) both have these two properties, so Vickrey's equivalence results may be viewed as a corollary of our equation. However, we shall see that Vickrey's auctions are not in general optimal for the seller.

1.2 Aims and Objectives

This thesis aims to:

- Develop mathematical models and characterise the properties of different algorithms that may be found in or may be built into Internet auction mechanisms, basing these models on a series of operational assumptions including the arrival rate of bids as well as the distribution from which the private values of buyers are sampled.

- Construct and run simulation experiments in the context of the mathematical models, checking the validity of the mathematical models and using the same assumptions.

- Scrape data from real-world auction websites and evaluate the extent to which the mathematical model agrees with the data. The information derived from this analysis is then used for parameter tuning in order to align the mathematical model with real-world data.

1.3 Contributions

Internet auctions exhibit characteristics which are not often shared with conventional auctions, e.g. auctions of fixed duration which encourage sniping (whereby bidders submit their bids moments before the close of an auction thereby preventing other bidders from submitting counter-bids), the acceptance of multiple bids in a single auction, and a maximum threshold whereby the auction will terminate at that price point. Due to lack of regulation, the size of the market and the volume of bidders and sellers, Internet auctions are better suited to incorporating algorithms of increased complexity as opposed to the more established procedures at traditional auction houses. For example, while *eBay* runs what essentially amounts to an English auction with a fixed duration, *Swoopo* runs what is known as a bidding fee auction, where each bid incurs a fee and also extends the length of the auction by a short amount (10–20 seconds).

This report provides a mathematical analysis of the characteristics and the properties of these different and unconventional types of Internet auctions using a series of operational assumptions which characterise the arrival rate of bids, as well as the distribution from which the private values of buyers are sampled. From this, closed-form expressions of several key performance metrics are determined, including the average selling price in a given auction, as well as the average auction duration. In cases where a seller may be selling a commodity and auctions repeat themselves with the same items for sale multiple times, the income per unit time may also be quantified.

Having such analysis will pave the way for the development of optimum and dominant strategies on the part of both the buyer and the seller and an understanding of how the sensitivity of different auction parameters affect the income and length of an auction will aid auction design. A seller can decide the optimal type of auction to host given his/her aims, with the performance of the chosen auction able to be measured with different metrics in accordance to the specific aims.

1.4 Publications

- Leung, Timothy L. Y., and Knottenbelt, W. J. (2012). Comparative Evaluation of Independent Private Values Distributions on Internet Auction Performance. *International Journal of E-Entrepreneurship and Innovation (IJEEI)*, 1(3):59–71. London, UK.

- Leung, Timothy L. Y., and Knottenbelt, W. J. (2011). Consumer-to-consumer Internet Auctions. *International Journal of Online Marketing (IJOM)*, 1(3):17–28. London, UK.

- Leung, Timothy L. Y., and Knottenbelt, W. J. (2011). The Effect of Private Value on E-Auction Revenues. In *Proceedings of the 2011 International Conference on Digital Enterprise and Information Systems (DEIS 2011)*. London, UK.

- Leung, Timothy L. Y., and Knottenbelt, W. J. (2011). Global B2C and C2C Online Auction Models. In *Proceedings of the 2011 Annual Conference on Innovations in Business and Management (CIBMP)*. London, UK.

- Leung, Timothy L. Y., and Knottenbelt, W. J. (2011). Stochastic Modelling and Optimisation of Internet Auction Processes. In *Proceedings of the 6th Workshop on Practical Applications of Stochastic Modelling (PASM 2011)*. Karlsruhe, Germany.

- Sakellari, Georgia, and Leung, Timothy L. Y., and Gelenbe, Erol. (2011). Auction-based Admission Control for Self-Aware Networks. In *Proceedings of the 26th International Symposium on Computer and Information Sciences (ISCIS 2011)*. London, UK.
- Leung, Timothy L. Y., and Knottenbelt, W. J. Analysis of Internet Auction Processes and Mechanisms. *ACM Transactions on Internet Technology*. (To be submitted).

1.5 Thesis Outline

In Chapter 2, existing literature that addresses the current landscape of Internet auctions is reviewed. This is divided into sections detailing Internet Auction Characteristics and Applications, Internet Auction Behaviour, and the Stochastic Modelling of Auction Processes.

Chapter 3 presents the algorithms that represent a variety of different Internet auction mechanisms, analysing them in a stochastic framework. In particular, an examination of the metrics, average auction duration and the average offer accepted, are analysed for each auction mechanism.

Chapter 4 extends the theory in the previous chapter to additional types of auctions found on the Internet. These include the Vickrey and reverse auction algorithms. A similar approach is taken to that in Chapter 3, with the metrics, average auction duration and the average offer accepted, being analysed for each auction mechanism.

Chapter 5 conducts surplus analysis and uses allocative efficiency in evaluating how Internet auctions perform.

Chapter 6 conducts experimental validation for the algorithms detailed in Chapters 3 and 4. Simulation experiments are performed, and comparisons are made between the theoretical predictions and experimental observations. Furthermore, real-world data scraped from eBay is used to compare with the model assumptions.

Chapter 7 deals with generalisations and extensions of the auction algorithms and outlines the stochastic behaviour of private values and auction performance and the invariance of distribution on private values. In addition, generalisation of the results to non-homogeneous Poisson bid arrival is undertaken.

The thesis concludes with Chapter 8 where a list of thesis achievements are given in addition to details on the applications of the work that constitutes the thesis and directions for future work.

Chapter 2

Literature Review

2.1 Introduction

Internet auctions have begun to pervade large sections of the Internet economy and there is an increasing amount of literature in this field. As a first step, we provide a detailed comparison of the characteristics of Internet auctions which are often not shared by conventional auctions in Table 2.1.

2.2 Internet Auction Characteristics and Applications

There has been substantial work done on auctions (see Figure 2-1), with several books written on the topic (Cramton et al., 2006; Klemperer, 2004; Krishna, 2002; Milgrom, 2004). As a branch of game theory, auctions have been given the classification, *D44*, by the *Journal of Economic Literature*. Though parallel in many ways, the volume of literature on Internet auctions, however, is significantly less. It is important to draw precise distinctions between traditional and Internet auctions in order to determine which aspects of auction theory are applicable to which.

The *Independent Private Values* model is often associated with auctions (Parsons et al., 2011). The characteristics of this model include the assumptions of privacy and

Internet Auctions	**Conventional Auctions**
Asynchronous; bidders need not be at the same place at the same time	Synchronous
Global competition across national boundaries	No competition across national boundaries
Vickrey auctions dominant	Tend to be first-price auctions
Can have absolute pre-determined end-time	Pre-determined fixed end-time usually not acceptable
Can run for days or weeks	Usually run for no more than minutes or hours
Concealed information; highest bid not known	Often without concealed information — highest bid known
Sniping	No sniping
Buy-It-Now (BIN) option common	Buy-It-Now option uncommon
Can support arbitrarily complex auction rules and algorithms (e.g. Timeshift, Reject without recall)	Unable to support such complexity
Seller Ratings; a key consideration	Seller Ratings — largely unimportant
Large-scale shilling possible through sellers systematically creating bidding accounts	Large-scale shilling not feasible
Timely submission of bids depends on network speed and reliability	No such dependency
Bidders may participate in several auctions simultaneously (via Desktops, Notebooks, Mobile Phones etc.)	Usually not possible or with difficulty; not the same degree of monitoring or control
Proxy bidding widely used	Proxy bidding not widely used
Bidders need to create accounts first and "reveal" their identity	Mystery bidders may sometimes just turn up without revealing their identities
Statistically determined or event triggered random end-time may be adopted	Random end-time usually not acceptable
Increasingly dominant as a mechanism for different types of business and commercial transactions	Auctions not extensively used for conventional business and commercial transactions and likely to remain so
Inspection of items not usually possible	Inspection of items usually possible
Multiple channels of delivery	Usually a single channel of delivery
Sellers can be a company	A company selling normal items through auctions is not cost-effective
Individuals can make systematic sales and profits using auctions without needing to set up a company (i.e. dedicated e-commerce site)	Hard for individuals to do regular business using only auctions

Table 2.1: Comparison between Internet auctions and conventional auctions.

independence where the value of the commodity in question is private to the individual buyers, and that different buyers do not know the values other buyers attach to the commodity. In addition, these values are drawn from a common distribution which is known to the buyers. In probabilistic terms, this essentially amounts to a series of values which are independent and identically distributed. A common distribution used is the uniform distribution (Katok and Kwasnica, 2008). In our subsequent analysis, we shall follow the independent private values model using the uniform distribution.

The online auction website, eBay, is a popular and recent implementation of the auction mechanism. It is classed as a consumer-to-consumer (C2C) auction and it runs open-bid second-price auctions that are of a fixed length. Due to its fixed length, eBay auctions are susceptible to sniping, which see bidders submit their bids moments before the close of an auction preventing other bidders from submitting counter-bids (Ockenfels and Roth, 2006). While this is seen to be problematic, eBay has always maintained the policy that a bidder should bid his private value. Since the winner pays the second price, there is little reason for a bidder to shade his bid. In order to counteract sniping, other online auction websites, such as Amazon, have employed auctions with a soft close, automatically extending the length of the auction. The investigation of different types of auction terminations has been undertaken in Ockenfels and Roth (2002), where it is found that late bidders in *eBay*-style auctions tend to be associated with highly experienced bidders, whereas those of the *Amazon*-type tend to be relatively inexperienced bidders. In Bajari and Hortacsu (2003), it is found that sniping often leads to winning, and it observes that many sellers tend to set the starting bid price unrealistically low to stimulate bidder participation.

In Kaghashvili (2009), the mechanism of online timeshift auctions is proposed, whereby auctions are qualitative modifications of the existing popular auctions with items offered for a fixed time period. Conducting a standard timeshift auction comprises of several steps: i) the seller defines an auction length and a specific point during the auction where the auction transitions to the timeshift interval before terminating, ii) following the highest bidder who had placed at least a single bid before the transition to the timeshift interval, obtains the item. For B2C online auctions, the analysis of their design, and the optimal design of online auction channel have been studied in Bapna et al. (2002, 2003).

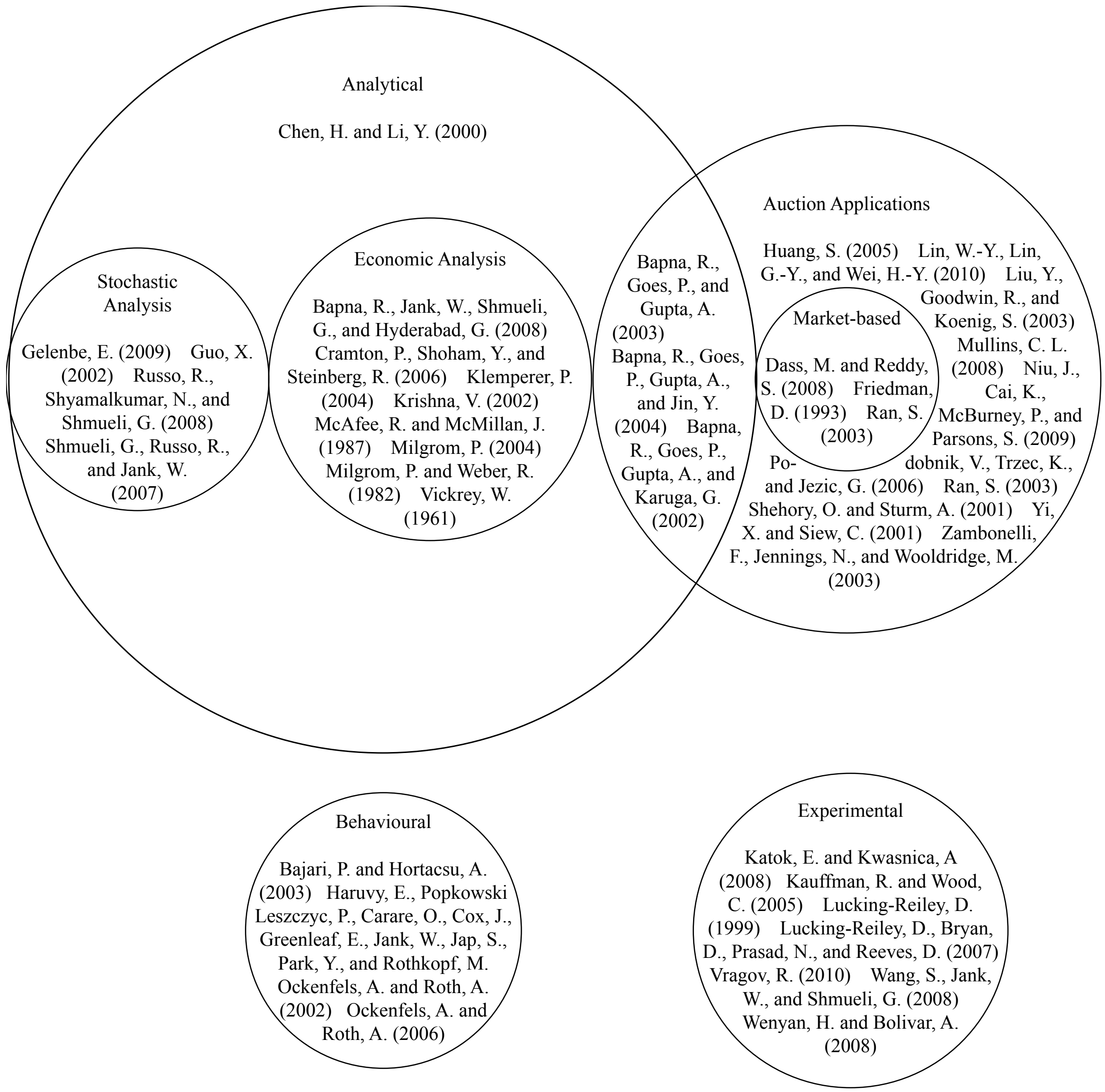

Figure 2-1: Venn diagram classification of literature covered in this literature review.

The user of auctions as a market mechanism has also been studied in various e-business and Web services contexts (Chen and Li, 2000; Ran, 2003; Huang, 2005), often for allocating different types of resources with different policies for pricing. In Huang (2005), a progressive resource allocation scheme which is the continuous version of the generalised Vickrey auction is used, where the auction mechanism is built into the service sharing prototype so that users and auctioneers rely on the software agents to exchange information in a distributed environment. More recent Internet auction applications include Cloud computing resource allocation (Lin et al., 2010), as typified by *Amazon*'s EC2 cloud computing service. The EC2 employs a continuous double auction (CDA) for cloud server space, which they label "Spot Instances", and which has been analysed within a framework that simulates dynamic demand through the peak/off-peak concept. In a recent patent (Mullins, 2008), Microsoft has extended this use of double auctions to incorporate different types of decentralised computing resources, which is partitioned into a two-tier pricing structure that accounts for peak and off-peak traffic, with the discounted price of the latter reflecting an increased risk of latency and network outage. In Lin et al. (2010), a dynamic auction mechanism for solving the allocation problem of computation capacity in cloud computing is proposed, which makes use of second-price auctions to regulate computational resource efficiency and to ensure a reasonable level of profit for the CSP (Cloud Service Provider).

2.3 Internet Auction Behaviour

Studies of Internet auction bidding behaviour have been undertaken in Ockenfels and Roth (2002, 2006); Wenyan and Bolivar (2008) In Wenyan and Bolivar (2008), different properties of online auctions such as consumer surplus, sniping, bidding strategy and their interactions are studied, and a significant correlation between sniping and surplus ratios is found. It also examines the efficiency of online auctions, where Pareto efficiency is used as the optimality criterion. In Ockenfels and Roth (2006), it is suggested that the strategic advantages of sniping are eliminated or severely eroded in auction mechanisms that apply an auction extension rule, and that there is noticeable difference between sniping on eBay and Amazon in proportion to user experience.

Experimental studies of Internet auction behaviour have been undertaken in Lucking-Reiley (1999); Vragov (2010); Katok and Kwasnica (2008). In Katok and Kwasnica (2008), it concentrates on the Dutch auction and first-price sealed bid auction formats, using laboratory experiments and human subjects, where values are drawn from the uniform distribution between 0 and 100, focusing primarily on the effect of clock speed on seller's revenue. In Vragov (2010), laboratory experiments with human subjects are also conducted, and the operational efficiency of Internet auctions is studied. Collusion behaviour such as shilling, in which the seller plays a part in the bidding process, is studied in Kauffman and Wood (2005), where two types of shilling strategies are examined, these deploy competitive bidding and reserve price mechanism and each of these exhibits a characteristic pattern of behaviour. While an auction can be defined as a market institution whereby offers are made only by the buyers, i.e. bids, or only by the sellers, i.e. asks, a double auction is one where both buyers and sellers are able to make offers (Friedman, 1993). Viewing the interlinking relationship between bidders and sellers as networks is proposed in Dass and Reddy (2008), and the competitions in auctions is investigated in Haruvy et al. (2008). Price variation characteristics and consumer surplus are studied in Bapna et al. (2008); Jank et al. (2006). The use of various types of curves for fitting price data for Internet auctions have been proposed in Hyde et al. (2007), in which monotone splines and beta functions are used. Empirical investigations of eBay auctions have also been undertaken in Lucking-Reiley et al. (2007) where the auction of coins is conducted. This makes use of regression models to estimate the price of items and examines the influence of seller ratings (which measures the reliability and services provided by the seller) on the final price. It has also found that the effect of positive and negative ratings is not symmetrical, with the latter having a much greater (adverse) influence on the price. It also suggests that longer auctions tend to have a beneficial effect in achieving a higher price. Moreover, in Dellarocas and Wood (2008), it was found that there is a reluctance on the part of users to give negative feedbacks compared with giving positive feedbacks. The K-means clustering algorithm has been employed in Bapna et al. (2004) which classifies bidders into five categories based on factors such as entry time, number of bids placed, and exit time. It also examines the use of automated agents in carrying out bidding as well as the different experience levels of bidders. The use of analogies from physics to study price movements have been applied in Hyde et al. (2007); Jank et al. (2008); Wang et al. (2008), which make use of the concepts of price-velocity to characterise the dynamics of price changes and may subsequently be exploited to produce forecasts.

2.4 Stochastic Modelling of Auction Processes

While the majority of the literature in the previous section considered auctions in their entirety, it is useful to look at cases where the bids are separated from each other and arrive following a certain distribution. A stochastic number of bidders is studied in McAfee and McMillan (1987), where first-price sealed-bid auctions having constant absolute risk aversion is analysed. As a result of the stochastic analysis, the authors conclude that the seller should conceal the number of bids in order to maximise the selling price. Stochastic models of bid arrival characteristics are studied in Shmueli et al. (2007); Russo et al. (2008), where the so-called BARISTA (Bid ARrivals InSTAges) model that makes use of non-homogeneous Poisson process is proposed. The probabilistic and statistical properties of these models are analysed and studied and the usefulness of these models for auction modelling is illustrated and discussed.

An interesting Internet auction mechanism is proposed in Guo (2002). It considers a seller who sets the lowest acceptable price for an item without revealing it, with bidders arriving at different times with their bids. If a bid is lower than the set lowest acceptable price, the seller immediately rejects the bid. However, if a bid is higher than the set lowest acceptable price, then the seller faces the decision of either accepting this bid or rejecting it and moving on to the next bidder, with the hope of achieving a higher price. In the latter case, it is assumed that the rejected bid disappears, never to return. Assuming that the seller is allowed to make a choice at any time, the goal is to maximise the sellers expected return by choosing the best bid. The situation is represented as an optimal stopping problem, and using techniques from convex analysis, an explicit solution that yields a simple algorithm for the seller is obtained.

A stochastic approach to Internet auctions is given in Gelenbe (2009), where bid arrivals also follow a Poisson process, with successive bids increasing in value and the seller's problem is to decide when it should accept a bid. The associated auction algorithm is illustrated in Figure 2-2. After each bid, the seller waits for some random decision time to determine whether to accept the offer. If a new bid arrives before that time expires, then the process is repeated for this new bid. However, if a new bid does not arrive before this time expires, then the seller accepts the current bid. If the seller accepts the offer too quickly, then the price obtained may be low with respect to the price that the seller would have received had he or she been more patient. On the other hand, if the seller waits a long time before accepting an offer, a higher price may be obtained but at the expense of wasting more time. After selling the good, the seller rests for some random time before initiating a new auction. Characteristics of this mechanism include decision time, rest time, and maximum bid value. These may be adjusted, which will affect the auction duration. Similarly, in our study, we model the bid arrivals as a Poisson process, and we follow the commonly adopted *Independent Private Values* approach in which the buyers valuations are drawn independently from a common distribution. Unlike the assumptions in Gelenbe (2009), however, there are no overheads relating to decision time or rest time in our model, which more realistically models real-life Internet auction mechanisms. We focus on the key performance metrics of income per auction, and the duration of an auction, from which we may determine the auction efficiency in terms of the income rate.

2.5 Agent-Mediated Auctions

Similar to conventional auctioning where human agents are often employed to serve the interests of different parties of an auction, autonomous software agents are sometimes used to ensure that the requirements of buyers and sellers are satisfactorily catered for. The use of trade agents for auctions is studied in Niu et al. (2009), and in Yi and Siew (2001) software agent technology is used together with cryptographic technology to enable the automation of online auction and develops a secure agent-

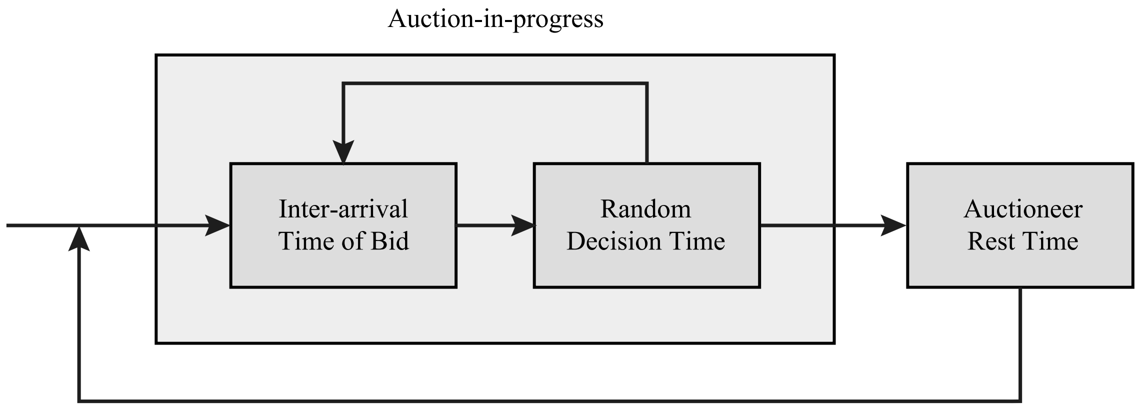

Figure 2-2: Gelenbe auction algorithm.

mediated online auction framework (see Figure 2-3). In the proposed framework, an online auctioneer generates an auction agent which acts as a mobile auctioneer traversing a list of online bidders, asking and collecting bids of each round to the online auctioneer. From the bid information so collected, the online auctioneer specifies a new minimum bid and sends out the software agent again. This process is repeated until a minimum bid remains unchanged for three consecutive times, after which the auctioneer would broadcast the auction outcome. Generalisation of such agent-mediated auctioning mechanism is indicated in Zambonelli et al. (2003) in which both buyers and sellers also employ agents in a multi-agent environment to achieve the overall application goal.

In Shehory and Sturm (2001), the implementation of an auction agent is described, which participates and bids in web-based auctions on behalf of its user. The user provides the agent with relevant details of product, price and bidding strategy before activating the agent. After being activated, the agent enters the auction site, locates the specific product, and monitors the site. The agent terminates its auction activity either when its buying strategy requires withdrawal or when the auction finishes. In Das et al. (2001), a series of laboratory experiments that employs human subjects to interact with software bidding agents in continuous double auctions are described, and it is found that agents consistently obtain significantly larger gains from

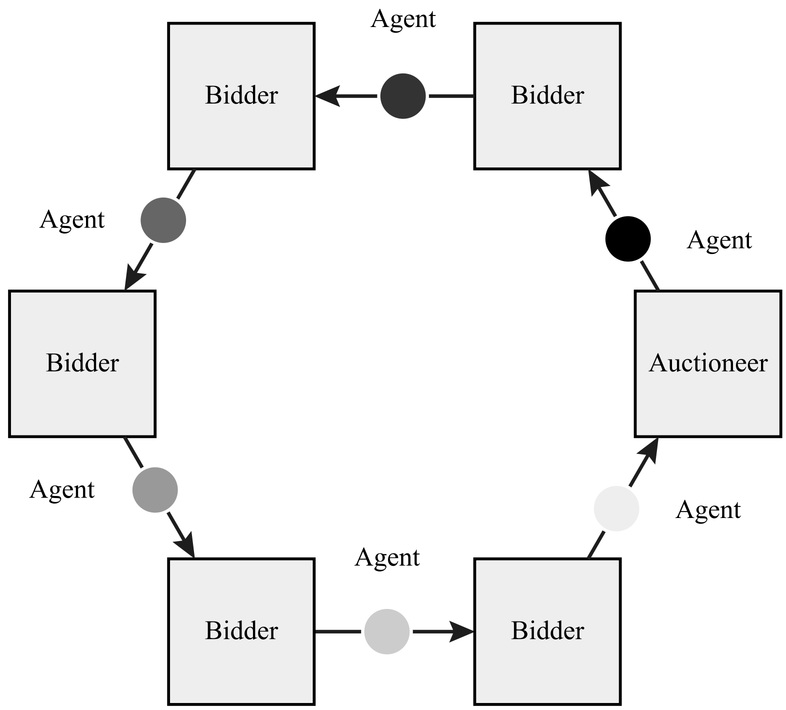

Figure 2-3: Illustration of the Agent-mediated online auction framework.

trade than their human counterparts, and that unexpected non-equilibrium trading is observed in such systems. In Podobnik et al. (2006), mediation between service requester's agents and sellers service provider's agents is studied, in which enables provider agents to dynamically and autonomously advertise semantic descriptions of available services through an auction model, called the Semantic Pay-Per-Click Agent (SPPCA) auction. Requester agents then use the advertised services to discover appropriate services. Then, information regarding the actual performance of service providers is considered in conjunction with the prices bid by service provider's agents in the SPPCA auction before a final set of advertised services is then chosen and proposed to the buyer agent as an answer to its request.

In Liu et al. (2003) a theory for auction agents forming part of a supply-chain management system combining concepts from auction theory, utility theory, and dynamic programming is proposed. It employs results from utility theory to obtain the optimal bidding strategy for risk-averse auction agents, both for first- and second-price sealed-bid auctions in the symmetric independent private values model, in addition to using dynamic programming techniques to integrate the resulting auction agents with a production-planning system. It also makes use of simulations in combining the auction and production-planning system to obtain crude approximations of the competitor's valuations distributions of the auctioned item. This theory aims to provide a framework for building more powerful auction agents geared to highly complex decision situations. In our study, the actual auction implementation mechanism may or may not make use of agents to achieve the desired goal. Using agents, however, may have the advantage of saving significant human efforts, especially when the auction takes place at times when the participating parties are unavailable. Similar to the study Liu et al. (2003), we also make use of simulation experiments, but unlike those, we also develop mathematical analysis of the different auction algorithms which mostly leads to closed-form results.

Chapter 3

Analysis of Auction Algorithms

This chapter contains the algorithms of a number of key auctions which differ from one another in a significant way. In the analyses that follow, both in this current chapter and the next, the different Internet auction algorithms may be categorised according to their properties and characteristics:

1. First-price Forward Auctions with Fixed Duration (Algorithms I, III)

2. First-price Forward Auctions with Variable Duration (Algorithms II, IV, V)

3. Vickrey Auctions with Fixed Duration (Algorithms VI, VIII)

4. Vickrey Auctions with Variable Duration (Algorithms VII, IX, X)

5. Reverse Auctions with Fixed Duration (Algorithms XI, XIII)

6. Reverse Auctions with Variable Duration (Algorithms XII, XIV, XV)

Specifically, this chapter examines the cases of the fixed-time first-price forward auction, which is similar to *eBay*, the first-price forward auction with variable duration, which is similar to the auctions that occur in auction houses such as *Christie's* and *Sotheby's*, the cases where there is a maximum threshold termination, i.e. the auction is terminated on reaching a specific bid level, and the case where the auction is terminated on reaching a specific number of bids.

The purpose of this chapter and the next is to deduce, given the simple set of assumptions detailed below, the closed-form solutions to a number of key metrics that can be used to compare the different auction types. These key metrics include the average auction income as well as the average auction duration, and in cases where possible, the average auction income per unit time.

While these closed form solutions are deduced using theoretical methods, Chapter 6 gathers real-world data from *eBay* auctions and tests the model with the collected data, computing how closely they compare with each other, and in so doing, determining how well the theoretical formulas describe real-world behaviour.

Internet auction mechanisms take on a variety of forms. Many of them are forward auctions, e.g. `http://www.ebay.com`, such as the English and Vickrey auctions, while others may be reverse auctions, e.g. `http://www.oltiby.com`. There are auctions that are of fixed duration, and there are those that allow an auction to terminate prematurely when a high enough bid is received. Usually only a single bid is accepted in each auction, but there also exist auctions that allow the acceptance of multiple bids in an auction, e.g. *Google* AdWords.

As in Gelenbe (2009), bids are assumed to arrive over time in a Poisson manner with rate λ, and that the bids $\{ Q_k \}$ are ascending ordered values taken from the uniform distribution over the interval $(0, L)$. The latter is a commonly-used distribution in auction analysis (Katok and Kwasnica, 2008). This property is further discussed in Chapter 7. If there are N bids, these ordered values are denoted by $Q_{(1)} < Q_{(2)} < \ldots < Q_{(N)}$.

3.1 Auction Algorithm I: Fixed-time First-price Forward Auction

A forward auction is an auction where buyers compete for lots and the price increases as time passes. An example of this type of auction is *eBay*. In this algorithm, the auction time is assumed to be fixed with duration T. Let N be the number of bids received, such that the largest bid $Q_{(N)}$ received over the time interval $(0, T)$ is accepted. Each bid comes at a certain time t, with a bid value of R, from a user with a specific `bidder_id`. A high value for T will produce a larger average accepted bid but the duration of the auction will be longer. For practical and meaningful operation of the auction, T should be significantly greater than the mean bid inter-arrival time, $1/\lambda$, i.e. $T \gg 1/\lambda$. At the close of the auction, the maximum bid $Q_{(N)}$ is accepted (see Algorithm I).

Algorithm 1 Fixed-time first-price forward auction.

```
Require: T > 0
Ensure: L > 0 and accept_id ≠ null
  accept_id ← null
  clock ← 0
  L ← 0
  while clock < T do
    while new_bid(t, R, bidder_id) do
      clock ← clock + t
      if clock < T and R > L then
        L ← R
        accept_id ← bidder_id
      end if
    end while
  end while
  return L, accept_id
```

From the results of order statistics (Barry C. Arnold et al., 2008), it can be shown that the conditional income per auction is

$$E[Q_{(N)}|\text{Number of bids} = N] = \frac{NL}{N+1}. \tag{3.1}$$

Since,

$$\frac{dE[Q_{(N)}|\text{Number of bids} = N]}{dN} = \frac{L}{(N+1)^2} > 0,$$

as the number of bids N increases, the corresponding average income per auction will also increase. Thus, when the bid arrival λ rate is high, the corresponding average income per auction is likewise be expected to be high. To determine the average income $E\left[Q_{(N)}\right]$, the condition on N is removed in Equation 3.1 using the Poisson probabilities, i.e.

$$\begin{aligned}
E\left[Q_{(N)}\right] &= \sum_{N=1}^{\infty} \frac{NL}{N+1} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= \sum_{N=1}^{\infty} \left[L - \frac{L}{N+1}\right] \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= \sum_{N=1}^{\infty} L \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} - \sum_{N=1}^{\infty} \frac{L}{N+1} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= L\left(1 - e^{-\lambda T}\right) - \frac{Le^{-\lambda T}}{\lambda T} \sum_{N=1}^{\infty} \frac{(\lambda T)^{N+1}}{(N+1)!} \\
&= L\left(1 - e^{-\lambda T}\right) - \frac{Le^{-\lambda T}}{\lambda T}\left(e^{\lambda T} - 1 - \lambda T\right). \qquad (3.2)
\end{aligned}$$

The term $N = 0$ is omitted above, since the income is zero upon $N = 0$. This gives an average income per auction of

$$E\left[Q_{(N)}\right] = \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right), \qquad (3.3)$$

and an income rate, or income per unit time, of

$$\frac{L}{\lambda T^2}\left(\lambda T + e^{-\lambda T} - 1\right). \qquad (3.4)$$

Figure 3-1 shows $E\left[Q_{(N)}\right]$ for different values of λ for $L = 100$, and $T = 5, 10, 15$. In the case where $T = 10$, the increase in bid rate up to $\lambda = 4$ produces a rather steep average auction income improvement. There seems to be a critical bid rate at around $\lambda = 6$, above which the improvement in income becomes less pronounced.

Figure 3-2 shows $E\left[Q_{(N)}\right]/T$, or the income rate, for different values of λ for $L = 100$, and $T = 5, 10, 15$. In the case where $T = 10$, the increase in bid rate up to $\lambda = 1$ produces a rather marked increase in auction income rate. There seems to be a critical bid rate at around $\lambda = 2$, above which the increase in auction income rate slows down. Moreover, the income rate drops as T increases, and the drop is much more significant for smaller values of T: the drop is much greater from $T = 5$

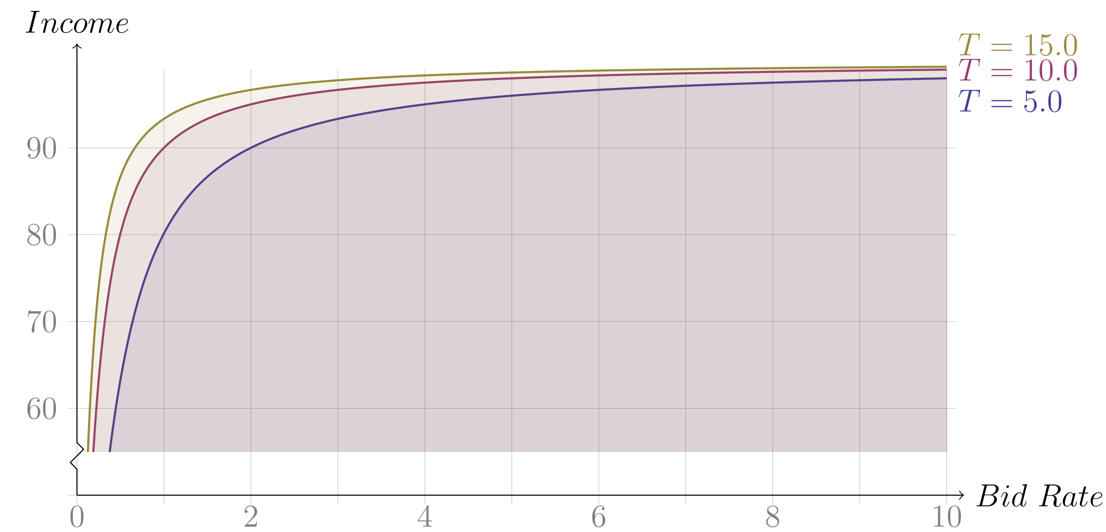

Figure 3-1: Average auction income of fixed-time first-price forward auction.

to $T = 10$ than from $T = 10$ to $T = 15$.

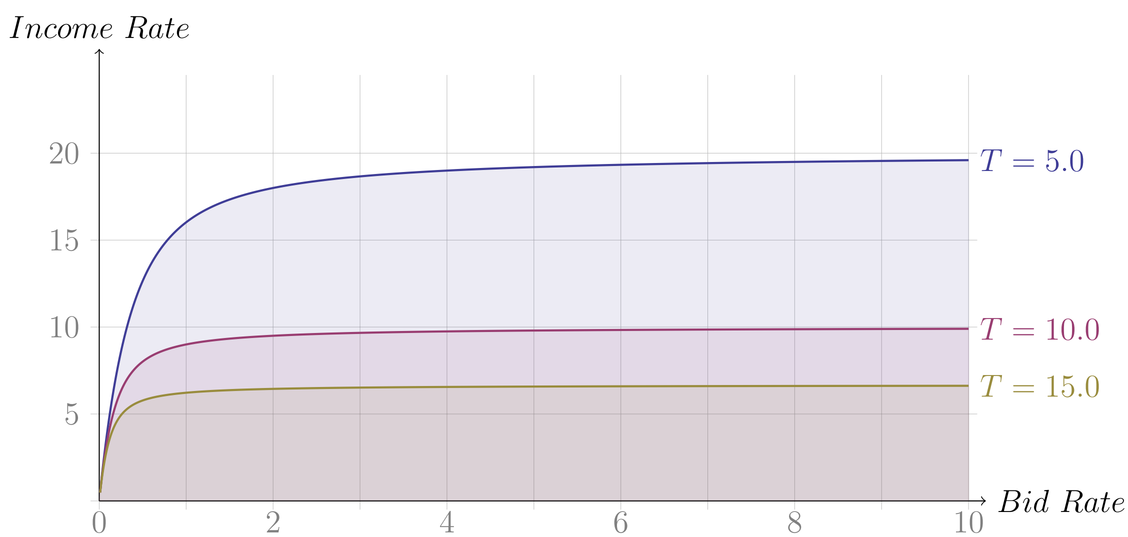

Figure 3-2: Average auction income rate of fixed-time first-price forward auction.

Differentiating the auction income with respect to T, yields

$$\frac{dE\left[Q_{(N)}\right]}{dT} = \frac{L}{\lambda T^2}\left[1 - \frac{1+\lambda T}{e^{\lambda T}}\right].$$

Since, $\lambda T > 0$, and

$$e^{\lambda T} = 1 + \lambda T + \frac{(\lambda T)^2}{2!} + \frac{(\lambda T)^3}{3!} + \ldots, \tag{3.5}$$

the inequality $(1 + \lambda T) < e^{\lambda T}$ holds, so that $(1 + \lambda T)/e^{\lambda T} < 1$, and thus

$$\left[1 - \frac{1 + \lambda T}{e^{\lambda T}}\right] > 0.$$

Consequently,

$$\frac{dE\left[Q_{(N)}\right]}{dT} > 0.$$

Thus, the auction income can be raised by increasing the auction duration. Similarly, differentiating the auction income with respect to λ and by similar argument using Equation 3.5,

$$\frac{dE\left[Q_{(N)}\right]}{d\lambda} = \frac{L}{\lambda^2 T}\left[1 - \frac{1 + \lambda T}{e^{\lambda T}}\right] > 0.$$

Thus, the auction income can also be augmented by increasing the incoming rate of bids.

3.2 Auction Algorithm II: Variable-time First-price Forward Auction with Fixed Inactivity Window

Here, unlike Algorithm I, the auction will terminate when there is no bid arrival for a fixed window of length α. On termination, the largest bid received will be accepted (see Algorithm II).

The fixed inactivity window α may be adjusted, and for meaningful operation, should not be significantly smaller than the mean bid inter-arrival time; a high value

Algorithm 2 Variable-time first-price forward auction with fixed inactivity window.

Require: $\alpha > 0$
Ensure: $L > 0$ **and** $accept\_id \neq null$

```
accept_id ← null
L ← 0
loop
  while new_b id(t, R, bidder_id) do
    if t < α then
      if R > L then
        L ← R
        accept_id ← bidder_id
      end if
    else
      return L, accept_id
    end if
  end while
end loop
```

for α will produce a higher income but the auction will take longer. Unlike Algorithm I, in which the auction duration is always bounded by T, here, it is possible for the auction duration to go on for an indefinite period without a predictable end point. In particular, if the average bid inter-arrival time $1/\lambda$ is significantly less than α, i.e. $\alpha \gg 1/\lambda$, or $\lambda\alpha \gg 1$, then there is little chance of having an interval of length α without any arrival.

Let $V(\alpha)$ be the average waiting time from the beginning of the auction to the (beginning of the) first occurrence of an inter-arrival time interval of greater than α (i.e. excluding the time interval α itself). If the first arrival interval is less than α, which happens with probability $(1 - e^{-\lambda\alpha})$, then the process effectively starts all over again, except for any time penalty incurred, i.e.

$$V(\alpha) \;\; = \;\; \int_0^\alpha \lambda t e^{-\lambda t} dt + (1 - e^{-\lambda\alpha})V(\alpha), \qquad (3.6)$$

where the first term represents the waiting time penalty for the first bid arrival averaged over an inter-arrival interval of less than α. Solving for $V(\alpha)$, Equation 3.6

becomes

$$V(\alpha) = e^{\lambda\alpha}\int_0^{\alpha}\lambda t e^{-\lambda t}dt = e^{\lambda\alpha}\left[\frac{1-e^{-\lambda\alpha}(\lambda\alpha+1)}{\lambda}\right] = \frac{e^{\lambda\alpha}-(\lambda\alpha+1)}{\lambda}$$

This gives an average auction duration of $\tau_\alpha = \alpha + V(\alpha)$, i.e.

$$\tau_\alpha = \frac{e^{\lambda\alpha}-1}{\lambda}. \tag{3.7}$$

It can be seen that

$$\frac{d\tau_\alpha}{d\alpha} = e^{\lambda\alpha} > 0,$$

so that τ_α grows as α increases. Figure 3-3 shows how τ_α varies with λ for $\alpha = 1, 2$ and 3. It can be seen that τ_α increases relatively gradually for small values of λ, but accelerates for large values. As the bid rate increases, there is reduced chance of a no-bid interval occurring, which lengthens the auction, and the difference between $\alpha = 1$ and $\alpha = 3$ also becomes more pronounced as $\lambda \to 2$.

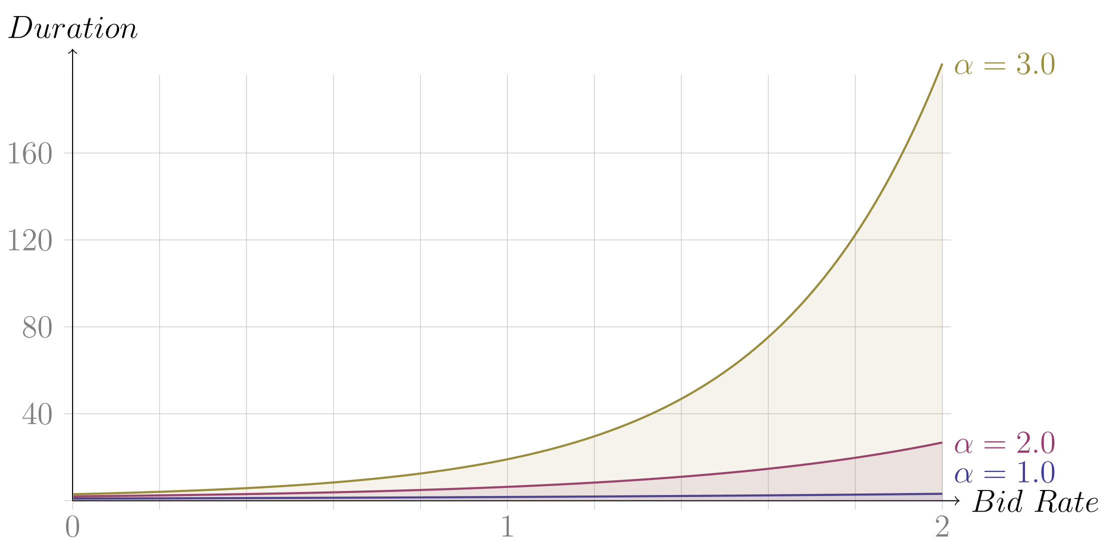

Figure 3-3: Average auction duration of variable-time first-price forward auction with fixed inactivity window.

To determine the average income $E\left[Q_{(N)}\right]$, the condition on N in Equation 3.1 is removed. Unlike the fixed-time case where Poisson probabilities could be used, however, the probability mass function for N must first be determined. Denoting the inter-arrival times by $\{T_k\}$, the number of bids is N if

$$T_1 < \alpha, T_2 < \alpha, \ldots, T_N < \alpha, T_{N+1} \geq \alpha$$

Now, the probability of $T_k < \alpha$ is $1 - e^{-\lambda\alpha}$ and the probability of $T_{N+1} \geq \alpha$ is $e^{-\lambda\alpha}$, and since the sequence $\{T_k\}$ is independent,

$$\Pr[T_1 < \alpha, T_2 < \alpha, \ldots, T_N < \alpha, T_{N+1} \geq \alpha] \;=\; (1 - e^{-\lambda\alpha})^N \times e^{-\lambda\alpha}. \tag{3.8}$$

Thus,

$$\begin{aligned}
E\left[Q_{(N)}\right] &= \sum_{N=1}^{\infty} \frac{NL}{N+1} \times (1 - e^{-\lambda\alpha})^N e^{-\lambda\alpha} \\
&= \sum_{N=1}^{\infty} \left[L - \frac{L}{N+1}\right] \times \left(1 - e^{-\lambda\alpha}\right)^N e^{-\lambda\alpha} \\
&= \sum_{N=1}^{\infty} L\left(1 - e^{-\lambda\alpha}\right)^N e^{-\lambda\alpha} - \frac{Le^{-\lambda\alpha}}{(1 - e^{-\lambda\alpha})} \sum_{N=1}^{\infty} \frac{\left(1 - e^{-\lambda\alpha}\right)^{N+1}}{N+1} \\
&= L\left(1 - e^{-\lambda\alpha}\right) - \frac{Le^{-\lambda\alpha}}{(1 - e^{-\lambda\alpha})} \sum_{N=1}^{\infty} \frac{\left(1 - e^{-\lambda\alpha}\right)^{N+1}}{N+1}
\end{aligned} \tag{3.9}$$

From the logarithmic series,

$$\sum_{N=1}^{\infty} \frac{\theta^N}{N} \;=\; -\log(1 - \theta), \tag{3.10}$$

and letting $\theta = 1 - e^{-\lambda\alpha}$, from Equation 3.9,

$$\begin{aligned}
E\left[Q_{(N)}\right] &= L\left(1 - e^{-\lambda\alpha}\right) - \frac{Le^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} \left[\left(\sum_{N=1}^{\infty} \frac{\left(1 - e^{-\lambda\alpha}\right)^N}{N}\right) - \left(1 - e^{-\lambda\alpha}\right)\right] \\
&= L\left(1 - e^{-\lambda\alpha}\right) - \frac{Le^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} \left[-\log\left(e^{-\lambda\alpha}\right) - \left(1 - e^{-\lambda\alpha}\right)\right],
\end{aligned}$$

which gives,

$$E\left[Q_{(N)}\right] = L\left(1-e^{-\lambda\alpha}\right) - \frac{L}{e^{\lambda\alpha}-1}\left[\lambda\alpha + e^{-\lambda\alpha} - 1\right].$$

On simplification,

$$E\left[Q_{(N)}\right] = \frac{L}{e^{\lambda\alpha}-1}\left[e^{\lambda\alpha} - \lambda\alpha - 1\right] = L\left[1 - \frac{\lambda\alpha}{e^{\lambda\alpha}-1}\right]. \tag{3.11}$$

Combining with Equation 3.7, this becomes

$$E\left[Q_{(N)}\right] = L\left[1 - \frac{\alpha}{\tau_\alpha}\right]. \tag{3.12}$$

Figure 3-4 gives the average auction income as a function of λ for $\alpha = 1, 2$ and 3. From the graph, income differences tend to narrow for high values of α.

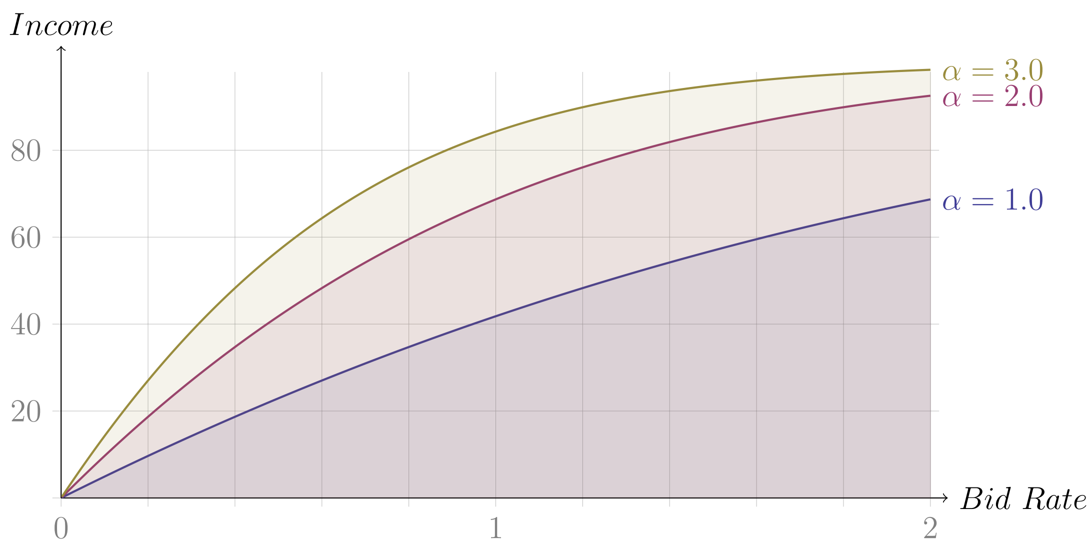

Figure 3-4: Average auction income of variable-time first-price forward auction with fixed inactivity window.

Differentiating the auction income with respect to α, yields

$$\frac{dE\left[Q_{(N)}\right]}{d\alpha} = \frac{\lambda L\left[e^{\lambda\alpha}(\lambda\alpha - 1) + 1\right]}{\left(e^{\lambda\alpha}-1\right)^2}.$$

For the running of the auction to be meaningful, there must be an average of at least one bid arrival per interval of length α, i.e. $\lambda\alpha \geq 1$, so that

$$\frac{dE\left[Q_{(N)}\right]}{d\alpha} = \frac{\lambda L\left[e^{\lambda\alpha}(\lambda\alpha - 1) + 1\right]}{\left(e^{\lambda\alpha} - 1\right)^2} > 0.$$

This shows that the auction income can be increased by augmenting the inactivity window, α. Similarly, differentiating the auction income with respect to λ and by similar argument,

$$\frac{dE\left[Q_{(N)}\right]}{d\lambda} = \frac{\alpha L\left[e^{\lambda\alpha}(\lambda\alpha - 1) + 1\right]}{\left(e^{\lambda\alpha} - 1\right)^2} > 0.$$

Thus, the expected auction income would also increase with higher bid rate. Also, it is interesting to compare the average auction duration of Algorithms 1 and 2. Equating the two auction durations from Equation 3.7 yields

$$T = \frac{e^{\lambda\alpha} - 1}{\lambda}.$$

Solving for α,

$$\alpha = \frac{\log_e(1 + \lambda T)}{\lambda}.$$

That is, when α equals the above value, then average auction durations of the two algorithms are equal. For example, $\lambda = 1$ and $T = 5$ gives $\alpha = \log_e 6 = 1.8$; $\lambda = 1$ and $T = 10$ gives $\alpha = \log_e 11 = 2.4$, and $\lambda = 1$ and $T = 15$ gives $\alpha = \log_e 16 = 2.8$. Thus, halving T would not require a proportionate reduction of α, since a relatively small reduction of α may lead to a relatively substantial reduction in the overall average auction duration. Conversely, a small increase in α may lead to a substantial increase in the auction duration since the bid arrival occurring in the interval increase, $\Delta\alpha$, may cause the entire auction process to be repeated all over again. As shown earlier, the average auction income for Algorithm II is an increasing function of α, thus, whenever

$$\alpha > \frac{\log_e(1+\lambda T)}{\lambda}, \tag{3.13}$$

the average auction income of Algorithm II will be higher than that of Algorithm I.

3.3 Auction Algorithm III: Fixed-time First-price Forward Auction with Maximum Threshold Termination

Similar to Algorithm I, this auction has an added termination condition which allows the auction to terminate before T time units have elapsed if a sufficiently high bid is received. In some Internet auctions, there is a Buy-It-Now (BIN) option which sets the price at an appropriately high level, such that the seller would be happy to accept that bid and to terminate the auction immediately without letting it run its full course. Let this maximum threshold be M, expressed in currency units, such that whenever an arriving bid is greater or equal to M, the auction mechanism terminates the auction immediately and the bid is accepted.

The threshold M set by the seller may be private to the seller or made public to the bidders. The advantage of keeping it private is to allow the possibility of receiving a bid higher than M. As shall be shown later, the average income from receiving such a bid will fall between M and L. In the first case, the value M is not disclosed to the bidders, and later, in the second case, the results are modified for the Open Buy-It-Now (OBIN) case where M is made known to the bidders.

In the case where the value M is not revealed to the bidders, it shall be referred to as the Closed Buy-It-Now (CBIN). In this dissertation, when neither OBIN or CBIN is specifically indicated, Buy-It-Now (BIN) will generally refer to the Closed Buy-It-Now.

Having a low maximum threshold M will allow the auction to terminate sooner.

The lower the value of M, the shorter will be the auction duration. Setting M much higher than $E\left[Q_{(N)})\right]$ will be unrealistic and will mean that early termination of the auction before T is not likely. Setting M to be very small would mean that the first bid arrival would likely be accepted, thereby triggering the immediate termination of the auction. In general, the average auction duration is less than T (see Algorithm III).

Algorithm 3 Fixed-time first-price forward auction with maximum threshold termination.

Require: $M > 0$ **and** $T > 0$
Ensure: $L > 0$ **and** $accept\_id \neq null$

```
accept_id ← null
clock ← 0
L ← 0
while clock < T do
  while new_bid(t, R, bidder_id) do
    clock ← clock + t
    if clock < T and R > L then
      L ← R
      accept_id ← bidder_id
      if L ≥ M then
        return L, accept_id
      end if
    end if
  end while
end while
return L, accept_id
```

Denote by p, the probability that a given arrival meets the maximum threshold, i.e. $p = \Pr[\text{Bid} \geq M] = 1 - M/L$. The sub-stream, which meets the maximum threshold requirement has arrival rate $\Lambda = \lambda p$, is also a Poisson stream. Likewise, denote by p' the probability that a given arrival does not meet the maximum threshold, i.e. $p' = \Pr[\text{Bid} < M] = M/L$. Therefore, there are two Poisson arrival streams with rates $\Lambda = \lambda p$ and $\Lambda' = \lambda p'$. The first arrival of the Λ sub-stream at time t before T will terminate the auction at t immediately, which happens with probability $\Lambda e^{-\Lambda t} dt$. Averaging over all such arrivals before T gives

$$\int_0^T \Lambda t e^{-\Lambda t} dt = \Lambda \left[\frac{1 - e^{-\Lambda T}(\Lambda T + 1)}{\Lambda^2} \right] = \frac{1 - e^{-\Lambda T}(\Lambda T + 1)}{\Lambda}.$$

On the other hand, if the first arrival from sub-stream Λ occurs after T, then the auction duration will be T, which happens with probability $e^{-\Lambda T}$. Therefore, averaging over both possibilities, the mean auction duration $E[T_M]$ for this case is

$$E\left[T_M\right] = \frac{1 - e^{-\Lambda T}(\Lambda T + 1)}{\Lambda} + Te^{-\Lambda T} = \frac{1 - e^{-\Lambda T}}{\Lambda} = \frac{1 - e^{-\lambda T\left(1 - \frac{M}{L}\right)}}{\lambda\left(1 - \frac{M}{L}\right)}$$

That is,

$$E\left[T_M\right] = \frac{L - Le^{-\lambda T\left(1 - \frac{M}{L}\right)}}{\lambda(L - M)}. \tag{3.14}$$

The limit $E[T_M] \rightarrow (1 - e^{-\lambda T})/\lambda$ as $M \rightarrow 0$ can be interpreted as: if there is an arrival before T, which happens with probability $1 - e^{-\lambda T}$, then the auction lasts for a duration of $1/\lambda$ which is just the time of the first arrival, and this arrival is immediately accepted because it must be greater than M. On the other hand, $E[T_M] \rightarrow T$ as $M \rightarrow L$ (which can be seen by application of L'Hôpital's rule).

To show that the mean auction duration in this case is less than T, i.e. $E\left[T_M\right] < T$, then $\left(1 - e^{-\Lambda T}\right)/\Lambda < T$ must be proven. This is equivalent to $e^{\Lambda T} < 1 + \Lambda T e^{\Lambda T}$ and upon expanding both sides, becomes $1 + \Lambda T + (\Lambda T)^2/2! + \ldots < 1 + \Lambda T + (\Lambda T)^2 + (\Lambda T)^3/2!$, which is seen to be true on a term-by-term comparison. Hence, $E\left[T_M\right] < T$.

Figure 3-5 plots the mean auction duration against the bid rate λ for values of $M = 20, 40, 60$ and 80, for $L = 100$ and $T = 10$. As the bid rate increases, the likelihood of receiving a bid higher than M also increases, and so the average auction duration will drop. The gap between larger values of M (between $M = 60$ and $M = 80$) tends to be greater than that of the smaller values (between $M = 20$ and $M = 40$) when the bid rate approaches unity. This algorithm is able to reduce the auction duration significantly for small to moderate values of M.

Also, differentiating $E\left[T_M\right]$ with respect to M in Equation 3.14, gives

$$\frac{dE\left[T_M\right]}{dM} = \frac{\lambda\left[1 - e^{-\Lambda T}(1 + \Lambda T)\right]}{L\Lambda^2}.$$

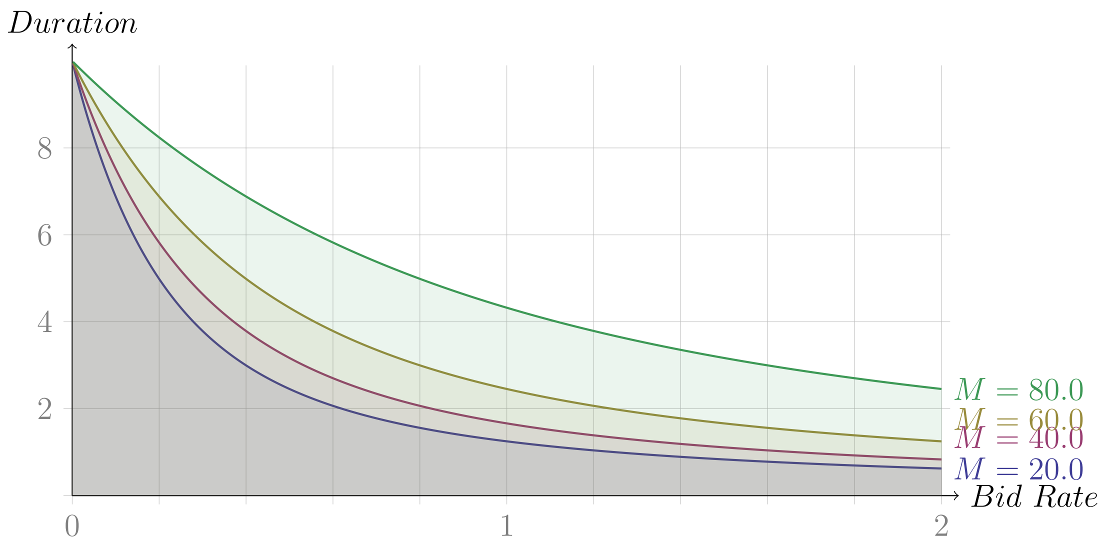

Figure 3-5: Average auction duration of fixed-time first-price forward auction with maximum threshold termination.

Since $(1 + \Lambda T)/e^{\Lambda T} < 1$, through using the expansion as in Equation 3.5, the numerator of the above is positive, showing that

$$\frac{dE\left[T_M\right]}{dM} = \frac{\lambda\left[1 - e^{-\Lambda T}(1 + \Lambda T)\right]}{L\Lambda^2} > 0.$$

Thus, increasing M will increase the duration of the auction. The auction income, on the other hand, given that the auction terminates before T, can be shown to be uniformly distributed over the interval (M, L). Since the auction income must be greater than or equal to M, the auction income is a sample from the following conditional density over the interval $[M, L)$,

$$f(x|x \geq M) = \begin{cases} \frac{1}{(L-M)} & \text{for } M \leq x \leq L, \\ 0 & \text{elsewhere,} \end{cases}$$

where $(L - M)/L$ is the probability that the bid falls in the interval $[M, L)$, and the mean of this density is $M + (L - M)/2 = (M + L)/2$. Thus, the mean income given the auction terminates before T is $(M + L)/2$, which happens with probability $1 - e^{-\Lambda T}$. There is, however, also a chance that none of the arriving bids attain the maximum M, which happens with probability $e^{-\Lambda T}$, and this then accepts the bid with average value $MN/(N+1)$, with N coming from the stream Λ'. Here, L is replaced by M since none of the associated bids are allowed to exceed M. Therefore, the average magnitude of the accepted bid $E[Q_M]$ operating under the given auction termination rule, from Equation 3.3, is

$$E[Q_M] = \frac{(M+L)(1-e^{-\Lambda T})}{2} + \frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + e^{-\Lambda' T} - 1\right), \qquad (3.15)$$

which is plotted in Figure 3-6, for $M = 20, 40, 60$ and 80, with $L = 100$ and $T = 10$.

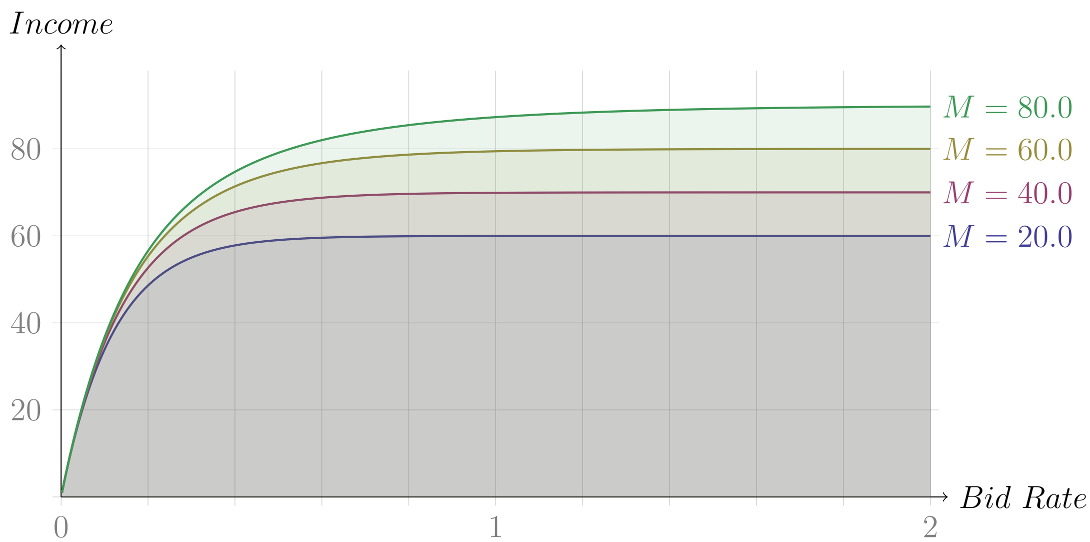

Figure 3-6: Average auction income of fixed-time first-price forward auction with maximum threshold termination.

While the differences in auction income for different M values are about the same for high bid rates, these differences tend to decline for high values of M when the bid rate is low. When $M \to L; \Lambda \to 0; \Lambda' \to \lambda$ then $E[Q_M] \to \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right)$, which is in agreement with the auction income in the non-threshold termination case.

In the Open Buy-It-Now (OBIN) situation, the threshold maximum M is made known to the bidders. While a bidder may be willing to pay an amount higher than M, he/she does not need to do so, as paying M is sufficient to secure the good. Hence, the above average auction income is reduced to, on replacing $(M+L)/2$ by M in the first income component,

$$E\left[Q_M\right] \;=\; M\left(1-e^{-\Lambda T}\right)+\frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T+e^{-\Lambda' T}-1\right).$$

which on simplification gives

$$E\left[Q_M\right] \;=\; M\left[1+\frac{e^{-\Lambda T}\left(e^{-\Lambda' T}-1\right)}{\Lambda' T}\right], \tag{3.16}$$

which is plotted in Figure 3-7 for $M = 20, 40, 60$ and 80, with $L = 100$ and $T = 10$.

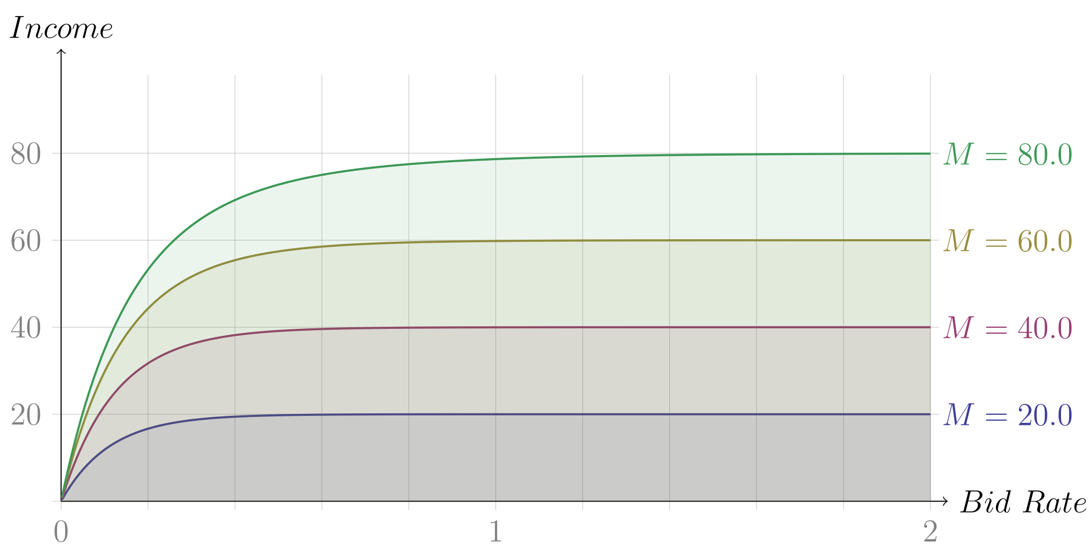

Figure 3-7: Average auction income of fixed-time first-price forward auction with maximum threshold termination and Open Buy-It-Now (OBIN) termination.

3.4 Auction Algorithm IV: Variable-time First-price Forward Auction with Fixed Inactivity Window and Maximum Threshold Termination

In this algorithm, the auction can terminate in two ways: either on the expiry of an inactivity window or on receiving a sufficiently high bid (see Algorithm IV). As in the previous section, the arrival stream contains two sub-streams: $\Lambda = \lambda p$ and $\Lambda' = \lambda p'$, with $\lambda = \Lambda + \Lambda'$, where the first sub-stream, Λ, meets the maximum threshold M, while the second sub-stream, Λ', fails to meet the maximum threshold.

Algorithm 4 Variable-time first-price forward auction with fixed inactivity window and maximum threshold termination.

```
Require: α > 0 and M > 0
Ensure: L > 0 and accept_id ≠ null
  accept_id ← null
  L ← 0
  loop
    while new_bid(t, R, bidder_id) do
      if t < α then
        if R > L then
          L ← R
          accept_id ← bidder_id
          if L ≥ M then
            return  L, accept_id
          end if
        end if
      else
        return  L, accept_id
      end if
    end while
  end loop
```

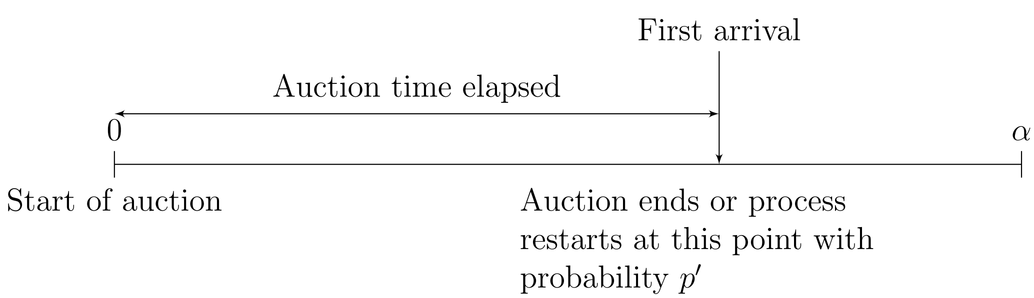

Figure 3-8: Events in the first interval of length $(0, \alpha)$.

Focusing on the first interval of length $(0, \alpha)$ (see Figure 3-8), denote the average auction duration by $E[\tau_M]$. Assuming there is an arrival in this interval, which has probability $1 - e^{-\lambda\alpha}$, its expected time of arrival is

$$\int_0^\alpha \frac{\lambda t e^{-\lambda t}}{1 - e^{-\lambda\alpha}} dt.$$

The denominator $1 - e^{-\lambda\alpha}$ is present since it is given that an arrival takes place within $(0, \alpha)$, and so the conditional density,

$$g(t|t < \alpha) = \begin{cases} \frac{\lambda e^{-\lambda t}}{1-e^{-\lambda\alpha}} & \text{for } 0 < t < \alpha, \\ 0 & \text{for } t \geq \alpha, \end{cases}$$

is used, and this conditional average is given by

$$\int_0^\alpha tg(t|t < \alpha)dt \;\; = \;\; \frac{1 - e^{-\lambda\alpha}(\lambda\alpha + 1)}{\lambda\left(1 - e^{-\lambda\alpha}\right)}.$$

Consider this first arrival: if it comes from the Λ stream, then the auction terminates immediately and no further auction time is incurred. If it comes from the Λ' stream, however, which happens with probability p', then further auction time will be incurred and the process effectively restarts. That is, the additional average auction time incurred will be $p'E\left[\tau_M\right]$. Thus, given there is an arrival in $(0, \alpha)$, the expected auction duration is

$$\frac{1 - e^{-\lambda\alpha}(\lambda\alpha + 1)}{\lambda\left(1 - e^{-\lambda\alpha}\right)} + p'E\left[\tau_M\right],$$

where the first term corresponds to the auction time incurred, irrespective of whether it is an arrival from the Λ stream or an arrival from the Λ' stream. Next, if there is no arrival in this interval, then the auction duration is simply α (i.e. an empty auction), which happens with probability $e^{-\lambda\alpha}$.

Thus, averaging over the two possibilities of

1. having an arrival in the first interval of length $(0, \alpha)$, and

2. having no arrival in that interval,

it can be determined that

$$E\left[\tau_M\right] \;\; = \;\; \left(1 - e^{\lambda\alpha}\right)\left\{\frac{1 - e^{-\lambda\alpha}(\lambda\alpha + 1)}{\lambda\left(1 - e^{-\lambda\alpha}\right)} + p'E\left[\tau_M\right]\right\} + \alpha e^{-\lambda\alpha}.$$

Solving for $E\left[\tau_M\right]$ yields

$$E\left[\tau_M\right] \;=\; \frac{1-e^{-\lambda\alpha}}{\lambda\left[p+p'e^{-\lambda\alpha}\right]}, \tag{3.17}$$

which is plotted in Figure 3-9 for $M = 20, 40, 60$ and 80, with $L = 100$ and $\alpha = 3$.

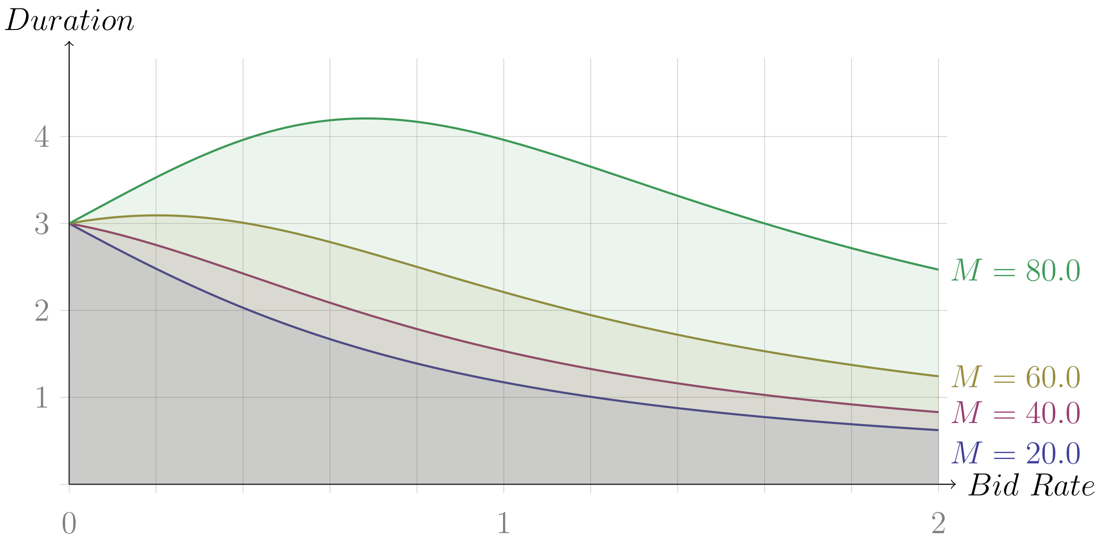

Figure 3-9: Average auction duration of variable-time first-price forward auction with fixed inactivity window and maximum threshold termination.

Note that as $M \to L$, $p' \to 1$,

$$E\left[\tau_M\right] \;\to\; \frac{1-e^{-\lambda\alpha}}{\lambda e^{-\lambda\alpha}} = \frac{e^{\lambda\alpha}-1}{\lambda},$$

which is in agreement with Equation 3.7.

To show that $E\left[\tau_M\right] < \tau_\alpha$, Equation 3.17 is multiplied by $e^{\lambda\alpha}$ both in the numerator and in the denominator, yielding

$$E\left[\tau_M\right] \;=\; \frac{e^{\lambda\alpha}-1}{\lambda\left(p'+pe^{\lambda\alpha}\right)}.$$

Since $e^{\lambda\alpha} > 1$ (see Equation 3.5),

$$p' + pe^{\lambda\alpha} \;>\; p' + p = 1.$$

Thus,

$$E\left[\tau_M\right] \quad = \quad \frac{e^{\lambda\alpha}-1}{\lambda\left(p'+pe^{\lambda\alpha}\right)} < \frac{e^{\lambda\alpha}-1}{\lambda},$$

giving $E\left[\tau_M\right] < \tau_\alpha$.

As for the average auction income, consider the final interval of length, $(0, \alpha)$. For the auction to terminate, there are the two possibilities of

1. no bid arrival in this interval, which occurs with probability $p_1 = e^{-\lambda\alpha}$, and

2. a bid arrival in this interval of a magnitude that falls between M and L, and that this bid is the first bid to arrive in this interval—if this bid is not the first bid to arrive, then this is no longer the final interval (let p_2 denote the probability of occurrence of this second possibility).

Focusing on the second possibility, let the inter-arrival time of the first sub-stream, Λ, which meets the maximum threshold and whose arrival will terminate the auction immediately, be $X \sim \Lambda e^{-\Lambda t}$, and the inter-arrival time of the second sub-stream, Λ', which fails to meet the maximum threshold and will allow the auction to continue, be $Y \sim \Lambda' e^{-\Lambda' t}$. Thus, the probability p_2 that the second probability occurs is:

$$p_2 \quad = \quad \Pr[X < \alpha] \times \left\{\Pr[X < Y < \alpha] + \Pr[Y \geq \alpha]\right\},$$

where $\Pr[X < \alpha]$ is the probability that a Λ-stream arrival occurs in the interval, $\Pr[X < Y < \alpha]$ is the probability that there is a Λ'-stream arrival also in the interval but it occurs after the Λ'-stream arrival, and $\Pr[Y \geq \alpha]$ is the probability that there is no Λ'-stream arrival in the interval. That is,

$$p_2 \quad = \quad \left(1 - e^{-\Lambda\alpha}\right)\left(\Pr[X < Y < \alpha] + e^{-\Lambda'\alpha}\right).$$

With reference to Figure 3-10, to evaluate $\Pr[X < Y < \alpha]$, let $t \in (0, \alpha)$ be the arrival time of the Λ-stream, then the probability of this is $\Lambda e^{-\Lambda t}dt$. Next, the probability of arrival of the Λ'-stream in the interval (t, α) is given by

$$\int_t^\alpha \Lambda' e^{-\Lambda' t} dt \;=\; e^{-\Lambda' t} - e^{-\Lambda' \alpha}.$$

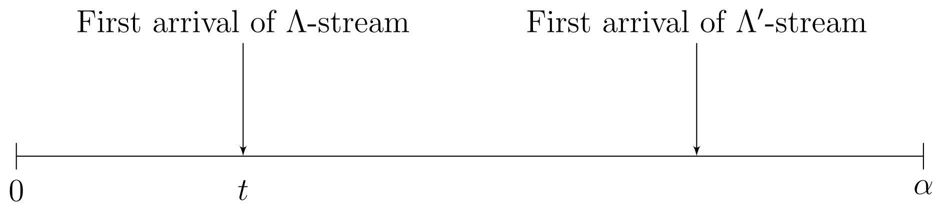

Figure 3-10: Events in the final interval of length $(0, \alpha)$.

Averaging over all $t \in (0, \alpha)$,

$$\begin{aligned}
\Pr[X < Y < \alpha] &= \int_0^\alpha \Lambda e^{-\Lambda t}\left(e^{-\Lambda' t} - e^{-\Lambda'\alpha}\right) dt \\
&= \int_0^\alpha \Lambda e^{-(\Lambda+\Lambda')t} dt - \int_0^\alpha \Lambda e^{-\Lambda t}\left(e^{-\Lambda'\alpha}\right) dt \\
&= \left[\frac{-\Lambda}{\Lambda+\Lambda'} e^{-(\Lambda+\Lambda')t}\right]_0^\alpha + e^{-\Lambda'\alpha}\left[e^{-\Lambda t}\right]_0^\alpha \\
&= \frac{\Lambda}{\Lambda+\Lambda'}\left(1 - e^{-(\Lambda+\Lambda')\alpha}\right) + e^{-\Lambda'\alpha}\left(e^{-\Lambda\alpha} - 1\right) \\
&= \frac{\Lambda}{\Lambda+\Lambda'}\left(1 - e^{-(\Lambda+\Lambda')\alpha}\right) + \left(e^{-(\Lambda+\Lambda')\alpha} - e^{-\Lambda'\alpha}\right) \\
&= \left(1 - \frac{M}{L} + \frac{M}{L} e^{-\lambda\alpha}\right) - e^{-\Lambda'\alpha}.
\end{aligned}$$

Upon simplification, the probability of occurrence of the second possibility p_2 is

$$p_2 \;=\; \left(1 - e^{-\Lambda\alpha}\right)\left(1 - \frac{M}{L} + \frac{M}{L} e^{-\lambda\alpha}\right).$$

Therefore, the auction income using Equation 3.11 replacing L by M and λ by Λ', and conditioning on these two possibilities becomes

$$\begin{aligned}
E\left[Q'_M\right] &= \frac{p_1\left[M\left(1 - \frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right)\right] + p_2\left(\frac{M+L}{2}\right)}{p_1 + p_2} \\
&= \frac{e^{-\lambda\alpha}\left[M\left(1 - \frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right)\right] + \left(1 - e^{-\Lambda\alpha}\right)\left(1 - \frac{M}{L} + \frac{M}{L}e^{-\lambda\alpha}\right)\left(\frac{M+L}{2}\right)}{e^{-\lambda\alpha} + \left(1 - e^{-\Lambda\alpha}\right)\left(1 - \frac{M}{L} + \frac{M}{L}e^{-\lambda\alpha}\right)},
\end{aligned} \tag{3.18}$$

which is plotted in Figure 3-11 for $M = 20, 40, 60$ and 80, with $L = 100$ and $\alpha = 3$.

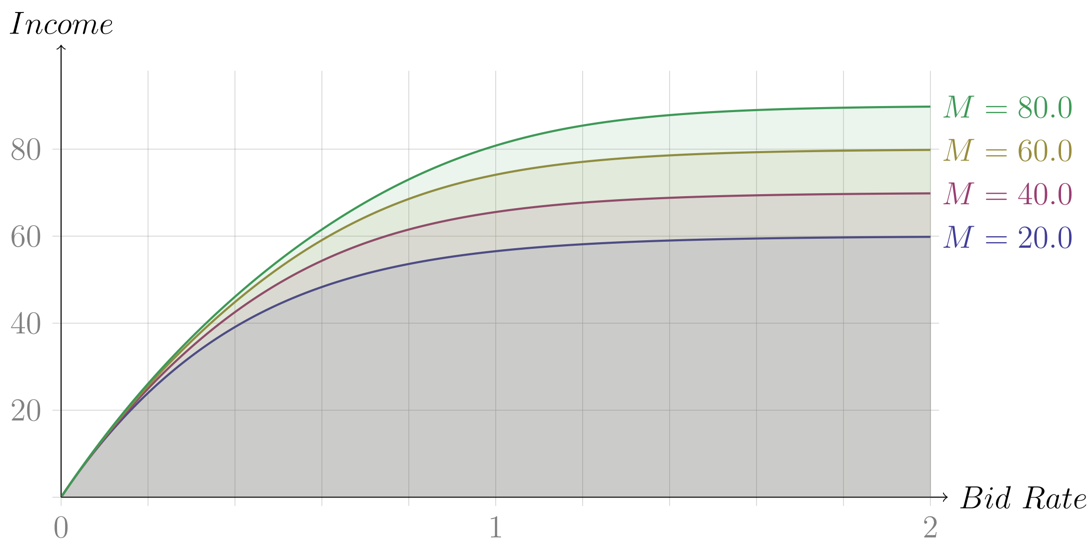

Figure 3-11: Average auction income of variable-time first-price forward auction with fixed inactivity window and maximum threshold termination.

Note that as $M \to L$; $\Lambda \to 0$; $\Lambda' \to \lambda$,

$$E\left[Q'_M\right] \quad \to \quad \frac{L}{e^{\lambda\alpha} - 1}\left(e^{\lambda\alpha} - \lambda\alpha - 1\right),$$

which is in agreement with Equation 3.11. The corresponding Open Buy-It-Now income is

$$E\left[Q'_M\right] \quad = \quad \frac{e^{-\lambda\alpha}\left[M\left(1 - \frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right)\right] + M\left(1 - e^{-\Lambda\alpha}\right)\left(1 - \frac{M}{L} + \frac{M}{L}e^{-\lambda\alpha}\right)}{e^{-\lambda\alpha} + \left(1 - e^{-\Lambda\alpha}\right)\left(1 - \frac{M}{L} + \frac{M}{L}e^{-\lambda\alpha}\right)}, \quad (3.19)$$

which is plotted in Figure 3-12 for $M = 20, 40, 60$ and 80, with $L = 100$ and $\alpha = 3$.

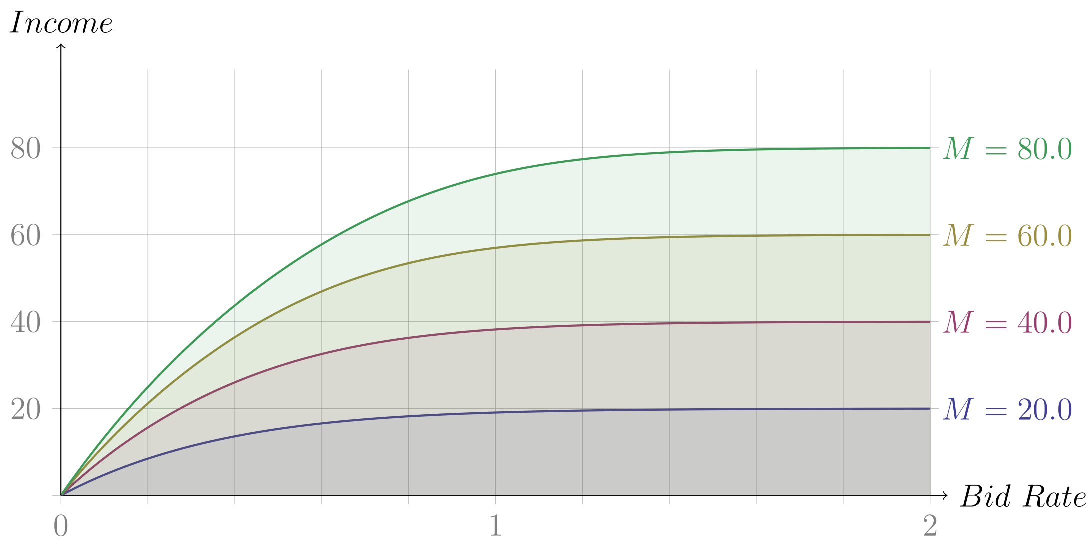

Figure 3-12: Average auction income of variable-time first-price forward auction with fixed inactivity window and maximum threshold termination and Open Buy-It-Now (OBIN) termination.

3.5 Auction Algorithm V: Variable-time First-price Forward Auction with Bid Enumeration Termination

In Algorithm I, it was mentioned that for effective auction operation, the condition, that $T \gg 1/\lambda$, should be met in order to ensure that there are enough bids in the auction. Despite this, sometimes empty auctions can occur, where there are no bid arrivals, and which can somewhat defeat the purpose of the auctioning mechanism. For example, for Algorithm I, having no bid arrival in $(0, T)$ can occur with probability $e^{-\lambda T}$, and for Algorithm II, the auction can close having no bid arrival with probability $e^{-\lambda\alpha}$. Thus, auction time is expended without achieving any income.

An algorithm that overcomes this empty auction problem is to terminate an auction only after a specified number of bids are received. If B is the required number of bid arrivals, then the auction will terminate on the arrival of the B-th bid, and the highest bid is chosen (see Algorithm V).

The average magnitude of the accepted bid, from results of order statistics (Barry

Algorithm 5 First-price forward auction with bid enumeration termination.

Require: $B > 0$
Ensure: $L > 0$ **and** $accept\_id \neq null$
  $accept\_id \leftarrow null$
  $L \leftarrow 0$
  while $B > 0$ **do**
    while $new_bid(t, R, bidder\_id)$ **do**
      $B \leftarrow B - 1$
      if $R > L$ **then**
        $L \leftarrow R$
        $accept\_id \leftarrow bidder\_id$
      end if
    end while
  end while
  return $L, accept\_id$

C. Arnold et al., 2008), is

$$E[Q_{(B)}] = \frac{BL}{B+1}, \tag{3.20}$$

and is plotted in Figure 3-13 for $B = 5, 10$ and 15. Here, the bid rate does not directly determine the auction income, but it has a direct impact on the duration of the auction.

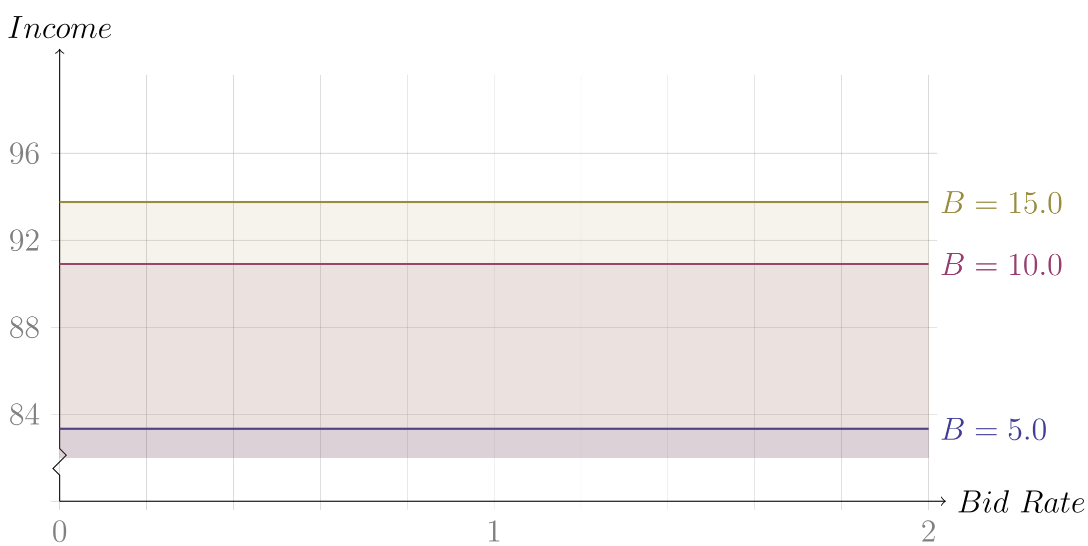

Figure 3-13: Average auction income of variable-time first-price forward auction with bid enumeration termination.

The average auction duration in this case is simply the sum of B average inter-arrival times, or B/λ, and is plotted in Figure 3-14 for $B = 5, 10$ and 15.

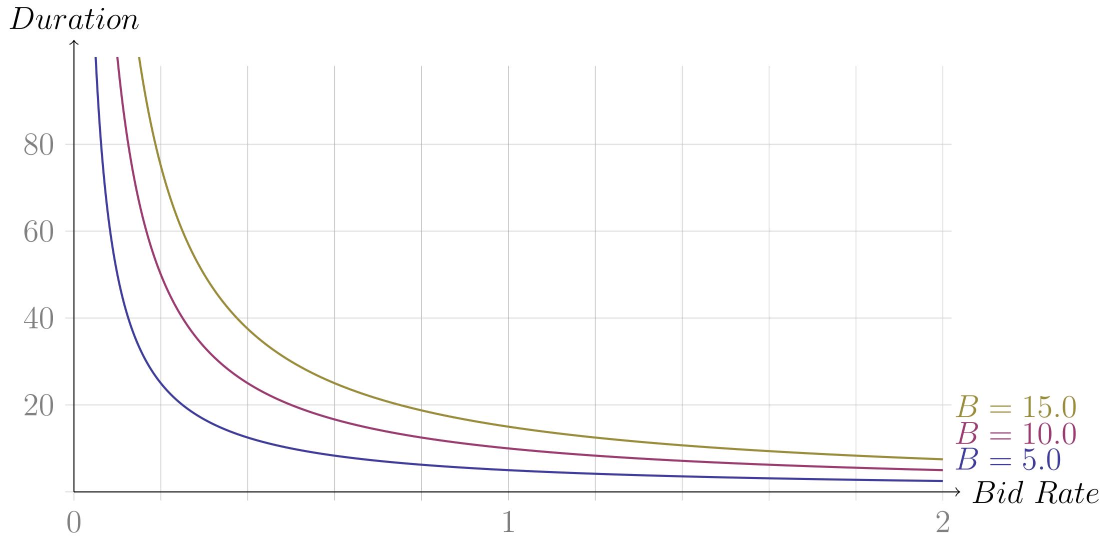

Figure 3-14: Average auction duration of variable-time first-price forward auction with bid enumeration termination.

Looking at duration, it is interesting to compare the average auction duration of Algorithm V with those of Algorithms 1 and 2. If $B/\lambda < T$ or $B < \lambda T$, then the average auction duration of Algorithm V will be shorter than that of Algorithm I. For the average duration of Algorithm 5 to be shorter than that of Algorithm II, the following condition must be fulfilled:

$$e^{\lambda\alpha} - 1 \quad > \quad B,$$

giving

$$\alpha \quad > \quad \frac{\log_e(1+B)}{\lambda}.$$

For example, for $B = 8$ and $\lambda = 0.5$, for the average auction duration of Algorithm V to be longer than that of Algorithm V, α must be larger than 4.4.

Looking at income, the average auction income of AAlgorithm V is an increasing function of B because

$$\frac{dE\left[Q_{(B)}\right]}{dB} \quad = \quad \frac{L}{(B+1)^2} > 0.$$

For the average auction income of Algorithm V to be less than that of Algorithm I, the following condition must be fulfilled:

$$\frac{BL}{B+1} \quad < \quad \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right),$$

giving

$$B \quad < \quad \frac{\lambda T}{1 - e^{-\lambda T}} - 1.$$

Similarly, for the average auction income of Algorithm V to be less than that of Algorithm II, the following condition must be fulfilled:

$$\frac{BL}{B+1} \quad < \quad \frac{L}{e^{\lambda\alpha} - 1}\left(e^{\lambda\alpha} - \lambda\alpha - 1\right),$$

giving

$$B \quad < \quad \frac{e^{\lambda\alpha} - 1}{\lambda\alpha} - 1.$$

3.6 Results Summary

This chapter has studied the properties of five Internet auction algorithms exhibiting different characteristics. Closed-form expressions are obtained for the average auction duration, and average auction income, which are the key measures of Internet auction performance. These results are summarised in Table 3.1.

For the fixed-time first-price forward auction, as expected, an increase in the number of bids results in a corresponding increase in average auction income. The size of the average bid rate is a strong determinant of the amount of income obtained in the auction and an increase in bid rate up to a certain point produces a rather marked increase in auction income rate, though there seems to be a critical bid rate above which the increase in auction income rate slows down.

For the variable-time first-price forward auction with fixed inactivity window, the inactivity window, α, should not be significantly smaller than the mean bid inter-arrival time, otherwise the auction will end without any or many bids. In the same way, if it is much larger than the mean bid inter-arrival time, the auction may drag on indefinitely with minimum impact on the income obtained in the auction.

For the fixed-time first-price forward auction with maximum threshold termination, similar to the Buy-It-Now (BIN) option in *eBay*, the threshold, M, needs to be set carefully. Setting M much higher than average auction income will mean that early termination of the auction before T is unlikely, while setting M to be very small would mean that the first bid is likely to be accepted, which is the same as not conducting an auction at all.

For the variable-time first-price forward auction with fixed inactivity window and maximum threshold termination, varying the bid rate makes for very interesting results. As the bid rate increases, there is an interplay of two forces that affect the duration of the auction: the auction duration lengthens as a result of an increased probability that a bid arrives in the α interval; the auction duration shortens as the probability of a bid hitting the maximum threshold is increased. This interplay can be seen in Figure 3-9.

Algorithm	Average Auction Duration	Average Offer Accepted Per Auction
I	T	$\frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right)$
II	$\frac{e^{\lambda\alpha}-1}{\lambda}$	$\frac{L}{e^{\lambda\alpha}-1}\left[e^{\lambda\alpha} - \lambda\alpha - 1\right]$
III	$\frac{L-Le^{-\lambda T\left(1-\frac{M}{L}\right)}}{\lambda(L-M)}$	$\frac{(M+L)\left(1-e^{-\Lambda T}\right)}{2} + \frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + e^{-\Lambda' T} - 1\right)$ **[CBIN]**
	$\frac{L-Le^{-\lambda T\left(1-\frac{M}{L}\right)}}{\lambda(L-M)}$	$M\left(1 - e^{-\Lambda T}\right) + \frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + e^{-\Lambda' T} - 1\right)$ **[OBIN]**
IV	$\frac{1-e^{-\lambda\alpha}}{\lambda\left[p+p'e^{-\lambda\alpha}\right]}$	$\frac{e^{-\lambda\alpha}\left[M\left(1-\frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right)\right]+\left(1-e^{-\Lambda\alpha}\right)\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)\left(\frac{M+L}{2}\right)}{e^{-\lambda\alpha}+(1-e^{-\Lambda\alpha})\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)}$ **[CBIN]**
	$\frac{1-e^{-\lambda\alpha}}{\lambda\left[p+p'e^{-\lambda\alpha}\right]}$	$\frac{e^{-\lambda\alpha}\left[M\left(1-\frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right)\right]+M\left(1-e^{-\Lambda\alpha}\right)\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)}{e^{-\lambda\alpha}+(1-e^{-\Lambda\alpha})\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)}$ **[OBIN]**
V	$\frac{B}{\lambda}$	$\frac{BL}{B+1}$

Table 3.1: First-price forward auctions.

Chapter 4

Analysis of Vickrey and Reverse Auction Algorithms

The previous chapter studied Internet auction algorithms which were first-price forward auctions. As Internet auctions can take on a variety of forms, this chapter studies other Internet auction algorithms which are prevalent on the Internet. There are two main classes of algorithms: the first is related to Vickrey auctions, and the second is related to reverse auctions.

Vickrey auctions are commonly used for the reason that it gives bidders an incentive to bid an item's true value without needing to worry about overpaying. For markets where liquidity is an issue, Vickrey auctions can be dangerous. An example of this is demonstrated in a New Zealand spectrum auction, where there were only to participants. While the first bid ran into the hundreds of thousands of dollars, the second bid was a token bid in the tens of dollars. As this was a Vickrey auction, the winner received a discount of four orders of magnitude, a far cry from the true value that the buyer was willing to pay.

Reverse auctions are used for for obtaining low prices. This is often used when supply outstrips demand. It is frequently used in procurement where a government or private sector organisation wishes to obtain goods or services and the sellers compete against each other in providing the goods or services at increasingly lower prices.

Note that this is different from a Dutch auction, which some also consider to be a reverse auction. In a Dutch auction, the auction still seeks to obtain as high a price as possible and involves an auctioneer starting at a very high asking price that is continually lowered until some bidder is willing to accept the offer. The advantage of this is that auctions can happen very fast as it only takes a single bid to end the auction; it plays off bidders' fear of losing out and their drive towards a lower price.

4.1 Auction Algorithm VI: Fixed-time Vickrey Forward Auction

Various forms of Vickrey auctions, which can also be called second-price auctions, are commonly found on the Internet. In *eBay*'s system of proxy bidding, for example, the auction winner does not pay the highest bid, but the value of the second-highest bid plus a bid increment. Variations exist for the increment charged by the auction mechanism which may be a fixed increment, or one that depends on the value of the relevant bid. An advantage of Vickrey auctions is that it gives bidders an incentive to bid an item's true value without needing to worry about overpaying. As in Algorithm I, the auction time here is fixed with duration T. For effective auction operation, one would need to have at least two bids, and thus T should be significantly greater than twice the mean inter-arrival time $1/\lambda$. If N is the number of bid arrivals in the time interval $(0, T)$, then at the close of the auction, the auctioning mechanism will pick the bidder of maximum bid $Q_{(N)}$ to be the winner while the amount the winner pays will be $Q_{(N-1)}$ (see Algorithm VI).

From the results of order statistics (Barry C. Arnold et al., 2008), it can be shown that the conditional income per auction, or the second-largest order statistic, is

$$E[Q_{(N-1)}|\text{Number of bids} = N] = \frac{(N-1)L}{N+1} \tag{4.1}$$

To remove the condition on N, three cases are considered:

1. $N = 0$. Here the income is zero.

Algorithm 6 Fixed-time Vickrey forward auction.

Require: $\delta > 0$ **and** $T > 0$
Ensure: $L2 + \delta > 0$ **and** $accept\_id1 \neq null$

```
accept_id1 ← null
accept_id2 ← null
clock ← 0
L1 ← 0
L2 ← 0
while clock < T do
   while new_b id(t, R, bidder_id) do
      clock ← clock + t
      if clock < T then
         if R > L1 then
            L2 ← L1
            L1 ← R
            accept_id2 ← accept_id1
            accept_id1 ← bidder_id
         else if R > L2 then
            L2 ← R
            accept_id2 ← bidder_id
         end if
      end if
   end while
end while
return L2 + δ, accept_id1
```

2. $N = 1$. Here the income is zero plus an appropriate bid increment δ_o.

3. $N \geq 2$. Here the Vickrey auction functions normally, and a bid increment δ will apply.

Disregarding the bid increments temporarily, since Equation 4.1 gives zero for $N = 1$, cases (ii) and (iii) can be combined and the equation can be summed from $N = 1$, i.e.

$$\begin{aligned} E\left[Q_{(N-1)}\right] &= \sum_{N=1}^{\infty} \frac{(N-1)L}{N+1} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\ &= \sum_{N=1}^{\infty} \left[L - \frac{2L}{N+1}\right] \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\ &= \sum_{N=1}^{\infty} L \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} - \sum_{N=1}^{\infty} \frac{2L}{N+1} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!}. \end{aligned}$$

Using the same technique as Equation 3.2,

$$\begin{aligned} E\left[Q_{(N-1)}\right] &= L\left(1 - e^{-\lambda T}\right) - \frac{2Le^{-\lambda T}}{\lambda T}\left(e^{\lambda T} - 1 - \lambda T\right) \\ &= \frac{L}{\lambda T}\left(\lambda T + \lambda T e^{-\lambda T} + 2e^{-\lambda T} - 2\right). \end{aligned} \tag{4.2}$$

Thus, incorporating the bid increments of the last two cases above, results in an income of

$$\lambda T e^{-\lambda T}\delta_o + \frac{L}{\lambda T}\left(\lambda T + \lambda T e^{-\lambda T} + 2e^{-\lambda T} - 2\right) + \delta, \tag{4.3}$$

where the first term corresponds to the case $N = 1$. This is plotted in Figure 4-1, for $T = 5, 10$ and 15, with $L = 100$ and $\delta_o = \delta = 0$.

It is useful to compare the average auction incomes of Algorithms 1 and 6. The difference between the two average auction incomes is

$$E\left[\Delta_1 Q\right] = E\left[Q_{(N)} - Q_{(N-1)}\right] = \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - \frac{L}{\lambda T}\left(\lambda T + \lambda T e^{-\lambda T} + 2e^{-\lambda T} - 2\right).$$

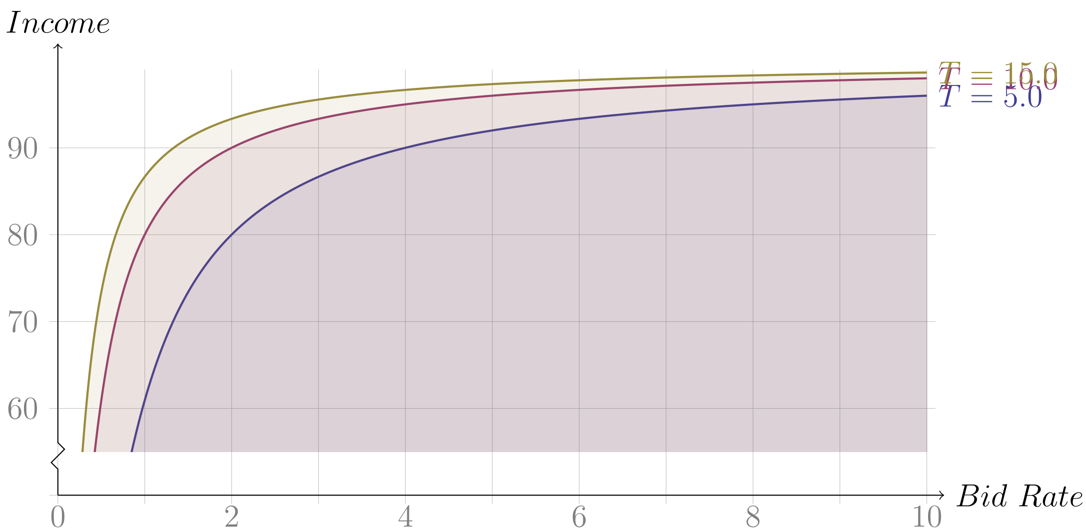

Figure 4-1: Average auction income of fixed-time Vickrey forward auction.

Simplifying gives

$$E\left[\Delta_1 Q\right] \quad = \quad \frac{L}{\lambda T}\left[1 - e^{-\lambda T}(1+\lambda T)\right]. \tag{4.4}$$

Since $(1+\lambda T)/e^{\lambda T} < 1$, which can be seen on expanding $e^{\lambda T}$ for $\lambda T > 0$, it can be seen that $E\left[\Delta_1 Q\right]$ is always greater than zero. As the average number of bids becomes very small, however, on application of L'Hôpital's rule on Equation 4.4, it can be seen that

$$E\left[\Delta_1 Q\right] \rightarrow 0 \text{ as } \lambda T \rightarrow 0$$

On the other hand, it is also interesting to see that as the average number of bids becomes very large, the same limit is approached.

$$E\left[\Delta_1 Q\right] \rightarrow 0 \text{ as } \lambda T \rightarrow \infty$$

To see this, Equation 4.4 can be rewritten as

$$E\left[\Delta_1 Q\right] \quad = \quad L\left[\frac{1}{\lambda T} - \frac{e^{-\lambda T}}{\lambda T} - e^{-\lambda T}\right]. \tag{4.5}$$

Thus, as $\lambda T \to \infty$, the above approaches zero. The reason for this is that, as the average number of bids becomes very large, the average auction income in both cases will approach the same limit L such that there is very little difference between them, i.e.

$$
\begin{aligned}
E\left[Q_{(N)}\right] &= \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) \to L \text{ as } \lambda T \to \infty \\
E\left[Q_{(N-1)}\right] &= \frac{L}{\lambda T}\left(\lambda T + \lambda T e^{-\lambda T} + 2e^{-\lambda T} - 2\right) \to L \text{ as } \lambda T \to \infty
\end{aligned}
$$

A comparison of $E\left[Q_{(N)}\right]$ and $E\left[Q_{(N-1)}\right]$ is plotted in Figure 4-2, for $T = 10$, with $L = 100$, $T = 10$ and $\delta_0 = \delta = 0$. The difference between the two is significant for low bid rates but narrows as the bid rate increases.

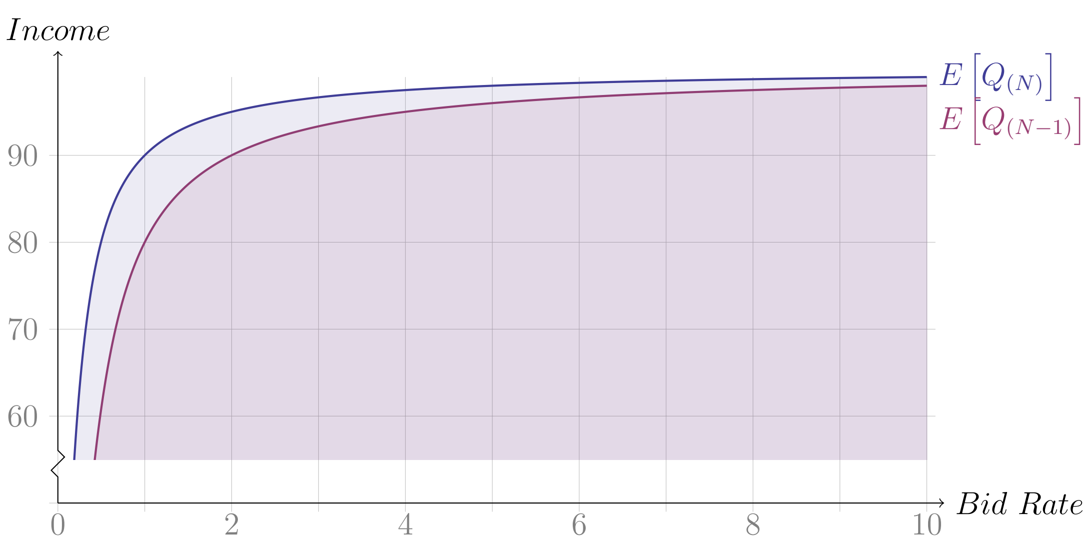

Figure 4-2: Average auction income comparison between the fixed-time first-price forward auction and the fixed-time Vickrey forward auction.

4.2 Auction Algorithm VII: Variable-time Vickrey Forward Auction with Fixed Inactivity Window

Here, like Algorithm II, the auction will terminate when there is no bid arrival for a fixed window of length α. On termination, the second-largest bid received will be accepted (see Algorithm VII).

Algorithm 7 Variable-time Vickrey forward auction with fixed inactivity window.

Require: $\alpha > 0$ **and** $\delta > 0$
Ensure: $L2 + \delta > 0$ **and** $accept\_id1 \neq null$

```
accept_id1 ← null
accept_id2 ← null
L1 ← 0
L2 ← 0
loop
  while new_bid(t, R, bidder_id) do
    if t < α then
      if R > L1 then
        L2 ← L1
        L1 ← R
        accept_id2 ← accept_id1
        accept_id1 ← bidder_id
      else if R > L2 then
        L2 ← R
        accept_id2 ← bidder_id
      end if
    else
      return L2 + δ, accept_id1
    end if
  end while
end loop
```

Ignoring the bid increments temporarily, the condition on N in Equation 4.1 is removed in order to determine the average income $E\left[Q_{(N-1)}\right]$. As in Algorithm II, the following probabilities are used:

$$\Pr[T_1 < \alpha, T_2 < \alpha, \ldots, T_N < \alpha, T_{N+1} \geq \alpha] = \left(1 - e^{-\lambda\alpha}\right)^N \times e^{-\lambda\alpha}$$

Summing from $N = 1$ as in the previous algorithm,

$$
\begin{aligned}
E\left[Q_{(N-1)}\right] &= \sum_{N=1}^{\infty} \frac{(N-1)L}{N+1} \times (1-e^{-\lambda\alpha})^N e^{-\lambda\alpha} \\
&= \sum_{N=1}^{\infty} \left[L - \frac{2L}{N+1}\right] \times \left(1-e^{-\lambda\alpha}\right)^N e^{-\lambda\alpha} \\
&= \sum_{N=1}^{\infty} L\left(1-e^{-\lambda\alpha}\right)^N e^{-\lambda\alpha} - \frac{2Le^{-\lambda\alpha}}{(1-e^{-\lambda\alpha})} \sum_{N=1}^{\infty} \frac{\left(1-e^{-\lambda\alpha}\right)^{N+1}}{N+1} \\
&= L\left(1-e^{-\lambda\alpha}\right) - \frac{2Le^{-\lambda\alpha}}{(1-e^{-\lambda\alpha})} \sum_{N=1}^{\infty} \frac{\left(1-e^{-\lambda\alpha}\right)^{N+1}}{N+1}
\end{aligned}
$$

Using the logarithmic series given in Equation 3.10, this gives,

$$
E\left[Q_{(N-1)}\right] = L\left(1-e^{-\lambda\alpha}\right) - \frac{2L}{e^{\lambda\alpha}-1}\left[\lambda\alpha + e^{-\lambda\alpha} - 1\right].
$$

On simplification and grouping terms, and on noting that $(e^x - e^{-x})/2$ may be simply written as $\sinh(x)$, we have

$$
E\left[Q_{(N-1)}\right] = \frac{2L}{e^{\lambda\alpha}-1}\left[\sinh\lambda\alpha - \lambda\alpha\right]. \tag{4.6}
$$

Incorporating the bid increments, this yields an expected auction income of

$$
\left(1-e^{-\lambda\alpha}\right) \times e^{-\lambda\alpha}\delta_o + \frac{2L}{e^{\lambda\alpha}-1}\left[\sinh\lambda\alpha - \lambda\alpha\right] + \delta. \tag{4.7}
$$

Figure 4-3 shows the auction income for different values of λ for $L = 100$, $\alpha = 1, 2$ and 3, $\delta_o = \delta = 0$. Note that the average auction income rises rather rapidly for small values of λ, but stabilises and yields only slight improvement as the bid rate increases beyond 2.

It is useful to compare the average auction incomes of Algorithms 2 and 7. The difference between the two average auction incomes is

$$
E\left[\Delta_2 Q\right] = E\left[Q_{(N)} - Q_{(N-1)}\right] = \frac{L}{e^{\lambda\alpha}-1}\left[e^{\lambda\alpha} - \lambda\alpha - 1\right] - \frac{2L}{e^{\lambda\alpha}-1}\left[\sinh\lambda\alpha - \lambda\alpha\right].
$$

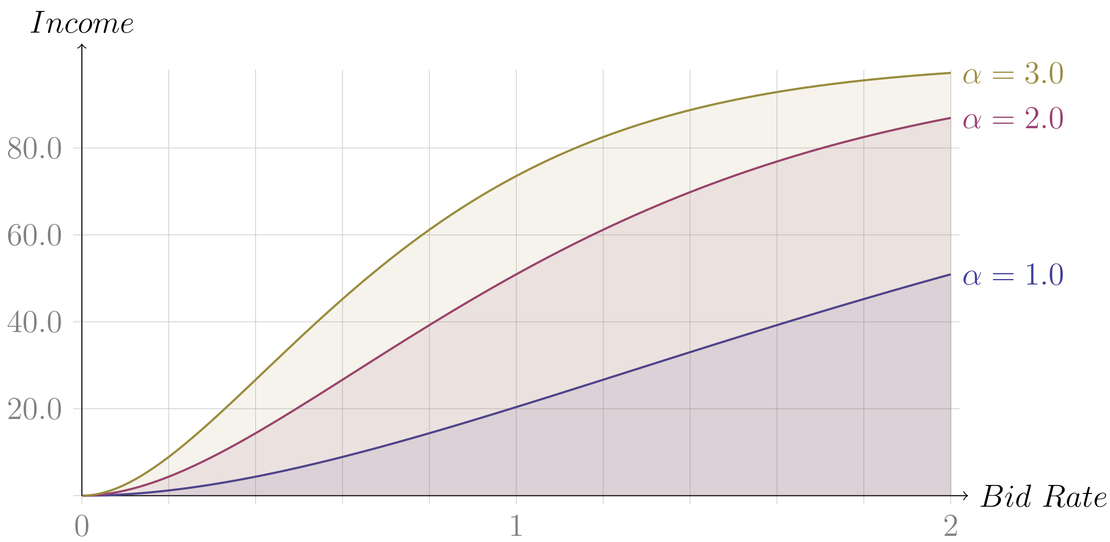

Figure 4-3: Average auction income of variable-time Vickrey forward auction with fixed inactivity window.

Simplifying gives

$$E[\Delta_2 Q] \;=\; E\left[Q_{(N)} - Q_{(N-1)}\right] = \frac{L}{e^{\lambda\alpha} - 1}\left[e^{-\lambda\alpha} + \lambda\alpha - 1\right]. \qquad (4.8)$$

Since $e^{\lambda\alpha} > 1$ and the numerator will be positive if

$$e^{-\lambda\alpha} + \lambda\alpha - 1 \;>\; 0,$$

which is equivalent to

$$\alpha \;>\; \frac{1 - e^{-\lambda\alpha}}{\lambda},$$

and this inequality has been shown earlier in Section 3.3 to be valid. Thus,

$$E[\Delta_2 Q] \;>\; 0.$$

As the average number of bids becomes very small, however, on application of

L'Hôpital's rule on Equation 4.8, it can be seen that

$$E\left[\Delta_2 Q\right] \to 0.$$

On the other hand, as the average number of bids per window $\lambda\alpha$ approaches infinity, the first and third terms of Equation 4.8 tends toward zero. As for the second term,

$$\frac{\lambda\alpha}{e^{\lambda\alpha}-1} = \frac{\lambda\alpha}{\lambda\alpha + \frac{(\lambda\alpha)^2}{2!} + \frac{(\lambda\alpha)^3}{3!} + \ldots} \to 0.$$

Thus, as $\lambda\alpha \to \infty$,

$$E\left[\Delta_2 Q\right] \to 0.$$

Therefore, as the number of bids in a window becomes large, there is little difference in auction income between the first-price and the corresponding Vickrey auctions.

This is plotted in Figure 4-4, for $\alpha = 2$, with $L = 100$, and $\delta_0 = \delta = 0$.

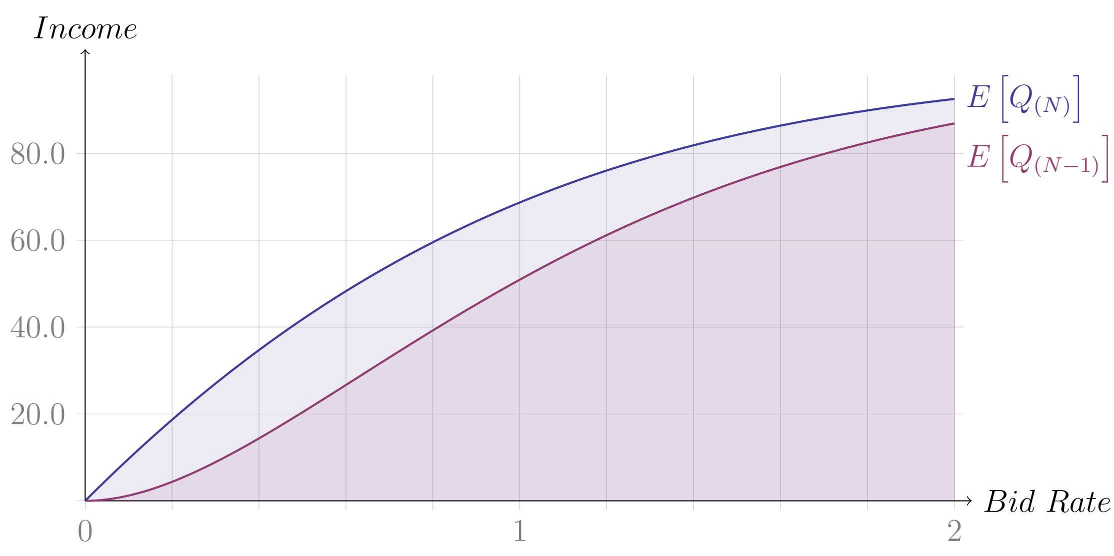

Figure 4-4: Average auction income comparison between the variable-time first-price forward auction with fixed inactivity window and the variable-time Vickrey forward auction with fixed inactivity window.

The auction duration for Algorithm VII is the same as that of Algorithm II and is given by Equation 3.7.

4.3 Auction Algorithm VIII: Fixed-time Vickrey Forward Auction with Maximum Threshold Termination

This is similar to Algorithm III, except that if the auction continues through to completion, the accepted bid will be the second highest bid (see Algorithm VIII).

Algorithm 8 Fixed-time Vickrey forward auction with maximum threshold termination.

Require: $\delta > 0$ **and** $M > 0$ **and** $T > 0$
Ensure: $L2 + \delta > 0$ **and** $accept\_id1 \neq null$

```
accept_id1 ← null
accept_id2 ← null
clock ← 0
L1 ← 0
L2 ← 0
while clock < T do
  while new_bid(t, R, bidder_id) do
    clock ← clock + t
    if clock < T then
      if R > L1 then
        L2 ← L1
        L1 ← R
        accept_id2 ← accept_id1
        accept_id1 ← bidder_id
        if L1 ≥ M then
          return L1, accept_id1
        end if
      else if R > L2 then
        L2 ← R
        accept_id2 ← bidder_id
      end if
    end if
  end while
end while
return L2 + δ, accept_id1
```

The average auction duration here is the same as Algorithm III, and is given by Equation 3.14. The mean bid accepted given the auction terminates before T is $\frac{M+L}{2}$, which happens with probability $\left(1 - e^{-\Lambda T}\right)$. If none of the arriving bids attain the maximum M, which happens with probability $e^{-\Lambda T}$, the bid with average value $L(N-1)/(N+1)$ is accepted, with N here coming from the stream Λ'. Here, L is replaced by M since none of the associated bids are allowed to exceed M. Therefore, the average magnitude of the accepted bid $E[Q_{Mv}]$ operating under the given auction termination rule, from Equation 4.2, is

$$E[Q_{Mv}] = \frac{(M+L)\left(1 - e^{-\Lambda T}\right)}{2} + \frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + \Lambda' T e^{-\Lambda' T} + 2e^{-\Lambda' T} - 2\right), \quad (4.9)$$

which is plotted in Figure 4-5, for $M = 20, 40, 60$ and 80, with $L = 100$, $T = 10$ and $\delta_0 = \delta = 0$.

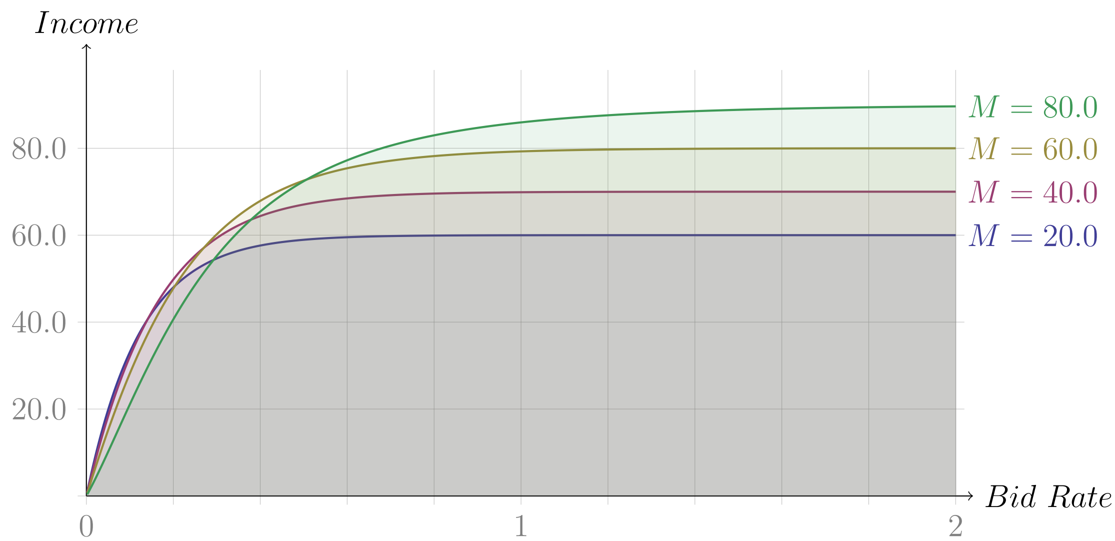

Figure 4-5: Average auction income of fixed-time Vickrey forward auction with maximum threshold termination.

The corresponding Open Buy-It-Now (OBIN) income is, on replacing $(M+L)/2$ by M in the first term,

$$E[Q_{Mv}] = M\left(1 - e^{-\Lambda T}\right) + \frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + \Lambda' T e^{-\Lambda' T} + 2e^{-\Lambda' T} - 2\right), \quad (4.10)$$

which is plotted in Figure 4-6, for $M = 20, 40, 60$ and 80, with $L = 100$ and

$T = 10$.

It is interesting to note that the curves on this graph cross-over. The reason for this is that, for small M, the income is likely to be $(M + L)/2$ which basically is like the first-price auction, while for large M, no bid exceeds M and hence the second price would have to be accepted. Thus, the income drops from $(M + L)/2$ to second price. Similar reasoning also applies to Figures 4-5, 4-6 and 4-8.

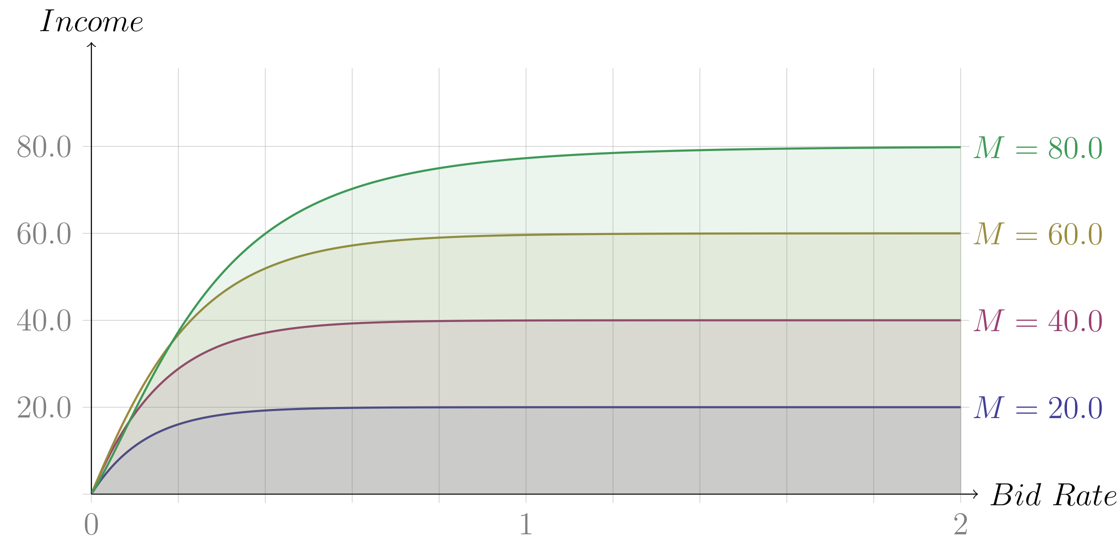

Figure 4-6: Average auction income of fixed-time Vickrey forward auction with maximum threshold termination and Open Buy-It-Now (OBIN) termination.

The auction duration for Algorithm VIII is the same as that of Algorithm III and is given by Equation 3.14.

4.4 Auction Algorithm IX: Variable-time Vickrey Forward Auction with Fixed Inactivity Window and Maximum Threshold Termination

This algorithm is similar to Algorithm IV, except that if the auction continues through to normal completion, the accepted bid will be the second-highest bid (see Algorithm IX).

Algorithm 9 Variable-time Vickrey forward auction with fixed inactivity window and maximum threshold termination.

Require: $\alpha > 0$ **and** $\delta > 0$ **and** $M > 0$
Ensure: $L2 + \delta > 0$ **and** $accept\_id1 \neq null$

```
accept_id1 ← null
accept_id2 ← null
L1 ← 0
L2 ← 0
loop
  while new_b id(t, R, bidder_id) do
    if t < α then
      if R > L1 then
        L2 ← L1
        L1 ← R
        accept_id2 ← accept_id1
        accept_id1 ← bidder_id
        if L1 ≥ M then
          return L1, accept_id1
        end if
      else if R > L2 then
        L2 ← R
        accept_id2 ← bidder_id
      end if
    else
      return L2 + δ, accept_id1
    end if
  end while
end loop
```

As per the derivation for Algorithm IV, the mean accepted bid $E[Q'_{Mv}]$, averaging over both termination possibilities, from Equation 4.6, is

$$E\left[Q'_{Mv}\right] = \frac{e^{-\lambda\alpha}\frac{2M}{exp(\Lambda'\alpha)-1}\left[\sinh(\Lambda'\alpha)-\Lambda'\alpha\right]+\left(1-e^{-\Lambda\alpha}\right)\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)\left(\frac{M+L}{2}\right)}{e^{-\lambda\alpha}+\left(1-e^{-\Lambda\alpha}\right)\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)} \tag{4.11}$$

which is plotted in Figure 4-7, for $M = 20, 40, 60$ and 80, with $L = 100$, $\alpha = 3$ and $\delta_0 = \delta = 0$.

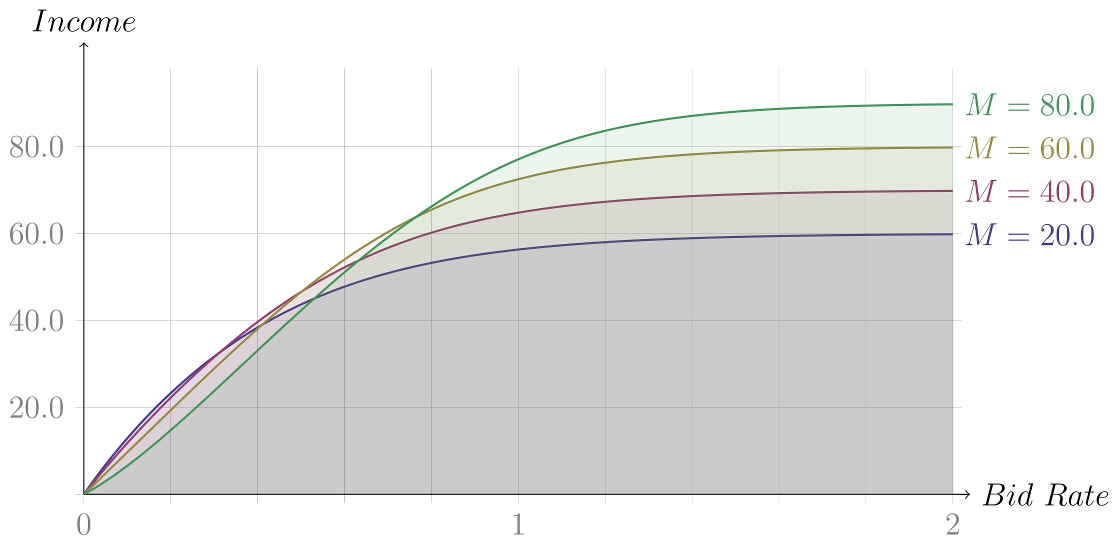

Figure 4-7: Average auction income of variable-time Vickrey forward auction with fixed inactivity window and maximum threshold termination.

In the Open Buy-It-Now (OBIN) situation, where M is made known to the bidders, the average auction income is, from Equation 4.11 on replacing $(M+L)/2$ by M in the second term,

$$E\left[Q'_{Mv}\right] = \frac{e^{-\lambda\alpha}\frac{2M}{exp(\Lambda'\alpha)-1}\left[\sinh(\Lambda'\alpha)-\Lambda'\alpha\right]+M\left(1-e^{-\Lambda\alpha}\right)\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)}{e^{-\lambda\alpha}+\left(1-e^{-\Lambda\alpha}\right)\left(1-\frac{M}{L}+\frac{M}{L}e^{-\lambda\alpha}\right)} \tag{4.12}$$

which is plotted in Figure 4-8, for $M = 20, 40, 60$ and 80, with $L = 100$, $T = 10$ and $\delta_0 = \delta = 0$.

The auction duration for Algorithm IX is the same as that of Algorithm IV and is given by Equation 3.17.

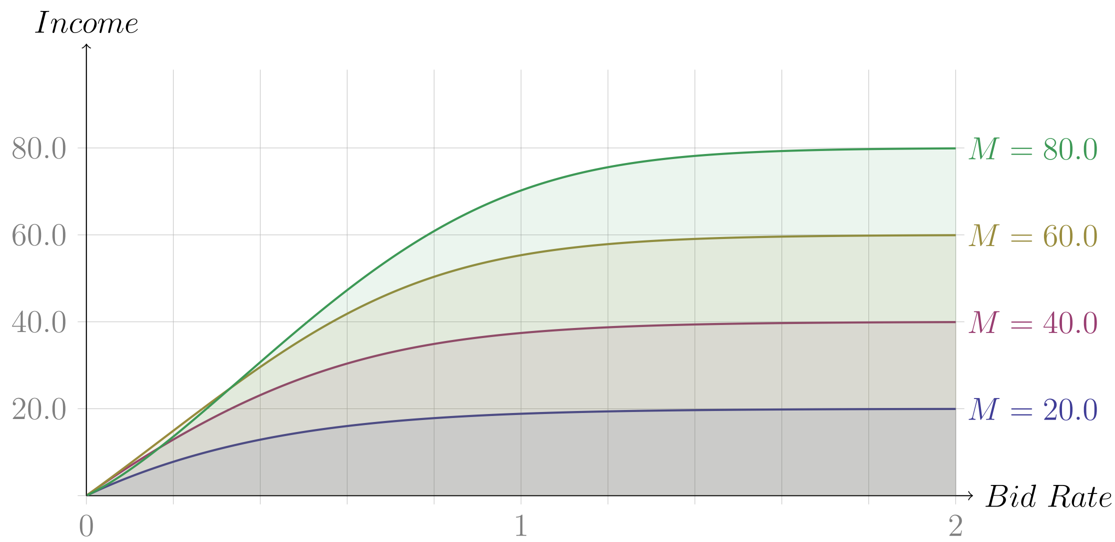

Figure 4-8: Average auction income of variable-time Vickrey forward auction with fixed inactivity window and maximum threshold termination and Buy-It-Now (BIN) termination.

4.5 Auction Algorithm X: Vickrey Forward Auction with Bid Enumeration Termination

This is similar to Algorithm V, except that the second highest bid is accepted (see Algorithm X)

The average magnitude of the accepted bid, from results of order statistics (Barry C. Arnold et al., 2008), is

$$E[Q_{(B-1)}] \quad = \quad \frac{(B-1)L}{B+1}. \tag{4.13}$$

The average auction duration in this case is simply the sum of B average inter-arrival times, or B/λ, which is identical to that of Algorithm V.

Algorithm 10 Vickrey forward auction with bid enumeration termination.

```
Require: B > 0
Ensure: L2 + δ > 0 and accept_id1 ≠ null
  accept_id1 ← null
  accept_id2 ← null
  L1 ← 0
  L2 ← 0
  while B > 0 do
    while new_bid(t, R, bidder_id) do
      B ← B − 1
      if R > L1 then
        L2 ← L1
        L1 ← R
        accept_id2 ← accept_id1
        accept_id1 ← bidder_id
      else if R > L2 then
        L2 ← R
        accept_id2 ← bidder_id
      end if
    end while
  end while
  return L2 + δ, accept_id1
```

4.6 Auction Algorithm XI: Fixed-time Last-price Reverse Auction

Some Internet auction processes follow the reverse auction mechanism. It is also frequently used in e-procurement where governments and private sectors wish to obtain goods or services (e.g. `http://www.esourcingsolutions.co.uk`). Typically, buyers indicate what they require, with the sellers competing against each other to provide the goods or services, and in doing so, the price is driven down. At the close of the reverse auction, the lowest bid is selected (see Algorithm XI).

Note that this is different from a Dutch auction, which some also consider to be a reverse auction. In a Dutch auction, the auction still seeks to obtain as high a price as possible and involves an auctioneer starting at a very high asking price that is continually lowered until some bidder is willing to accept the offer. The advantage of this is that auctions can happen very fast as it only takes a single bid to end the auction; it plays off bidders' fear of losing out and their drive towards a lower price.

Algorithm 11 Fixed-time last-price reverse auction.

Require: $T > 0$
Ensure: $L > 0$ **and** $accept\_id \neq null$

```
accept_id ← null
clock ← 0
L ← ∞
while clock < T do
  while new_bid(t, R, bidder_id) do
    clock ← clock + t
    if clock < T and R < L then
      L ← R
      accept_id ← bidder_id
    end if
  end while
end while
return L, accept_id
```

Here, we are interested in $E[Q_{(1)}]$, and the aim is to minimise expenditure rather than maximising income. Again, from (Barry C. Arnold et al., 2008), it can be shown that

$$E[Q_{(1)}|\text{Number of bids} = N] = \frac{L}{N+1} \tag{4.14}$$

Unlike the forward auction situation, however, where for $N = 0$, the auction income is zero, here, when $N = 0$, the maximum expenditure of L has to be paid, so the term $N = 0$ cannot be ignored. Thus, following the techniques of previous analysis, we have, for the unconditional average

$$\begin{aligned} E[Q_{(1)}] &= \sum_{N=0}^{\infty} \frac{L}{N+1} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\ &= \frac{Le^{-\lambda T}}{\lambda T} \sum_{N=0}^{\infty} \frac{(\lambda T)^{N+1}}{(N+1)!} \\ &= \frac{L}{\lambda T}\left[1 - e^{-\lambda T}\right], \end{aligned} \tag{4.15}$$

which is plotted in Figure 4-9, for $T = 5, 10$ and 15, with $L = 100$ and $\delta_0 = \delta = 0$.

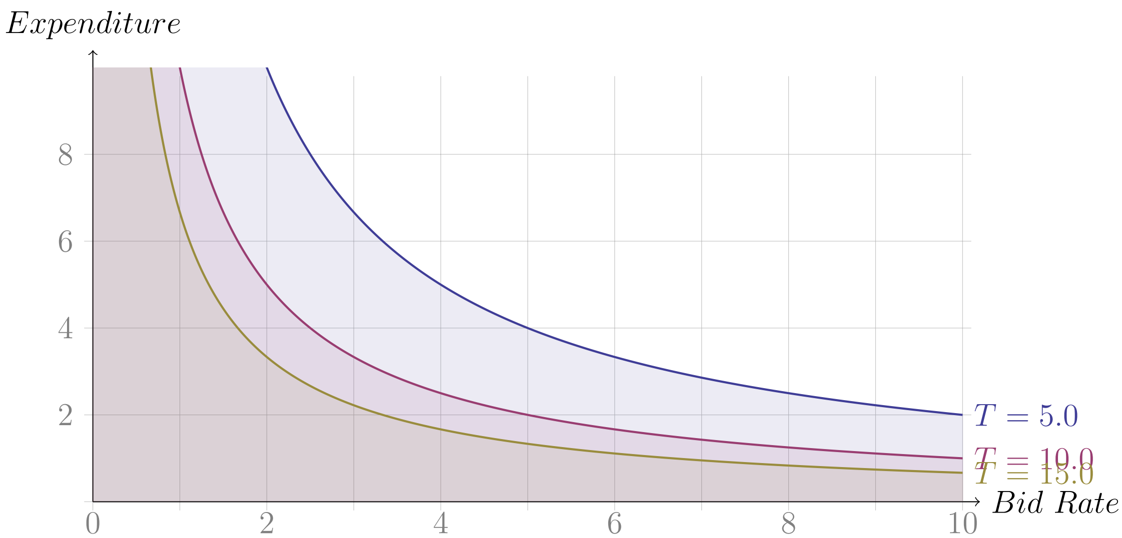

Figure 4-9: Average auction expenditure of fixed-time last-price reverse auction.

This gives an expenditure of

$$\frac{L}{\lambda T^2}\left[1 - e^{-\lambda T}\right]. \tag{4.16}$$

4.7 Auction Algorithm XII: Variable-time Last-price Reverse Auction with Fixed Inactivity Window

Similar to Algorithm II, here the auction will terminate when there is no bid arrival for a fixed window of length α. On termination, the smallest bid received will be chosen (see Algorithm XII).

Here, the average auction duration is the same as that of Algorithm II, and is given by Equation 3.7. Following similar arguments and using the same de-conditioning probabilities as in Section 3.2, we have, from Equation 4.14.

$$E[Q_{(1)}] = \sum_{N=0}^{\infty} \frac{L}{N+1} \times \left(1 - e^{-\lambda\alpha}\right)^N e^{-\lambda\alpha} = \frac{Le^{-\lambda\alpha}}{1 - e^{-\lambda\alpha}} \sum_{N=0}^{\infty} \frac{(1 - e^{-\lambda\alpha})^{N+1}}{N+1} \tag{4.17}$$

Algorithm 12 Variable-time last-price reverse auction with fixed inactivity window.

Require: $\alpha > 0$
Ensure: $L > 0$ **and** $accept\_id \neq null$

```
  accept_id ← null
  L ← ∞
  loop
    while new_bid(t, R, bidder_id) do
      if t < α then
        if R < L then
          L ← R
          accept_id ← bidder_id
        end if
      else
        return  L, accept_id
      end if
    end while
  end loop
```

This gives, on making use of the logarithmic series in Equation 3.10,

$$E[Q_{(1)}] = \frac{L\lambda\alpha}{e^{\lambda\alpha} - 1}, \tag{4.18}$$

which is plotted in Figure 4-10, for $\alpha = 1, 2$ and 3, with $L = 100$ and $\delta_0 = \delta = 0$.

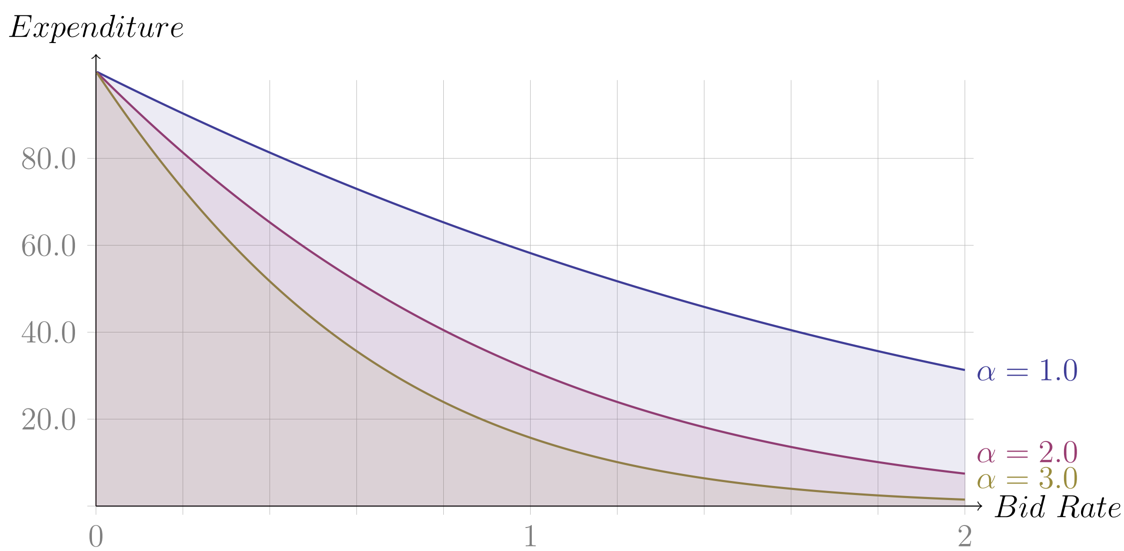

Figure 4-10: Average auction expenditure of variable-time last-price reverse auction with fixed inactivity window.

4.8 Auction Algorithm XIII: Fixed-time Last-price Reverse Auction with Minimum Threshold Termination

This is similar to Algorithm III, except that the auction may terminate before the expiry of T, if a sufficiently low bid is received. We let this minimum threshold be m, so that whenever an arriving bid is below or equal to m, then the auction terminates immediately and that bid is accepted. Thus, if a low enough bid is received, one would not wish to waste any more time in waiting for another offer, which may or may not result in an improvement (see Algorithm XIII)

Algorithm 13 Fixed-time last-price reverse auction with minimum threshold termination.

```
Require: M > 0 and T > 0
Ensure: L > 0 and accept_id ≠ null
  accept_id ← null
  clock ← 0
  L ← ∞
  while clock < T do
    while new_bid(t, R, bidder_id) do
      clock ← clock + t
      if clock < T and R < L then
        L ← R
        accept_id ← bidder_id
        if L ≤ M then
          return L, accept_id
        end if
      end if
    end while
  end while
  return L, accept_id
```

Analogous to the Buy-It-Now (BIN) situation in forward auctions, in the Sell-It-Now (SIN) situation, if a bid arrives with a value that is less than or equal to m, it is immediately accepted. As in the forward auctions, the value m may be kept private or made known to the public.

Here, we let $p = \Pr[\text{Bid} \leq m] = m/L$, $p' = \Pr[\text{Bid} > m] = 1 - m/L$. Thus, for each arrival, there is a probability p that it will meet the minimum threshold requirement. There are two Poisson arrival streams with rates $\Lambda = \lambda p$ and $\Lambda' = \lambda p'$, with $\lambda = \Lambda + \Lambda'$. The first sub-stream Λ meets the minimum threshold and its arrival will terminate the auction immediately, while the second sub-stream Λ' fails to meet the minimum threshold and will allow the auction to continue. The first arrival of the Λ sub-stream at time t before T would terminate the auction at t immediately, which happens with probability $\Lambda e^{-\Lambda t} dt$. Averaging over all such arrivals before T gives

$$\int_0^T \Lambda t e^{-\Lambda t} dt \quad = \quad \frac{1 - e^{\Lambda t}(\Lambda T + 1)}{\Lambda}.$$

On the other hand, if the first arrival from this sub-stream Λ occurs after T, then the auction duration would be T, which happens with probability $e^{-\Lambda T}$. Therefore, averaging over both possibilities, we have for the mean auction duration $E[T_m]$ for this case

$$E[T_m] = \frac{1 - e^{\Lambda t}(\Lambda T + 1)}{\Lambda} + T e^{-\Lambda T} = \frac{1 - e^{-\Lambda T}}{\Lambda} = \frac{1 - e^{-\lambda m T/L}}{\lambda m/L}$$

That is,

$$E[T_m] \quad = \quad \frac{L - L e^{-\lambda m T/L}}{\lambda m}. \tag{4.19}$$

We see that $E[T_m] \to \left(1 - e^{-\lambda T}\right)/\lambda$ as $m \to L$, which may be interpreted as: if there is an arrival before T, which happens with probability $\left(1 - e^{-\lambda T}\right)$, then the auction lasts for a duration of $1/\lambda$ which is just the time of the first arrival, and this arrival is immediately accepted because it must be less than m. On the other hand, as $m \to 0$, $E[T_m] \to T$ (which can be seen by application of L'Hôpital's rule). Again, as shown previously in Section 3.3, $\left(1 - e^{-\Lambda T}\right)/\Lambda < T$, which shows that $E[T_m] < T$.

Figure 4-11 plots the mean auction duration against the bid rate λ for values of $m = 25, 50, 75$ for $L = 100$ and $T = 4$. We see that, as the bid rate increases, the likelihood of receiving a bid less than m also increases, and so the average auction duration drops. This algorithm is able to reduce the auction duration from $T = 4$ for moderate to large values of m quite substantially.

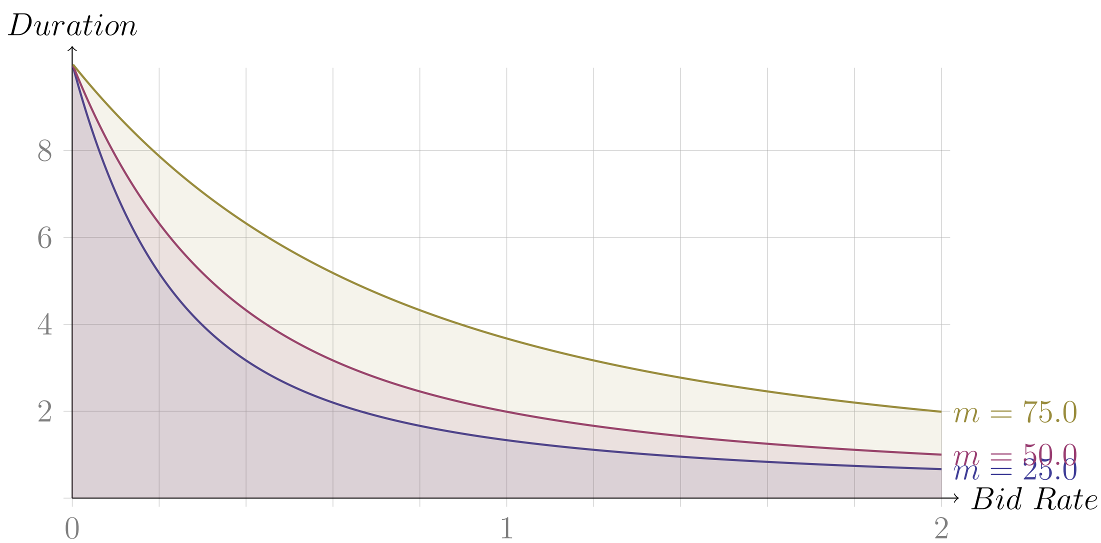

Figure 4-11: Average auction duration of fixed-time last-price reverse auction with minimum threshold termination.

In determining the average auction expenditure, the symmetry between the current algorithm and Algorithm III is used. Considering the interval $(0, L)$, and noting that the distance or magnitude between the auction income and L in Algorithm III is the same as that between the auction expenditure here and 0, which is precisely the average auction expenditure in the present algorithm. Thus, the current average auction expenditure is given by

$$L - E\left[Q_M\right],$$

and on replacing M by $L - m$, the average auction expenditure can be obtained:

$$E\left[Q_m\right] \quad = \quad L - \left[\frac{(2L-m)\left(1-e^{-\Lambda T}\right)}{2} + \frac{(L-m)e^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + e^{-\Lambda' T} - 1\right)\right], \tag{4.20}$$

where $\Lambda' = \lambda m/L$ and $\Lambda = \lambda(1 - m/L)$ and which is plotted in Figure 4-12, for $m = 20, 40, 60$ and 80, with $L = 100$, $T = 10$ and $\delta_0 = \delta = 0$.

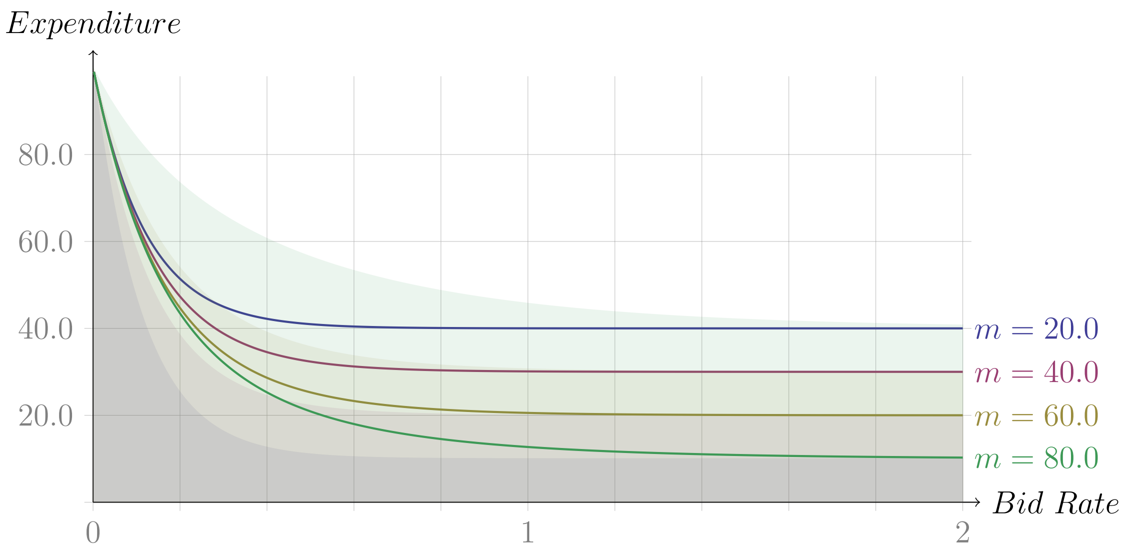

Figure 4-12: Average auction expenditure of fixed-time last-price reverse auction with minimum threshold termination and Closed Sell-It-Now (CSIN).

For the Open Sell-It-Now (OSIN) case, the average auction expenditure is

$$E\left[Q_m\right] \;=\; L-\left[(L-m)\left(1-e^{-\Lambda T}\right)+\frac{(L-m)e^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T+e^{-\Lambda' T}-1\right)\right] \tag{4.21}$$

which is plotted in Figure 4-13, for $m = 20, 40, 60$ and 80, with $L = 100$, $T = 10$ and $\delta_0 = \delta = 0$.

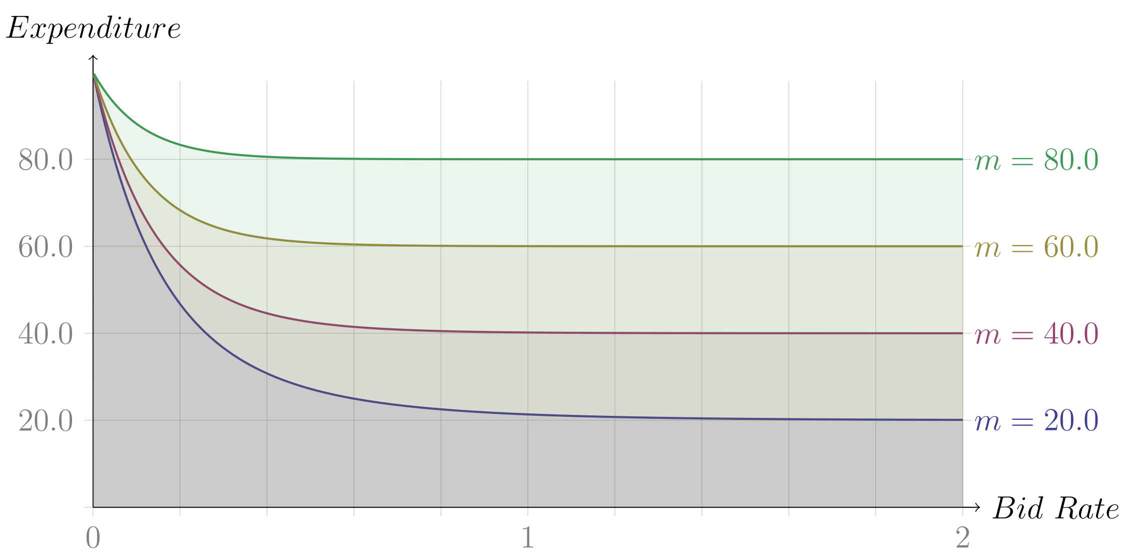

Figure 4-13: Average auction expenditure of fixed-time last-price reverse auction with minimum threshold termination and Open Sell-It-Now (OSIN).

4.9 Auction Algorithm XIV: Variable-time Last-price Reverse Auction with Fixed Inactivity Window and Minimum Threshold Termination

This is similar to Algorithm XII with an additional termination condition that the auction mechanism will accept a bid if it is sufficiently low (see Algorithm XIV). Here, the auction can be terminated either

1. by having none of the bid arrivals meeting the minimum threshold and the auction finishes on the expiry of a no bid interval of length α, or
2. by having an arrival which meets the minimum threshold m before the auction terminates through (1).

Algorithm 14 Variable-time last-price reverse auction with fixed inactivity window and minimum threshold termination.

Require: $\alpha > 0$ **and** $M > 0$
Ensure: $L > 0$ **and** $accept\_id \neq null$

```
accept_id ← null
L ← ∞
loop
  while new_bid(t, R, bidder_id) do
    if t < α then
      if R < L then
        L ← R
        accept_id ← bidder_id
        if L ≤ M then
          return L, accept_id
        end if
      end if
    else
      return L, accept_id
    end if
  end while
end loop
```

As in the previous algorithm, the auction expenditure here is best determined by observing the symmetry between the auction expenditure here and that in Algorithm IV. Thus, the average auction expenditure here is:

$$L - E\left[Q'_M\right], \tag{4.22}$$

which can be written as, on replacing M by $(L-m)$,

$$L - \frac{(L-m)e^{-\lambda\alpha}\left[1-\frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right]+\left(1-e^{-\Lambda\alpha}\right)\left[1-\frac{(L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right]\left(\frac{2L-m}{2}\right)}{e^{-\lambda\alpha}+(1-e^{-\Lambda\alpha})\left(1-\frac{L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right)}, \tag{4.23}$$

which is plotted in Figure 4-14, for $m = 20, 40, 60$ and 80, with $L = 100$, $\alpha = 3$ and $\delta_0 = \delta = 0$.

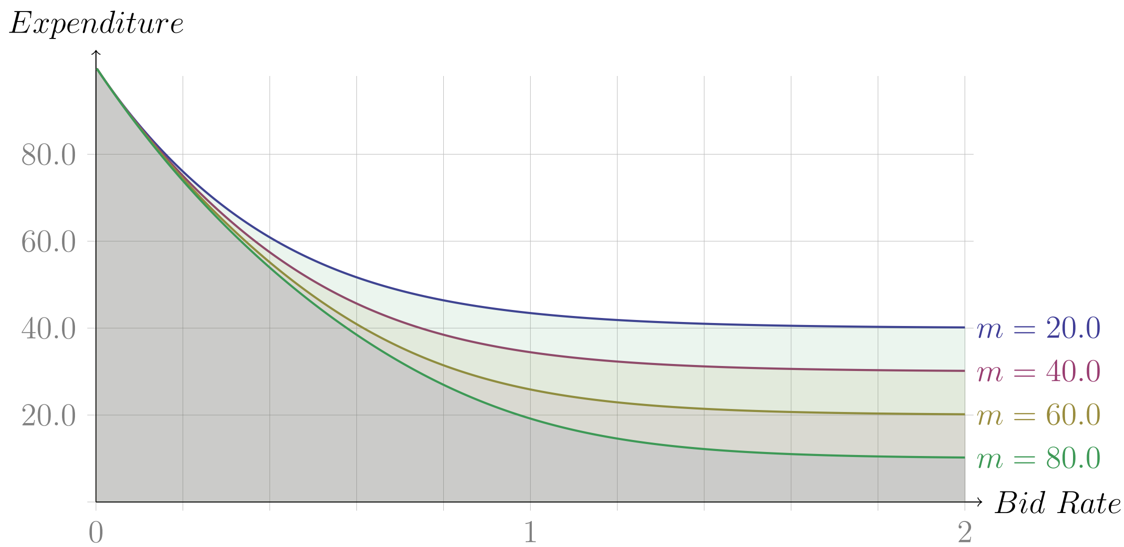

Figure 4-14: Average auction expenditure of variable-time last-price reverse auction with fixed inactivity window and minimum threshold termination and Closed Sell-It-Now (CSIN).

For the Open Sell-It-Now (OSIN) case, the average auction expenditure is

$$L - \frac{(L-m)e^{-\lambda\alpha}\left[1-\frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right]+(L-m)\left(1-e^{-\Lambda\alpha}\right)\left[1-\frac{(L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right]}{e^{-\lambda\alpha}+(1-e^{-\Lambda\alpha})\left(1-\frac{L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right)} \tag{4.24}$$

which is plotted in Figure 4-15, for $M = 20, 40, 60$ and 80, with $L = 100$, $\alpha = 3$ and $\delta_0 = \delta = 0$.

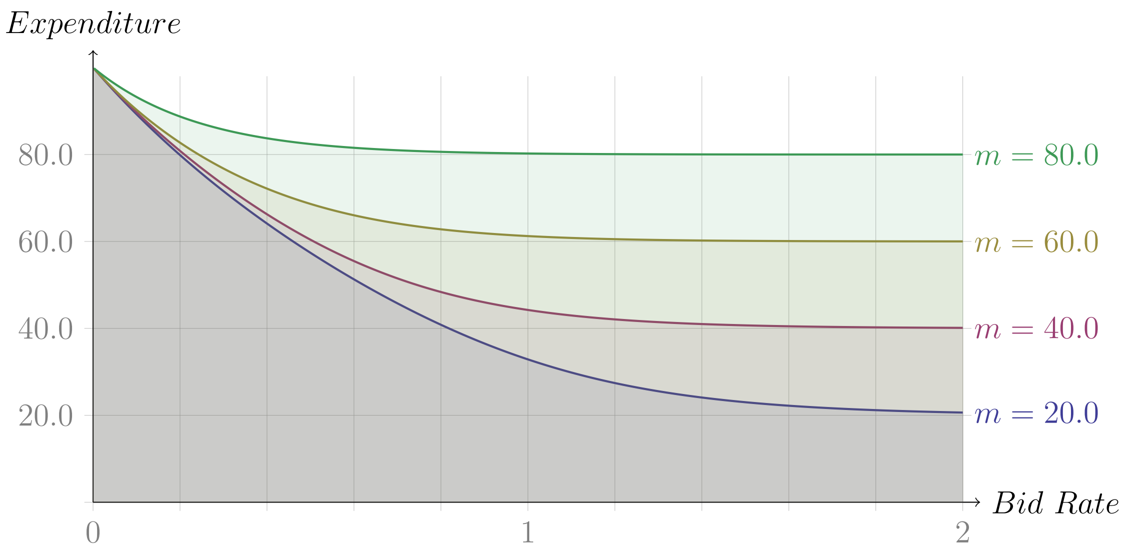

Figure 4-15: Average auction expenditure of variable-time last-price reverse auction with fixed inactivity window and minimum threshold termination and Open Sell-It-Now (OSIN).

As for the auction duration, from Algorithm IV, the following equation can be written for the average auction duration:

$$E\left[\tau_m\right] = \left(1-e^{-\lambda\alpha}\right)\left\{\frac{1-e^{-\lambda\alpha}(\lambda\alpha+1)}{\lambda\left(1-e^{-\lambda\alpha}\right)}+\left(1-\frac{m}{L}\right)E\left[\tau_m\right]\right\}+\alpha e^{-\lambda\alpha}.$$

This gives,

$$E\left[\tau_m\right] = \frac{1-e^{-\lambda\alpha}}{\lambda\left[\frac{m}{L}+\left(1-\frac{m}{L}\right)e^{-\lambda\alpha}\right]}. \tag{4.25}$$

4.10 Auction Algorithm XV: Last-price Reverse Auction with Bid Enumeration Termination

This is similar to Algorithm X, except that the lowest bid is accepted (see Algorithm XV).

Thus, the average magnitude of the accepted bid, from the results of order statis-

Algorithm 15 last-price reverse auction with bid enumeration termination.

```
Require: B > 0
Ensure: L > 0 and accept_id ≠ null
  accept_id ← null
  L ← ∞
  while B > 0 do
    while new_bid(t, R, bidder_id) do
      B ← B − 1
      if R < L then
        L ← R
        accept_id ← bidder_id
      end if
    end while
  end while
  return L, accept_id
```

tics is,

$$E\left[Q_{(B)}\right] = \frac{L}{B+1} \tag{4.26}$$

4.11 Results Summary

Tables 4.1 and 4.2, summarise the main properties of the different algorithms.

For the Vickrey auction, as expected, the auction income is less than the income obtained from a first-price auction for the same bid rate. Since the Vickrey auction encourages bidder participation due to the winner only paying a single bid increment over the second price, the bid rate may be significantly higher, counteracting any loss in income. The reverse auction, on the other hand, is a good counterpoint to the auction algorithms detailed in Chapter 3.

Algorithm	Average Auction Duration	Average Offer Accepted Per Auction
VI	T	$\lambda T e^{-\lambda T}\delta_o + \frac{L}{\lambda T}\left(\lambda T + \lambda T e^{-\lambda T} + 2e^{-\lambda T} - 2\right) + \delta$
VII	$\frac{e^{\lambda\alpha}-1}{\lambda}$	$\left(1 - e^{-\lambda\alpha}\right) \times e^{-\lambda\alpha}\delta_o + \frac{2L}{e^{\lambda\alpha}-1}\left[\sinh\lambda\alpha - \lambda\alpha\right] + \delta$
VIII	$\frac{L-Le^{-\lambda T\left(1-\frac{M}{L}\right)}}{\lambda(L-M)}$	$\frac{(M+L)\left(1-e^{-\Lambda T}\right)}{2} + \frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + \Lambda' T e^{-\Lambda' T} + 2e^{-\Lambda' T} - 2\right)$ **[CBIN]**
	$\frac{L-Le^{-\lambda T\left(1-\frac{M}{L}\right)}}{\lambda(L-M)}$	$M\left(1 - e^{-\Lambda T}\right) + \frac{Me^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + \Lambda' T e^{-\Lambda' T} + 2e^{-\Lambda' T} - 2\right)$ **[OBIN]**
IX	$\frac{e^{-\Lambda\alpha}\left[1-e^{-\Lambda'\alpha}(\Lambda'\alpha+1)\right]}{\Lambda'\left[1-e^{-\Lambda\alpha}\left(1-e^{-\Lambda'\alpha}\right)\right]} + \frac{1-e^{-\Lambda\alpha}}{\Lambda}$	$\frac{\left(1-e^{-\Lambda\alpha}\right)(M+L)}{2} + \frac{2Me^{-\Lambda\alpha}}{e^{\Lambda'\alpha}-1}\left[\sinh\Lambda'\alpha - \Lambda'\alpha\right]$ **[CBIN]**
	$\frac{e^{-\Lambda\alpha}\left[1-e^{-\Lambda'\alpha}(\Lambda'\alpha+1)\right]}{\Lambda'\left[1-e^{-\Lambda\alpha}\left(1-e^{-\Lambda'\alpha}\right)\right]} + \frac{1-e^{-\Lambda\alpha}}{\Lambda}$	$M\left(1 - e^{-\Lambda\alpha}\right) + \frac{2Me^{-\Lambda\alpha}}{e^{\Lambda'\alpha}-1}\left[\sinh\Lambda'\alpha - \Lambda'\alpha\right]$ **[OBIN]**
X	$\frac{B}{\lambda}$	$\frac{(B-1)L}{B+1}$

Table 4.1: Vickrey auctions.

Algorithm	Average Auction Duration	Average Offer Accepted Per Auction
XI	T	$\frac{L}{\lambda T}\left(1 - e^{-\lambda T} - \lambda T e^{-\lambda T}\right)$
XII	$\frac{e^{\lambda\alpha}-1}{\lambda}$	$\frac{L}{e^{\lambda\alpha}-1}\left[\lambda\alpha + e^{-\lambda\alpha} - 1\right]$
XIII	$\frac{L-Le^{-\lambda mT/L}}{\lambda m}$	$L - \left[\frac{(2L-m)\left(1-e^{-\Lambda T}\right)}{2} + \frac{(L-m)e^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + e^{-\Lambda' T} - 1\right)\right]$ **[CBIN]**
	$\frac{L-Le^{-\lambda mT/L}}{\lambda m}$	$L - \left[(L-m)\left(1 - e^{-\Lambda T}\right) + \frac{(L-m)e^{-\Lambda T}}{\Lambda' T}\left(\Lambda' T + e^{-\Lambda' T} - 1\right)\right]$ **[OBIN]**
XIV	$\frac{e^{-\Lambda\alpha}\left[1-e^{-\Lambda'\alpha}(\Lambda'\alpha+1)\right]}{\Lambda'\left[1-e^{-\Lambda\alpha}\left(1-e^{-\Lambda'\alpha}\right)\right]} + \frac{1-e^{-\Lambda\alpha}}{\Lambda}$	$L - \frac{(L-m)e^{-\lambda\alpha}\left[1-\frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right]+\left(1-e^{-\Lambda\alpha}\right)\left[1-\frac{(L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right]\left(\frac{2L-m}{2}\right)}{e^{-\lambda\alpha}+(1-e^{-\Lambda\alpha})\left(1-\frac{L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right)}$ **[CBIN]**
	$\frac{e^{-\Lambda\alpha}\left[1-e^{-\Lambda'\alpha}(\Lambda'\alpha+1)\right]}{\Lambda'\left[1-e^{-\Lambda\alpha}\left(1-e^{-\Lambda'\alpha}\right)\right]} + \frac{1-e^{-\Lambda\alpha}}{\Lambda}$	$L - \frac{(L-m)e^{-\lambda\alpha}\left[1-\frac{\Lambda'\alpha}{e^{\Lambda'\alpha}-1}\right]+(L-m)\left(1-e^{-\Lambda\alpha}\right)\left[1-\frac{(L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right]}{e^{-\lambda\alpha}+(1-e^{-\Lambda\alpha})\left(1-\frac{L-m}{L}+\frac{L-m}{L}e^{-\lambda\alpha}\right)}$ **[OBIN]**
XV	$\frac{B}{\lambda}$	$\frac{L}{B+1}$

Table 4.2: Reverse auctions.

Chapter 5

Analysis of Surplus and Economic Benefits

The concept of allocative efficiency (or sometimes called operational efficiency) is often employed to evaluate how Internet auctions perform (Vragov, 2010) in terms of economic benefits to the participants concerned. The seller surplus is the difference between the transaction price and the seller's costs, while the difference between the buyer's valuation and the transaction price gives the buyer surplus or consumer surplus (see Figure 5-1). The seller's costs is sometimes generically called production cost, which apart from the actual cost of production would also include other such costs as those associated with distribution, transportation, packaging, delivery, or insurance. The total surplus is the seller surplus plus the buyer surplus, and the allocative efficiency is given by the total actual realised surplus expressed as a fraction of the total possible surplus (Vragov, 2010). For simplicity, these quantities are represented in Figure 5.1 as linear functions, but the basic ideas remain the same if one or more of these are non-linear. Since the total surplus $(L-I)+(I-C)=L-C$ is constant, an increase in seller surplus would mean a reduction in buyer surplus and vice versa. In the next section, we shall first study the surplus analysis and design options relating to hard close auctions.

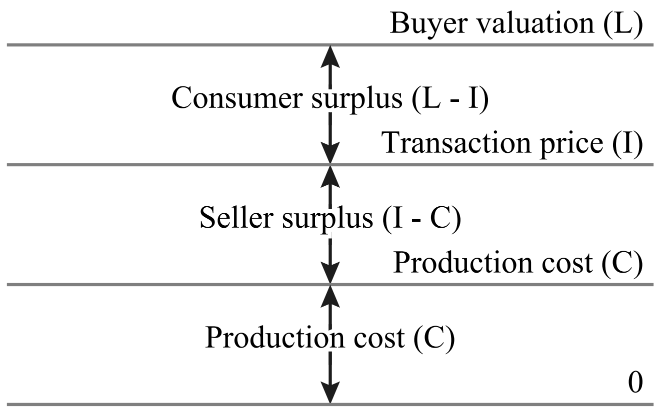

Figure 5-1: Measures of auction surpluses.

5.1 Surplus Analysis and Economic Benefits of Hard Close Auctions

From the buyers' point of view, their valuation of the auction item is indicated by the maximum price L that they are willing to pay. As indicated earlier, the buyer surplus is given by the difference in buyer valuation and the transaction price. Thus, for the fixed time first-price forward auction the buyer surplus β is given by the difference in buyer valuation and the transaction price, which from Chapter 3, is

$$\beta(z) = L - \frac{L}{\lambda T}(\lambda T + e^{-\lambda T} - 1) = \frac{L}{\lambda T}(1 - e^{-\lambda T}) = \frac{L}{z}(1 - e^{-z}) > 0 \tag{5.1}$$

Here, we have used $z = \lambda T$, which gives the average number of bids arriving in the interval $(0, T)$. Thus, the above equation can be interpreted as follows: the consumer surplus is the private value evenly divided by the number of bids times the probability of having a non-empty auction (i.e. an auction where there is at least one arriving bid). We see that the higher the value of L, the greater is the buyer surplus, and also the greater the probability of having a non-empty auction $(1 - e^{-z})$, the higher is the expected buyer surplus. Bidder collusion behaviour—where bidders collude in order to lower the transaction price—may be incorporated by having a lower value of L. On the other hand, shilling behaviour—where the seller artificially inflates the transaction price through disguising as bidders—may be reflected by a higher value for $z = \lambda T$, Differentiating the consumer surplus with respect to z, we have

$$\frac{d\beta(z)}{dz} = -\frac{L}{z^2}(1 - e^{-z} - ze^{-z}) = -\frac{L}{z^2}[1 - \frac{1+z}{e^z}]. \tag{5.2}$$

Since $z > 0$ for non-empty auctions, and from the series expansion of e^z, we have $(1 + z) < e^z$, so that

$$\frac{1+z}{e^z} < 1 \tag{5.3}$$

and thus

$$\frac{d\beta(z)}{dz} = -\frac{L}{z^2}[1 - \frac{1+z}{e^z}] < 0 \tag{5.4}$$

Thus, we see that shilling will reduce the buyer surplus while raising the seller surplus. From the seller's point of view, another mechanism available for the seller to increase the seller surplus is to adjust the auction duration. Since

$$\frac{d\beta(z)}{dT} = -\frac{\lambda L}{z^2}[1 - \frac{1+z}{e^z}] < 0 \tag{5.5}$$

by extending the auction duration, the consumer surplus will be eroded and accordingly the seller surplus will increase.

From the seller's perspective, it would be of interest to see how the auction income is approached. For example, if it is approached much faster than the uniform rate of increase, then there may be scope for adjusting the auction duration downwards. Here, we may use an index to measure this similar to the computation of the Gini coefficient in measuring the unevenness of income distribution (Firebaugh, 2003). In Figure 5-2, we compare the auction income

$$Q(z) = L[1 - \frac{1 - e^{-z}}{z}] \tag{5.6}$$

against the uniform rate of increase as a function of z.

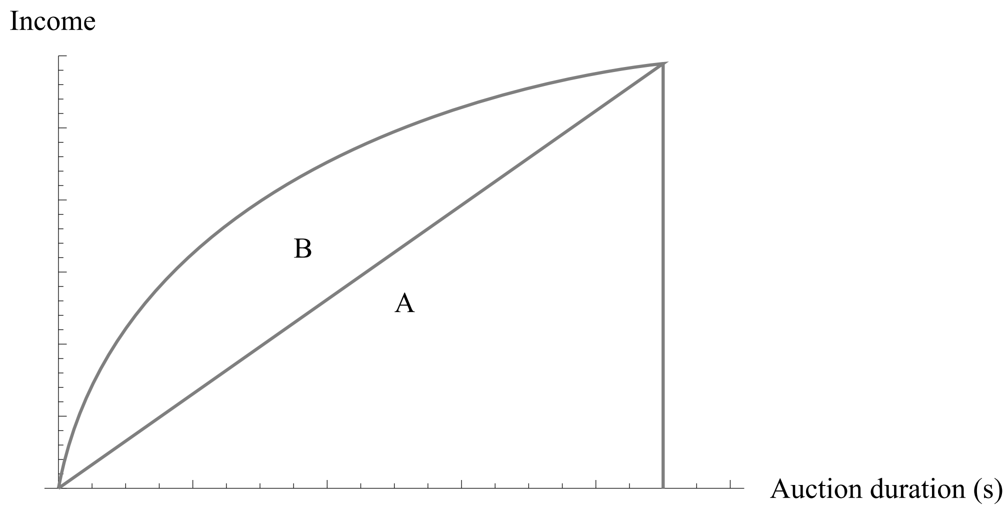

Figure 5-2: The Gini auction ratio.

We may regard the auction curve in Equation 5.6 as analogous to the Lorenz curve in the Gini coefficient. The equation of the uniform straight line curve for auction termination at $z = k$ is given by

$$U(z) = \frac{Lz}{k}[1 - \frac{1 - e^{-k}}{k}] \tag{5.7}$$

which starts from zero and ends at the same point at $Q(k)$. The areas A and B in Figure 5-2 is given respectively by

$$A = \int_0^k U(z)dz \tag{5.8}$$

and

$$B = \int_0^k Q(z)dz - \int_0^k U(z)dz \tag{5.9}$$

Analogous to the Gini coefficient, we calculate

$$G = \frac{A+B}{A} = \frac{\int_0^k Q(z)dz}{\int_0^k U(z)dz} \tag{5.10}$$

which we may call the Gini auction ratio. Since

$$A = \int_0^k U(z)dz = \frac{Lk}{2}[1 - \frac{1-e^{-k}}{k}] \tag{5.11}$$

and

$$A + B = \int_0^k Q(z)dz = L[k - \log_e k - \Gamma(0,k) - \gamma] \tag{5.12}$$

where $\Gamma(a, x)$ is the incomplete gamma function, and γ is the Euler-Mascheroni constant (from Wolfram Alpha), the Gini auction ratio is evaluated to be

$$G = \frac{2[k - \log_e k - \Gamma(0,k) - \gamma]}{k - 1 + e^{-k}} \tag{5.13}$$

Having a Gini auction ratio significantly greater than one would indicate that a relatively high auction income is fetched compared with the uniform rate of increase, and therefore there is scope for reducing the auction duration without sacrificing the auction income too much. For example, for $L = 100$, $k = 10$, we have

$$G = \frac{712.02}{450} = 1.58, \tag{5.14}$$

which signifies that there is scope for reducing the auction duration. For $\lambda = 0.5$, this corresponds to an auction duration of $T = 2k = 20$, and the size of the above ratio suggests that the auction duration can possibly be adjusted to below 20 without causing a large reduction in the auction income.

In many of the Internet auctions, Vickrey auctions are used, and sellers frequently have to consider the choice between holding first-price auctions and Vickrey auctions. Although Vickrey auctions clearly have the advantage in assuring the bidders that they would not overpay and hence encourage wider bidder participation, doing so would seem to erode the seller surplus. The buyer surplus in the Vickrey auction is, from Chapter 4,

$$\beta_v(z) = L - \frac{L}{z}(z + ze^{-z} + 2e^{-z} - 2) \tag{5.15}$$

Upon simplification, this becomes

$$\beta_v(z) = \frac{2L}{ze^z}[e^z - (1 + \frac{z}{2})] \tag{5.16}$$

Since, from Equation 3.5, for $z > 0$,

$$e^z > 1 + z > (1 + \frac{z}{2}) \tag{5.17}$$

we conclude from Equation 5.16 that $\beta_v(z) > 0$.

Since the buyer pays a lower price in Vickrey auctions, it is useful to compare the buyer surplus of a first-price auction with that of a Vickrey auction. From Equations 5.1 and 5.16 and simplifying, we have for the buyer surplus ratio

$$\frac{\beta_v(z)}{\beta(z)} = 2 - \frac{z}{e^z - 1} \tag{5.18}$$

which shall be referred to as the Vickrey surplus ratio. We see that, by application of L'Hôpital's rule,

$$\frac{z}{e^z - 1} \rightarrow 1 \mathrm{as} z \rightarrow 0 \tag{5.19}$$

so that the Vickrey surplus ratio

$$\frac{\beta_v(z)}{\beta(z)} \rightarrow 1 \mathrm{as} z \rightarrow 0 \tag{5.20}$$

Also, since

$$\frac{z}{e^z - 1} \rightarrow 0 \mathrm{as} z \rightarrow \infty \tag{5.21}$$

which can be seen on expanding e^z , we have

$$\frac{\beta_v(z)}{\beta(z)} \rightarrow 2\text{as}z \rightarrow \infty \tag{5.22}$$

which shows that the buyer surplus advantage for Vickrey auction is maintained in such situation even when the number of bids is large.

On the other hand, the seller surplus for the first-price auction is

$$\zeta(z) = \frac{L}{z}(z + e^{-z} - 1) - C \tag{5.23}$$

and the seller surplus for the Vickrey auction is

$$\zeta_v(z) = \frac{L}{z}(z + ze^{-z} + 2e^{-z} - 2) - C \tag{5.24}$$

By holding a Vickrey auction instead of a first-price auction, the seller surplus will be reduced, and the drop in seller surplus is

$$\zeta(z) - \zeta_v(z) = \frac{L}{z}(1 - e^{-z} - ze^{-z}) = \frac{L}{z}[1 - (\frac{1+z}{e^z})] \tag{5.25}$$

Since $e^z > (1 + z)$, we have therefore

$$\zeta(z) - \zeta_v(z) \leq \frac{L}{z} \tag{5.26}$$

which $\rightarrow 0$, as $z \rightarrow \infty$. That is, when the expected number of bids is large, the drop in seller surplus becomes insignificant. Consequently, for such a situation, the seller may choose to hold a Vickrey auction instead of a first-price one since the difference in seller surplus is small. Very often, buyers would prefer a Vickrey auction over a first-price one because the former would give them the psychological assurance that they are not overpaying in relation to other bidders. This would motivate wider participation from the bidding community. For this reason, it may be argued that Vickrey auctions should be held in any case in preference over first-price auctions, since with first-price auctions, z may be moderate or small, while with

Vickrey auctions, z may shoot up to much higher values so that there would be only insignificant drop, if any, in seller surplus.

Taking this a step further, it is possible that the drop in surplus may become negative due to the effect of Vickrey auctions on z. In fact, letting z_1 be the expected number of bids for a first-price auction, and z_2 be the expected number of bids for the corresponding Vickrey auction, the difference in seller surplus is

$$\zeta(z_1) - \zeta_v(z_2) = \frac{L}{z_1}(z_1 + e^{-z_1} - 1) - \frac{L}{z_2}(z_2 + z_2 e^{-z_2} + 2e^{-z_2} - 2) \tag{5.27}$$

This implies that the seller surplus would go negative if

$$\frac{1}{z_1}(e^{-z_1} - 1) < e^{-z_2} + \frac{2}{z_2}(e^{-z_2} - 1) \tag{5.28}$$

Since the first term on the right hand side $\exp(-z_2)$ is always positive, a sufficient condition for the above inequality is

$$\frac{1}{z_1}(e^{-z_1} - 1) < \frac{2}{z_2}(e^{-z_2} - 1) \tag{5.29}$$

Multiplying both sides by -1 and reversing the direction of the inequality, we have

$$\frac{1}{z_1}(1 - e^{-z_1}) > \frac{2}{z_2}(1 - e^{-z_2}) \tag{5.30}$$

or

$$\frac{z_2}{z_1} > \frac{2(1 - e^{-z_2})}{1 - e^{-z_1}} \tag{5.31}$$

Since $[1 - \exp(-z_2)]$ is the probability of non-empty auction, it is less than one, and so a sufficient condition of the above becomes

$$\frac{z_2}{z_1} > \frac{2}{1 - e^{-z_1}} \tag{5.32}$$

That is, we can conclude that if

$$z_2 > \frac{2z_1}{1 - e^{-z_1}} \tag{5.33}$$

the seller surplus of the Vickrey auction would be higher than the seller surplus of the corresponding first-price auction. For example, if $z_1 = 1$, this condition would be met if $z_2 > 3.16$; and if $z_1 = 5$, this condition would be met if $z_2 > 10.07 \simeq 2z_1$. For $L = 100$, $C = 20$, Table 5.1 compares the seller surpluses of the first-price auctions with those of the Vickrey auctions for cases when the lower bound in Equation 5.33 is used, and when $z_2 = 2z_1$. We see that the seller surpluses of Vickrey auctions can be noticeably higher than those of first-price auctions when the average number of bids is suitably raised in the former. Eventually, however, this difference in seller surpluses diminishes as the number of arriving bids grows. We also observe that using the approximation $z_2 = 2z_1$ instead of the bound in Equation 5.33 although does not work well when the average number of arriving bids is small, it is acceptable for higher values.

Earlier, we have shown that the buyer surplus ratio between the Vickrey and the first-price auctions is maintained at around two when the number of bids is large. This is assuming we have the same value for z in both cases. As we have just seen, the Vickrey auctions may stimulate greater bidder participation, so that it is useful to compare the buyer surplus ratio when the z values are different for the two types of auctions. Again, if z_1 is the expected number of bids for a first-price auction, and z_2 is the expected number of bids for the corresponding Vickrey auction, the buyer surplus ratio becomes, from equations (1) and (7) and simplifying

$$\frac{\beta_v(z_2)}{\beta(z_1)} = (\frac{z_1}{z_2})[\frac{2(1 - e^{-z_2}) - z_2 e^{-z_2}}{1 - e^{-z_1}}] \tag{5.34}$$

We wish to determine the condition for which the Vickrey auction buyer surplus falls below that of the first-price auction surplus, which is equivalent to

$$(\frac{z_1}{z_2})[\frac{2(1 - e^{-z_2}) - z_2 e^{-z_2}}{1 - e^{-z_1}}] < 1 \tag{5.35}$$

z_1	z_2 lower bound	**Average first-price seller surplus**	**Average Vickrey seller surplus (from Equation 5.33)**	**Average Vickrey seller surplus ($z_2 = 2z_1$)**
1	3.16	16.79	23.69	7.07
1.5	3.86	28.21	31.40	21.63
2	4.63	36.77	38.17	32.75
2.5	5.45	43.28	43.87	40.94
3	6.31	48.33	48.56	47.00
3.5	7.22	52.29	52.39	51.55
4	8.15	55.46	55.49	55.04
4.5	9.10	58.02	58.04	57.79
5	10.07	60.13	60.14	60.01
5.5	11.05	61.89	61.89	61.82
6	12.03	63.37	63.38	63.33
6.5	13.02	64.64	64.64	64.62
7	14.01	65.73	65.73	65.71
7.5	15.01	66.67	66.67	66.67
8	16.01	67.50	67.50	67.50

Table 5.1: Comparison of seller surplus when the number of arriving bids is different.

or, upon simplification,

$$z_2 > [\frac{2z_1(1 - e^{-z_2}) - z_1 z_2 e^{-z_2}}{1 - e^{-z_1}}] \tag{5.36}$$

Since $[1 - \exp(-z_2)]$ is a probability and therefore lies between zero and one, and the second term is positive, therefore a sufficient condition of the above becomes

$$z_2 > [\frac{2z_1}{1 - e^{-z_1}}] \tag{5.37}$$

Thus, under this condition, the buyer surplus of a Vickrey auction with an average number of bids z_2 would be lower than the buyer surplus of a first-price auction with an average number of bids z_1. Apart from surplus considerations, the seller is often interested in the rate at which the average auction income is earned. Since $z = \lambda T$, the income rate for first-price auction is given by

$$\frac{\lambda E[Q_{(N)}]}{z} = \frac{\lambda L}{z}[1 - \frac{1-e^{-z}}{z}] \tag{5.38}$$

From the seller's point of view, sometimes his/her auctions cannot be run continuously, i.e. it may not be possible to arrange things so that, as soon as one auction finishes, another one of the same seller's auction will start immediately, and there is a time delay between the auctions from the same seller. For example, in the model of (Gelenbe, 2009), there is a random rest time before the next auction from the seller is initiated. Such additional time delay between auctions will result in a lower auction income rate for the seller. Such a delay may be due to queueing, since there may be a time elapsed between the request for an auction and the time when the auction is actually run. If requests for auctions arrive as a Poisson stream with rate λ', and all auctions take a fixed amount of time T to run, then this situation may be modelled as an M/D/1 queue, and the average waiting time for an auction to commence is given by the Pollaczek-Khintchine formula (Gross et al., 2008)

$$E(W') = \frac{\rho T}{2(1-\rho)} \tag{5.39}$$

where $\rho = \lambda' T$ is the traffic intensity. This gives an overall average total time for completing an auction, as measured from the time of making a request to run the auction

$$E(W) = E(W') + T = \frac{T(2-\rho)}{2(1-\rho)} \tag{5.40}$$

The income rate for the first-price auction therefore becomes

$$\frac{E[Q_{(N)}]}{E(W)} = \frac{\lambda L}{z}[1 - \frac{1-e^{-z}}{z}][\frac{2(1-\rho)}{2-\rho}] \tag{5.41}$$

Compared with Equation 5.38, the income rate is reduced by a factor of

$$f = [\frac{2(1-\rho)}{2-\rho}] = 1 - \frac{\rho}{2-\rho} \tag{5.42}$$

Since for a stable queueing system, $\rho < 1$, so that the above factor is less than one. This reduction is less rapid than a linear decline if

$$f > 1 - \rho \tag{5.43}$$

or

$$\frac{2(1-\rho)}{2-\rho} > 1 - \rho. \tag{5.44}$$

This gives

$$\frac{2}{2-\rho} > 1, \tag{5.45}$$

which is true for $0 < \rho < 1$. Thus the average income reduction is slower than the linear income reduction rate. Figure 5-3 plots the income rate reduction factor f as a function of ρ, which also shows a comparison with the linear income reduction rate.

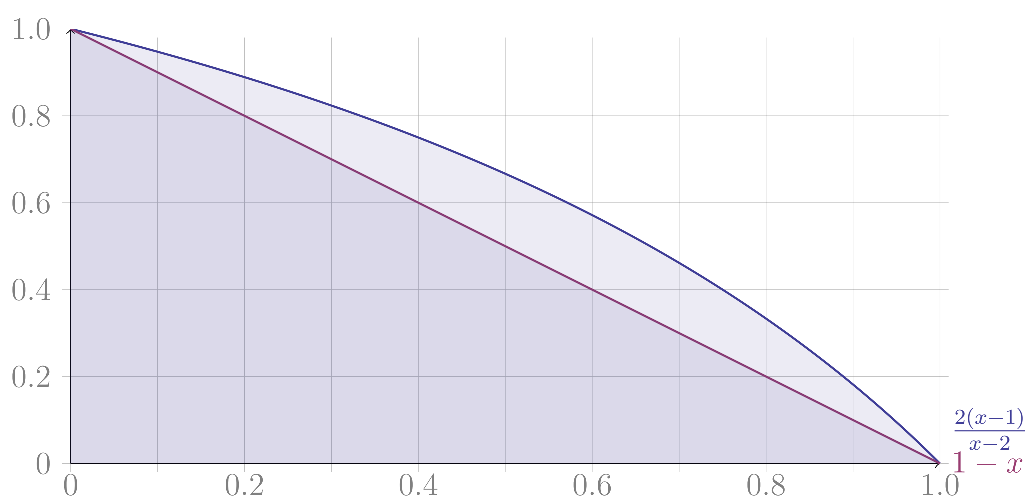

Figure 5-3: Income reduction factor for different traffic intensity values

We see that the income rate reduction is slight under relatively light traffic conditions, (e.g. when the traffic intensity is below 0.2), but can drop to below 10% of the original value under heavy traffic conditions (e.g. when the traffic intensity is over 0.9). We also see that the income rate decline is significantly slower than the linear rate which indicates a relatively subdued reduction behaviour. When the traffic intensity is 2/3, the reduction in auction income rate is approximately 50%, compared with the no queueing situation.

For Vickrey auctions, the income rate is

$$\frac{\lambda E[Q_{(N-1)}]}{z} = \frac{\lambda L}{z^2}(z + ze^{-z} + 2e^{-z} - 2). \tag{5.46}$$

Following similar arguments, the reduced auction income rate is

$$\frac{E[Q_{(N-1)}]}{E(W)} = \frac{\lambda f L}{z^2}(z + ze^{-z} + 2e^{-z} - 2) \tag{5.47}$$

which will decline in the same pattern as depicted in Figure 5-3.

In situations where different auction requests take different amounts of (fixed) time to run, then this will be an M/G/1 queue instead of an M/D/1 queue. If the distribution of the fixed auction durations is uniform over the integers $\{1, 2, \ldots, D\}$, then the mean service time is

$$\frac{1}{D}\sum_{k=1}^{D} k = \frac{D+1}{2} \tag{5.48}$$

and the second moment of the service time is

$$\frac{1}{D}\sum_{k=1}^{D} k^2 = \frac{(D+1)(2D+1)}{6} \tag{5.49}$$

Thus, from the Pollaczek-Khintchine formula, the average waiting time for an auction to commence is

$$E(V') = \frac{\rho'(2D+1)}{6\,(1-\rho')}, \tag{5.50}$$

where the traffic intensity here is $\rho' = \lambda'(D+1)/2$.

For a particular known fixed auction duration $T \in \{1, 2, \ldots, D\}$, the average total time for completing an auction measured from the time of making a request to run the auction

$$E(V) \;=\; E(V') + T = \frac{\rho'(2D+1)}{6\,(1-\rho')} + T. \tag{5.51}$$

resulting in an average first-price auction income rate of

$$\frac{E\left[Q_{(N)}\right]}{E(V)} \;=\; L\left[1 - \frac{1-e^{-z}}{z}\right] / \left[\frac{\rho'(2D+1)}{6\,(1-\rho')} + T\right]. \tag{5.52}$$

Compared with the income rate when there is no queueing delay in auction commencement, the average auction income rate reduction factor is

$$f' \;=\; T/\left[\frac{\rho'(2D+1)}{6\,(1-\rho')} + T\right] \tag{5.53}$$

A stable system requires $\rho' < 1$ or $\lambda' < 2/(D+1)$. For $D = 5$, the allowable range of values for λ' to ensure system stability is (0, 0.33), and for $T = 1$, this results in the following rate reduction factor

$$f' \;=\; \frac{2-6\lambda'}{2+5\lambda'}. \tag{5.54}$$

The above analysis is exact for situations where a website offers a single auction channel, or when certain categories of auctions (e.g. consumer auctions vs business auctions) are directed to one particular auction channel. In situations where a website supports multiple simultaneous auctions irrespective of auction categories, then the underlying queue may be regarded as a multi-channel queue and should be modelled by an $M/G/m$ queue. However, for this queue, there is in general no closed-form solution available. Even for the $M/D/m$ queue, there is no neat solution for the mean waiting time (Gross et al., 2008). However, the above may be regarded as a type of approximation—here we split a single $M/G/m$ queue into several $M/G/1$

queues, by evenly dividing the traffic to each queue. That is, if there are m servers, and if the overall auction traffic arrival rate is λ^*, then the auction arrival rate for each queue is $\lambda' = \lambda^*/m$. In this approximation, there is a slight over-estimation of the waiting time compared with the single $M/G/m$ case, since the allocation of servers to customers may be less efficient in the multiple single server queues. In particular, some single server queues may be idle even though others may have long queues of waiting customers—this situation would not arise in the $M/G/m$ case where all servers would be busy whenever there are waiting customers.

5.2 Accepting Multiple Bids and Auction Fees

From the seller's point of view, the aim of auction is to attain improvements in seller surplus through expending more time to achieve a higher price or income. If one simply accepts the first bid that comes along, then its average magnitude is $E[Q_i] = L/2$. By holding an auction, the average gain in surplus per bid acceptance due to one auction, assuming there are N bids, is

$$E\left[Q_{(N)}|N\right] - E[Q_i|N] = \frac{LN}{N+1} - \frac{L}{2} \tag{5.55}$$

$$= \frac{(N-1)L}{2(N+1)}. \tag{5.56}$$

If there are two or more identical items for sale, to speed things up one might accept the two highest bids, instead of the just the highest one. Accepting more than one bid per auction is quite common in Internet auctions; e.g. Google's ad auctions often accept several bids. From (Barry C. Arnold et. al. 2008), it is shown that the k-th order statistic of N samples from a uniform distribution distributed over a given interval is $k/(N+1)$ of the length of the interval. Thus, in accepting the two highest bids in one auction, the average gain in seller surplus per bid acceptance is:

$$\frac{1}{2}\left\{\left(E\left[Q_{(N)}|N\right] - E[Q_i|N]\right) + \right. \tag{5.57}$$

$$\left.\left(E\left[Q_{(N-1)}|N\right] - E[Q_i|N]\right)\right\} \tag{5.58}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(E\left[Q_{(N)}|N\right] - E\left[Q_{(N-1)}|N\right]\right) - E\left[Q_i|N\right] && (5.59)\\
&= \frac{L}{2}\left[\frac{N}{N+1} + \frac{N-1}{N+1}\right] - \frac{L}{2} && (5.60)\\
&= \frac{(N-2)L}{2(N+1)}, && (5.61)
\end{aligned}
$$

which is less than the gain in seller surplus in the case where only one bid is accepted, but it takes only one instead of two auction times and associated costs to achieve two acceptances. Correspondingly, the average buyer surplus will increase, since the transaction price of the second item is lower.

In general, the average of the highest K bids, given there are N bids, is

$$
\begin{aligned}
\frac{1}{K}\sum_{j=0}^{K-1} E\left[Q_{(N-j)}|N\right] &= \frac{L}{K}\sum_{j=0}^{K-1}\frac{N-j}{N+1} && (5.62)\\
&= \frac{(2N-K+1)L}{2(N+1)}. && (5.63)
\end{aligned}
$$

Thus, the conditional average gain in surplus per acceptance is

$$
\begin{aligned}
&\frac{1}{K}\sum_{j=0}^{K-1}\left\{E\left[Q_{(N-j)}|N\right] - E\left[Q_i|N\right]\right\} && (5.64)\\
&= \frac{1}{K}\left\{\sum_{j=0}^{K-1} E\left[Q_{(N-j)}|N\right]\right\} - E\left[Q_i|N\right] && (5.65)\\
&= \frac{1}{K}\sum_{j=0}^{K-1} E\left[Q_{(j)}|N\right] - \frac{L}{2} && (5.66)\\
&= \frac{(2N-K+1)L}{2(N+1)} - \frac{L}{2} && (5.67)\\
&= \frac{(N-K)L}{2(N+1)}, && (5.68)
\end{aligned}
$$

which from Equation 5.56 is always below the gain in surplus resulting from accepting a single bid per auction. The average total income in accepting the top K bids, from Equation 5.63, is

$$
\sum_{j=0}^{K-1} E\left[Q_{(N-j)}|N\right] = \frac{(2N-K+1)LK}{2(N+1)}. \qquad (5.69)
$$

Consider a variation of the basic model, in which the K highest bids are accepted in one auction. Note that accepting K highest bids requires that there are at least K arrivals (and of course at least K items for sale), and for meaningful operation, this requires that $T \gg K \times$Mean Inter-arrival Time or $\lambda T \gg K$. Removing the condition on N, and noting that $N \geq K$, we have, for the average total income in accepting the top K bids,

$$
\begin{aligned}
& \sum_{j=0}^{K-1} E\left[Q_{(N-j)}\right] && (5.70)\\
= & \sum_{N=K}^{\infty} \frac{LK(2N-K+1)}{2(N+1)} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} && (5.71)\\
= & \frac{LK}{2} \sum_{N=K}^{\infty} \left[2 - \frac{K+1}{N+1}\right] \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} && (5.72)\\
= & LK \sum_{N=K}^{\infty} \frac{e^{-\lambda T}(\lambda T)^N}{N!} - \frac{LK}{2\lambda T} \sum_{N=K}^{\infty} \left[\frac{K+1}{N+1}\right] \\
& \times \frac{e^{-\lambda T}(\lambda T)^{N+1}}{N!} && (5.73)\\
= & LK \sum_{N=K}^{\infty} \frac{e^{-\lambda T}(\lambda T)^N}{N!} - \\
& \frac{LK(K+1)}{2\lambda T} \sum_{N=K}^{\infty} \frac{e^{-\lambda T}(\lambda T)^{N+1}}{(N+1)!} && (5.74)\\
= & LK \left\{1 - \sum_{j=0}^{K-1} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\} - \\
& \frac{LK(K+1)}{2\lambda T} \left\{1 - \sum_{j=0}^{K} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\}. && (5.75)
\end{aligned}
$$

That is, we have for the expected income I_K when we choose to accept K top bids in a single auction

$$
\begin{aligned}
I_K = & LK \left\{1 - \sum_{j=0}^{K-1} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\} - \\
& \frac{LK(K+1)}{2\lambda T} \left\{1 - \sum_{j=0}^{K} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\}. && (5.76)
\end{aligned}
$$

We see that for $K = 1$, the above reduces to Equation 3.4, and for the important special case $K = 2$, we have

$$I_2 \quad = \quad \frac{L}{\lambda T}\left(2\lambda T + \lambda T e^{-\lambda T} - \frac{(\lambda T)^2 e^{-\lambda T}}{2} + 3e^{-\lambda T} - 3\right). \tag{5.77}$$

Consider the cost Ω of holding an auction, which may be related to the auction time and associated costs such as fees paid to the auction site, and payments to financial intermediaries. We assume that these costs are otherwise not incurred if the item is sold through other channels. Let the price of a unit of the good be C. If no auctions are held, then the expected seller surplus would simply be $(L/2 - C)$, where, as indicated from the arguments above, $L/2$ represents the average value of the first offer, and we assume that it will be accepted. By holding an auction, the seller surplus S becomes:

$$S \quad = \quad \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - (C + \Omega). \tag{5.78}$$

Thus, the break-even point of holding an auction is given by the improvement in surplus offset by the auction cost

$$\frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - \frac{L}{2} = \Omega. \tag{5.79}$$

As the auction extends in time, the expected income rises. Supposing one wishes to attain a certain level of seller surplus S_o, then the minimal auction duration T^* is given by the solution to the following equation

$$S_0 \quad = \quad \frac{L}{\lambda T^*}\left(\lambda T^* + e^{-\lambda T^*} - 1\right) - (C + \Omega). \tag{5.80}$$

While we may solve for the above using numerical methods, we may obtain closed-form solutions by using an approximation. An approximation will enable the quick estimation of auction parameters and economic benefits without going through the laborious process of numerical solution. We shall use an approximation based on Equation 3.1. If we remove the condition on N there by replacing N by its average (from the Poisson distribution) of $z = \lambda T$, we have approximately

$$E\left[Q_{(N)}\right] \cong \frac{Lz}{z+1}. \tag{5.81}$$

Figure 5-4 compares the average auction income from Equations 3.1 and the above for $L = 100$ for different values of $z = \lambda T$. We see that the approximation is quite good for moderate to large values of z. For very large values of z, the exact formula and the approximation are virtually indistinguishable. Using this approximation, the above becomes

$$S_0 = \frac{Lz'}{z'+1} - (C + \Omega), \tag{5.82}$$

giving

$$z' = \frac{S_0 + C + \Omega}{L - (S_0 + C + \Omega)}, \tag{5.83}$$

and this will provide a reasonable approximation for $z' \gg 1$. Thus, the approximate optimal auction duration T' is

$$T' = \frac{S_0 + C + \Omega}{\lambda\left[L - (S_0 + C + \Omega)\right]}, \tag{5.84}$$

Letting $S_0 + C + \Omega = 90$, $L = 100$, and $\lambda = 1$, and numerically solving Equation 5.80 provides the exact minimum T^* in order to achieve a minimum surplus of S_0 which in this case is found to be $T^* = 10$ (see Figure 5-5). As can be seen from Figure 5-5, any value of $T > 10$ will yield at least a surplus of S_0. The corresponding approximate solution gives $T' = 90/(100 - 90) = 9$, which yields an error of just under 10%. From the seller's point of view, in order to quickly determine the optimal T^*, while avoiding the elaborate procedure of finding a numerical solution to Equation 5.80, one can simply first solve for T', and then add a safety factor to ensure that the resultant surplus $\geq S_0$, which in the present case may be 10%. A higher safety factor may be used to ensure greater certainty of achieving the required level of minimum surplus.

Next, the overall surplus in accepting K bids per auction is

$$I_K - KC - \Omega \tag{5.85}$$

While the overall surplus in selling K items through K separate auctions would be

$$\frac{LK}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - K(C + \Omega). \tag{5.86}$$

Thus, it would be more profitable to sell K items in K separate auctions instead of selling them in a single auction if the expected surplus of the latter is higher, i.e.

$$\frac{LK}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - K(C + \Omega) \quad > \quad I_K - KC - \Omega \tag{5.87}$$

or

$$\frac{LK}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - (K - 1)\Omega \quad > \quad I_K \tag{5.88}$$

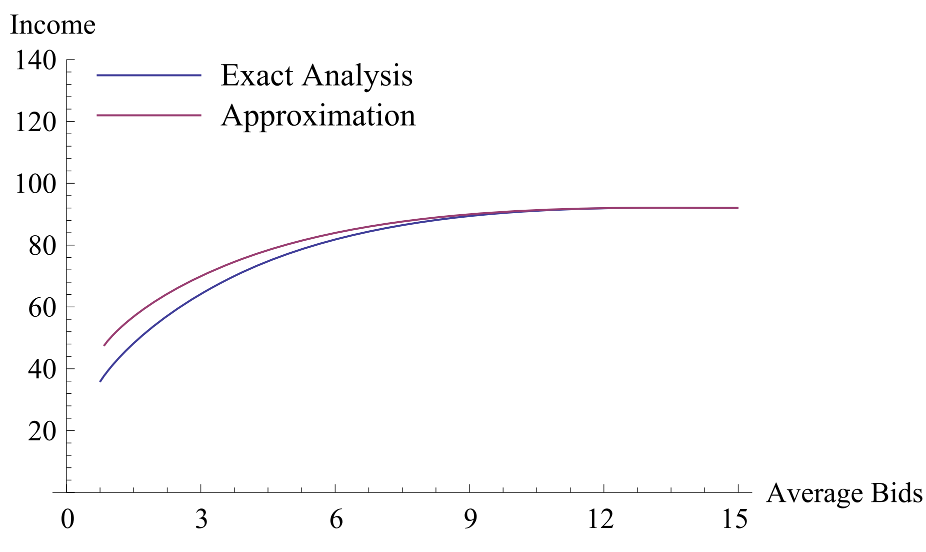

Figure 5-4: Comparison of exact analysis and approximation.

For the important special case $K = 2$, the above becomes

$$\frac{2L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - \Omega \quad > \quad I_2 \tag{5.89}$$

and making use of Equation 5.77, this condition simplifies to

$$e^z\left(1 - \frac{z\Omega}{L}\right) \quad > \quad 1 + z - \frac{z^2}{2}. \tag{5.90}$$

We see that in terms of magnitude, the left hand side increases exponentially in z, while the right hand side increases quadratically. Thus, for sufficiently large z, the left hand side will go negative with a large magnitude, while the right hand side will also go negative with a comparatively smaller magnitude; consequently the above inequality will not hold for large z. Thus, when the number of bids is large, it is always preferable to sell the items in single auctions.

Sometimes, the auction fee structure is such that the auction website would charge for a certain percentage of the income payment, which for instance is the common practice of eBay. Denoting by ξ such a percentage, then the overall surplus in accepting K bids per auction is

$$(1 - \xi)I_K - KC. \tag{5.91}$$

In adopting the same approximation as before by suitably replacing N by λT, then from Equation 5.69, we have

$$I_K \simeq \frac{(2\lambda T - K + 1)LK}{2(\lambda T + 1)}. \tag{5.92}$$

Using this approximation for the special case $K = 2$, we have

$$I_2 \simeq \frac{(2\lambda T - 1)L}{\lambda T + 1}, \tag{5.93}$$

so that the overall surplus is approximately

$$\frac{L(1-\xi)(2\lambda T-1)}{\lambda T+1}-2C. \tag{5.94}$$

Thus it would be preferable to accept two bids per auction rather than to accept a single bid in two separate auctions if

$$2(1-\xi)I_1-2C \quad < \quad (1-\xi)I_2-2C, \tag{5.95}$$

or

$$\frac{2\lambda TL}{\lambda T+1} \quad < \quad \frac{2\lambda TL-L}{\lambda T+1}, \tag{5.96}$$

which is never the case for $L > 0$. Thus, for this particular auction fee structure, unlike the previous case, it would always be preferable to run two separate auctions rather than a single auction given that the number of bids is large. In fact, even when the number of bids is not large, the general validity of this choice can be seen from Equation 5.69, where the total income from running a single auction is

$$(1-\xi)I_{K|N} \quad = \quad \frac{LK(1-\xi)(2N-K+1)}{2(N+1)}, \tag{5.97}$$

where $I_{K|N}$ signifies the total income conditioning on N. The corresponding quantity in running K separate auctions is

$$KI_{1|N} \quad = \quad \frac{(1-\xi)NLK}{N+1} \tag{5.98}$$

Thus, it is preferable to run separate auctions if

$$\frac{N}{N+1} \quad > \quad \frac{2N-K+1}{2(N+1)} \tag{5.99}$$

which will be valid whenever $K > 1$. Thus, for this particular auction fee structure, unlike the previous one, it is always more advantageous for the seller to sell the items in separate auctions.

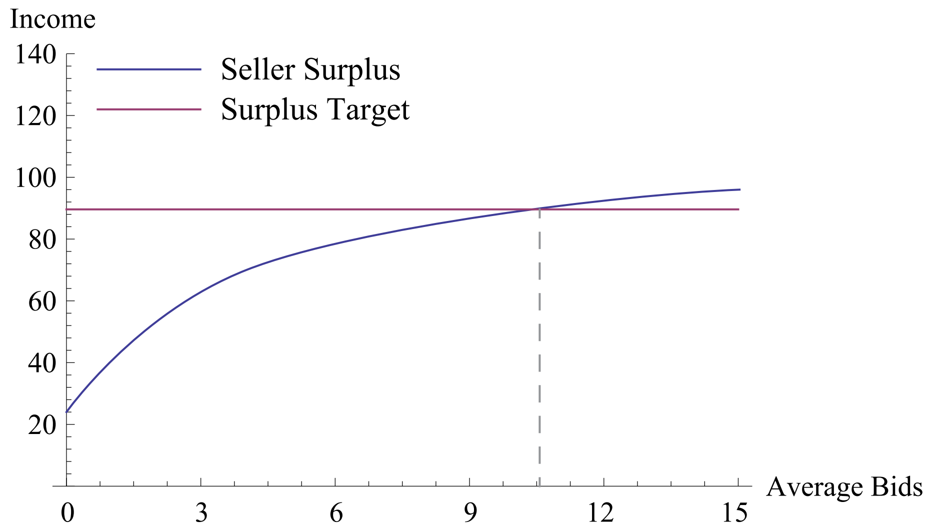

Figure 5-5: Numerical determination of optimal auction duration.

5.3 Surplus Analysis of Soft Close Auctions

For these auctions, a key parameter is $s = \lambda\alpha$, which is the expected number of bid arrivals in a window of length α. For the soft close auctions of Algorithm II, the buyer surplus, from equation (3.16), would be

$$\beta(s) = L - \frac{L}{e^{\lambda\alpha} - 1}[e^{\lambda\alpha} - \lambda\alpha - 1] = \frac{sL}{e^s - 1} \tag{5.100}$$

Since

$$\frac{d\beta(s)}{ds} = -\frac{L[e^s(s-1)+1]}{(e^s-1)^2} < 0 \tag{5.101}$$

for $s \geq 1$, which signifies that there is an average of at least one arrival in the window of length α, we see that as s increases, the buyer surplus tends to diminish.

Similarly, for the soft close auctions of Algorithm VII, the buyer surplus, from

Chapter 4, is

$$\beta_v(s) = L - \frac{2L}{e^s - 1}[\sinh(s) - s] = \frac{L(e^{-s} + 2s - 1)}{e^s - 1} \tag{5.102}$$

Now

$$\frac{d\beta_v(s)}{ds} = \frac{L[e^{-s} - 4 - e^s(2s - 3)]}{(e^s - 1)^2} \tag{5.103}$$

Since $\exp(-s) < 1$, for $s > 0$, and the last term is negative if $s \geq 2$, combining these, we conclude that

$$\frac{d\beta_v(s)}{ds} < 0 \; for \; s \geq 2 \tag{5.104}$$

The condition $s \geq 2$ is a reasonable one, since s represents the average number of bid arrivals in a window of length α, and for a Vickrey auction, it takes at least two bids to make it meaningful. This condition implies that as s increases, the buyer surplus tends to decline.

Since the buyer pays a lower price in Vickrey auctions, it is useful to compare the buyer surplus of a first-price auction with that of a Vickrey auction. From equations 5.100 and 5.102 and simplifying, we have for the Vickrey surplus ratio for this case

$$\frac{\beta_v(s)}{\beta(s)} = \frac{e^{-s} + 2s - 1}{s} \tag{5.105}$$

We see that, by application of L'Hôpital's rule,

$$\frac{\beta_v(s)}{\beta(s)} \to 1 \; as \; s \to 0 \tag{5.106}$$

Also, since the Vickrey surplus ratio can be written as

$$\frac{1}{se^s} + 2 - \frac{1}{s} \tag{5.107}$$

Thus, we see that

$$\frac{\beta_v(s)}{\beta(s)} \to 2 \; as \; s \to \infty \tag{5.108}$$

The case $s \to \infty$ corresponds to a situation where there are a very large number of bid arrivals in the window of length α, which suggests that the probability of having no arrivals in that window is very small. Consequently, this means that the auction will be kept going indefinitely, and thus will last for an effectively infinite amount of time. This is unlike the fixed auction time situations, where the auction will definitely finish in a pre-determined amount of finite time.

On the other hand, the seller surplus for the first-price auction is

$$\xi(s) = \frac{L[e^s - s - 1]}{e^s - 1} - C \tag{5.109}$$

and the seller surplus for the Vickrey auction is

$$\xi_v(s) = \frac{2L}{(e^s - 1)}[\sinh(s) - s] - C \tag{5.110}$$

Thus, the seller surplus reduction by holding a Vickrey auction instead of first-price auction is

$$\xi(s) - \xi_v(s) = \frac{L}{(e^s - 1)}[e^{-s} + s - 1] \tag{5.111}$$

For $s \to 0$, we see that, by application of L'Hôpital's rule, that the reduction in seller surplus $\to 0$. Also, for as $s \to \infty$, we see that each term of Equation 5.111 on dividing by $(e^s - 1)$ tends to 0, and so the reduction in seller surplus also $\to 0$.

In Equation 5.108, we see the buyer surplus ratio between the Vickrey and the first-price auctions is maintained at around two when the number of bids is large, when there is the same average number of bids s in both cases. However, since Vickrey auctions tend to encourage greater bidder participation, so that it is useful to compare the buyer surplus ratio when the average number of bids in the Vickrey auctions is greater than the average number of bids in the first-price auction. We denote by s_1 the expected number of bids for a first-price auction, and s_2 the expected number of bids for the corresponding Vickrey auction. Figure 5-6 plots the buyer surplus for different average number of bids for the first-price auction and Vickrey auction with $L = 100$. For ease of reference, the s value for the first-price auction shall be denoted by s_1, and the s value of the Vickrey auction shall be denoted by s_2.

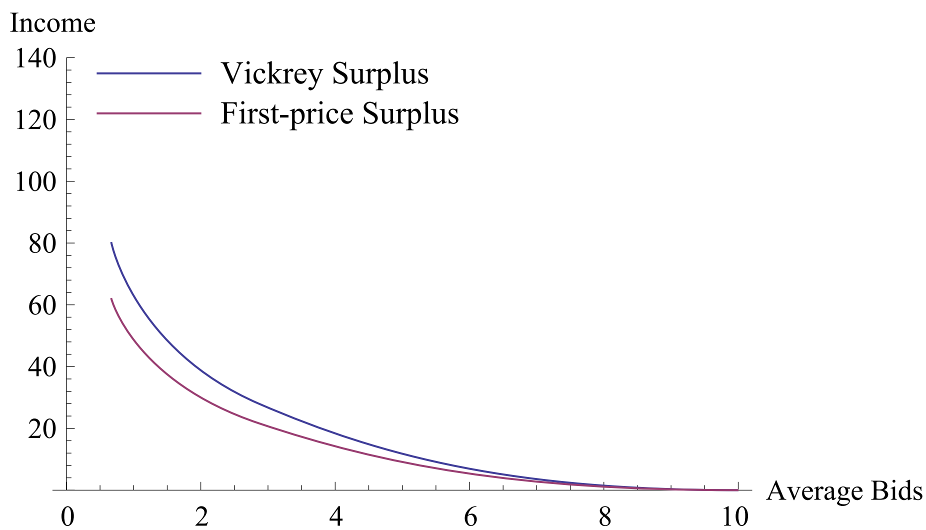

Figure 5-6: Comparison of buyer auction surpluses.

We see that when $s_1 = 2$, the first-price auction buyer surplus is around 30, which is greater than the Vickrey auction buyer surplus when $s_2 = 4$ of around 13. Table II shows the difference between the average number of bids for the same buyer surpluses for the first-price and Vickrey auctions. We see that for the same amount of surplus, the average number of bids for these two types of auctions tend to differ by around 0.7 to 0.75 for this situation. Consequently, we see that here the buyer surplus will be lower in Vickrey auctions if these are able to attract an additional one bid in the window of $(0, \alpha)$. Thus, from the buyers' point of view, it is preferable to go for first-price auctions with fewer bidders than Vickrey auctions with many bidders. This suggests that sometimes, if applicable, it may even be worthwhile to pay a surcharge (which may take on a variety of forms such as tax, membership fee, or additional delivery charge) to enter an auction in the hope that it would serve as an inhibiting factor for other bidders so that the surplus gained would more than cover the surcharge. In addition, individually, it is worth noting that with either of these two auction types, the buyer surplus drops steeply at first, and the drop slows down when the average number of bids becomes large.

Buyer Surplus	**Average Number of bids (Vickrey)**	**Average Number of bids (First-Price)**	**Difference in the Average Number of Bids**
50	1.97	1.26	0.71
45	2.15	1.43	0.72
40	2.35	1.62	0.73
35	2.57	1.83	0.74
30	2.81	2.06	0.75
25	3.09	2.34	0.75
20	3.41	2.66	0.75
15	3.81	3.06	0.75
10	4.36	3.61	0.75
5	5.25	4.51	0.74

Table 5.2: Comparison of the average number of bids for equal surplus.

From the seller's point of view, the seller surplus between the first-price auction and the corresponding Vickrey auction would be

$$\xi(s_1) - \xi_v(s_2) < 0 \tag{5.112}$$

if

$$\frac{e^{s_1} - s_1 - 1}{e^{s_1} - 1} < \frac{2[\sinh(s_2) - s_2]}{(e^{s_2} - 1)} \tag{5.113}$$

Figure 5-7 plots the seller surplus for both types of auctions for $C = 20$. We see that the two surplus curves tend to be parallel, separating by below one bid, initially. For example, to attain a surplus of 50, we see that $s_1 = 2.06$ and $s_2 = 2.81$ with $s_2 - s_1 = 0.75$. Hence, if the seller is confident that by running a Vickrey auction, an average of one additional bid is attracted to the auction, then the expected surplus of the a Vickrey auction would be higher, and thus it would be better to run a Vickrey auction instead of a first-price auction.

As for Algorithm V, if the buyer is aiming for a surplus of ξ, then B should be

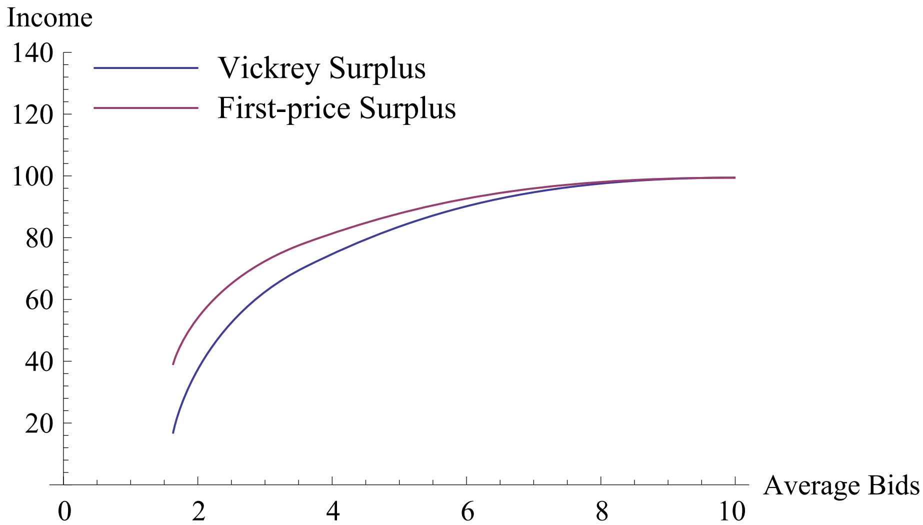

Figure 5-7: Comparison of seller auction surpluses.

chosen such that

$$C + \xi = \frac{BL}{B+1} \tag{5.114}$$

Solving for B gives

$$B = \lceil \frac{C+\xi}{L-(c+\xi)} \rceil \tag{5.115}$$

Since the above is generally not an integer, the ceiling function

$$\lceil \frac{C+\xi}{L-(c+\xi)} \rceil \tag{5.116}$$

should yield a surplus not less than ξ, since we have shown earlier that the auction income is an increasing function of B.

Similarly, for Algorithm X, the parameter B should satisfy

$$C + \xi = \frac{(B-1)L}{B+1} \tag{5.117}$$

giving

$$B = [\frac{L + C + \xi}{L - (C + \xi)}] \tag{5.118}$$

which would yield a surplus not less than ξ. For example, if $C + \xi = 80$, $L = 100$, then B in Algorithm V should be set to 4, while it should be set to 9 in Algorithm X.

From the buyers' perspective, the buyer surplus for the first-price auction is

$$\beta = L - \frac{BL}{B+1} = \frac{L}{B+1} \tag{5.119}$$

while that for the Vickrey auction is

$$\beta_v = L - \frac{(B-1)L}{B+1} = \frac{2L}{B+1} \tag{5.120}$$

showing again that the buyer surplus for the Vickrey auction is twice that for the first-price auction.

5.4 Surplus Analysis of Auctions with Threshold Termination

For auctions with threshold termination, we have shown earlier that their average duration is generally shorter than the corresponding auctions without the threshold termination. For these auction algorithms, the arriving bids are probabilistically the same as for the corresponding auctions without threshold termination. The only difference between the two is that it would terminate prematurely upon receiving a certain bid, otherwise the auction would continue through to usual completion. As we have shown earlier, longer auctions tend to yield higher incomes, and since auctions without threshold termination tend to run longer, their auction income is also higher (of course, in the event that there is no arrival from the Λ-stream, the two auctions would be identical in both duration and income), a property also evident from the income graphs of earlier chapters.

The setting of the threshold M by the sellers is related to the sellers' view on the magnitude of an acceptable surplus. If a seller regards ξ is an acceptable level of minimum surplus, then the seller may set $M = C + \xi$. If the auction is finished through threshold termination, then since the conditional income for this situation is $(M + L)/2$, which is higher than M, then the seller would expect to fetch a surplus exceeding ξ. However, as we have just indicated, such a surplus will still on average be lower than the expected surplus were the auction be allowed to run its full course, since the latter possibility will result in a longer auction and hence attaining higher average income. For example, if the acceptable level of minimum surplus ξ is 30, then for $C = 20$, we have $M = 50$, and the expected auction income for $L = 100$, $\alpha = 3$, is plotted in Figure 5-9. We see that the minimum acceptable surplus is attained when the arrival rate is 0.5, and when the arrival rate is 1, the surplus reaches 50. From Chapter 3, the average auction income of Algorithm IV,

$$E[Q'_M] = \frac{p_1 M[1 - \frac{\Lambda'\alpha}{(e^{\Lambda'\alpha}-1)}] + p_2[\frac{M+L}{2}]}{P_1 + P_2} \tag{5.121}$$

is a convex combination of two component incomes. The first one

$$M[1 - \frac{\Lambda'\alpha}{(e^{\Lambda'\alpha} - 1)}] \tag{5.122}$$

is bounded above by M, since

$$\frac{\Lambda'\alpha}{(e^{\Lambda'\alpha} - 1)} < 1 \tag{5.123}$$

which can be seen on expanding $\exp(\Lambda'\alpha)$. On the other hand, the second one

$$[\frac{M + L}{2}] \tag{5.124}$$

is bounded below by M, so that the second income is always greater than the first income. Therefore the average auction income is

$$E[Q'_M] \leq \frac{p_1[\frac{M+L}{2}] + p_2[\frac{M+L}{2}]}{p_1 + p_2} = [\frac{M+L}{2}] \tag{5.125}$$

which in this case is (50+100)/2=75. We see from Figure 5-9 that the auction income is always below 75. Indeed, as the bid rate increases, $s \to \infty$ and so $p_1 = e^s \to 0$, the average auction income gradually approaches this bound. This is intuitively reasonable and is to be expected since when there is a large number of bids arriving, the chance of one of them meeting the threshold requirement would accordingly be high, and so the auction terminates at $(M + L)/2$.

As for Algorithm III, the average auction income, from Chapter 3, may be written as

$$E[Q_M] = e^{-\Lambda T} M[1 - \frac{1 - e^{-\Lambda' T}}{\Lambda' T}] + (1 - e^{-\Lambda T})[\frac{M+L}{2}] \tag{5.126}$$

In order for the first component income to be less than M, the condition

$$\frac{(1 - e^{-\Lambda' T})}{\Lambda' T} < 1 \tag{5.127}$$

has to be satisfied, which is equivalent to

$$(e^{\Lambda' T} - 1) < \Lambda' T e^{\Lambda' T} \tag{5.128}$$

Expanding both sides, we have

$$\Lambda' T + \frac{(\Lambda' T)^2}{2!} + \frac{(\Lambda' T)^3}{3!} + ... < \Lambda' T + (\Lambda' T)^2 + \frac{\Lambda' T^3}{2!} + \frac{\Lambda' T^4}{3!} + \dots \tag{5.129}$$

which is seen to be true on term by term comparison and noting that $\Lambda' T > 0$. Thus the first component income in $E\left[Q_{(M)}\right]$ is less than M; obviously, the second income component is greater than M, and so

$$E[Q_M] \leq e^{-\Lambda T}[\frac{M+L}{2}] + (1 - e^{-\Lambda T})[\frac{M+L}{2}] = [\frac{M+L}{2}] \tag{5.130}$$

A plot of the average auction income for $C = 30$, $\xi = 40$, $L = 100$, $T = 10$, is shown in Figure 5-8. We see that the acceptable surplus is reached when the bid rate is around 0.34, and it converges to, but bounded by, 85 as the bid rate increases.

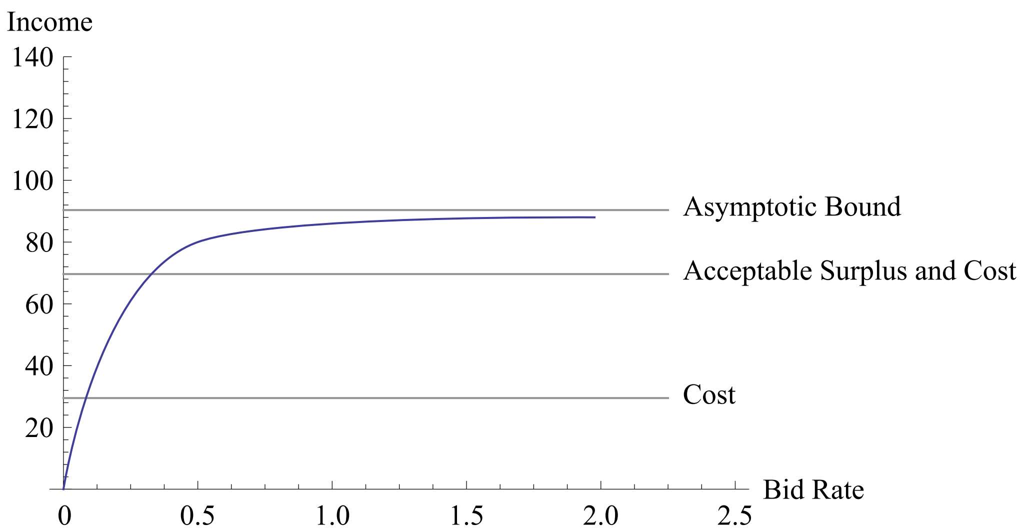

Figure 5-8: Seller Surplus for Algorithm III.

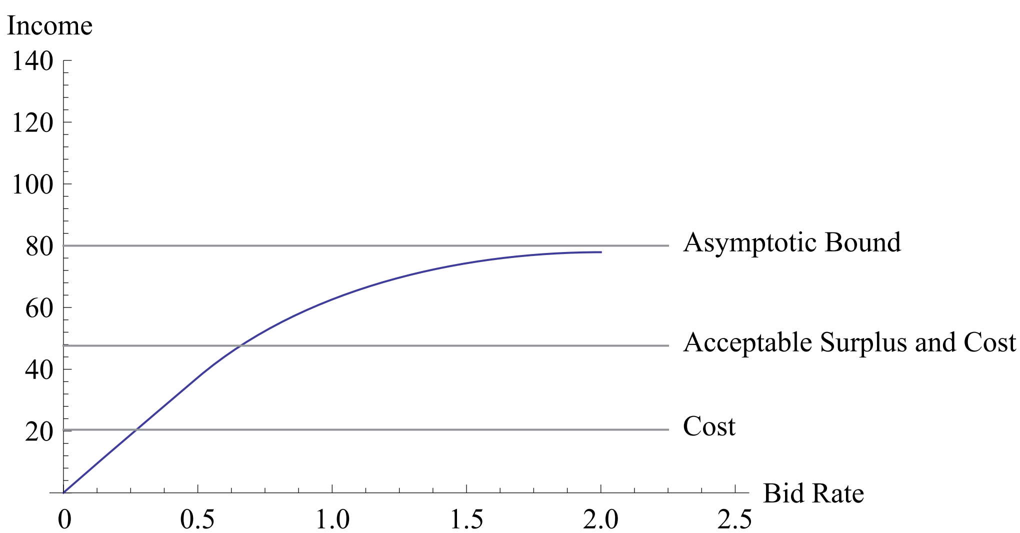

Figure 5-9: Seller Surplus for Algorithm IV.

As for Vickrey auctions with threshold termination, the same auction income bound of $(M + L)/2$ also applies since the income component from non-threshold termination here is even lower than that of the first-price auction. By the same argument, when the bid rate is large, the chance of termination by reaching M would be high, and so the income upper bound of $(M + L)/2$ is likely to be attained. Thus, for auctions with reasonably strong participation, there is little difference between the first-price auctions and Vickrey auctions in terms of surplus.

5.5 Summary

From both the buyers and sellers points of view, apart from successfully buying or selling the goods or services, a key objective is to reap maximum benefits from auctions in the form of economic surpluses. Compared with buyers, sellers tend to have greater flexibility and control over how an auction is being conducted. They can choose the duration parameters of the auctions, as well as whether to run them as first-price or Vickrey auctions. In addition, they can choose to incorporate the buy-it-now option to possibly shorten the auction duration once a required level of surplus is reached. We have analysed and compared the surpluses from both the buyers and sellers perspectives for the different auction algorithms. From the analysis performed, it is found that it would often be advantageous for sellers to run Vickrey auctions rather than first-price auctions. Another auction design option available to sellers when multiple items are for sale is whether or not to accept more than one bid per auction. Doing so will obviously save time and effort, but will also lower the average seller surplus. However, depending on the auction fee structure, accepting more than one bid in an auction can sometimes yield greater economic benefits than holding separate auctions. A factor which neither the sellers nor buyers have any real control is the bid rate. From the buyers' point of view, having a high bidder participation rate is always disadvantageous for buyer surplus. Thus, it is sometimes preferable for buyers to participate in less crowded first-price auctions than crowded Vickrey auctions. Indeed, inhibiting participation from other bidders is so important that it may sometimes pay to opt for those auctions with surcharge (if such choice is applicable), in the expectation that the gain in surplus will more than cover the surcharge. Of course, if such information is unavailable, then the buyers should always opt for the Vickrey auction which generally yields twice the surplus compared with first-price auctions.

Chapter 6

Experimental Validation & Design

6.1 Simulation Experimemts

To enable the comparison between observed and theoretical values and to validate the mathematical models, an auction process simulator that implements the pseudo-code listed in Algorithms I–XV, has been constructed in C++. In order to sample values from the uniform and exponential distributions for the private value of bidders and the rate of bids respectively, the Boost C++ Library is used. In particular, we use the `variate_generator` with the `uniform_01` and `exponential_distribution` headers, which is implemented on top of the `mersenne_twister` psuedo-random number generator. The result is outputted as a space-delimited text string that states lambda, which is the incoming rate of bids and usually the variable we change, the duration of that auction, and the revenue generated from that auction. Ten-thousand trials are run for each lambda which is sampled in 0.01 intervals in the units concerned over the desired interval. The precision of results is set to ten digits. Usage of the simulator is given in Figure 6-1.

A class diagram of the auction process simulator is presented in Figure 6-2. An

```
auction [-A algorithm] [-L L] [-l lambda] [-T T] [-a alpha]
    [-M M] [-B B] [-d delta] [-t trials]
```

Figure 6-1: Auction process simulator usage.

Internet auction website contains a number of buyers and sellers and runs multiple auctions, each of which contains a subset of the auction website's buyers and sellers exchanging money for lots of goods and services. Each auction has a set of parameters which equate to the parameters of the various algorithms.

Buyers have a utility function for each lot that is present at the auction website, and if the utility gained from buying a lot at a specific price is greater than that derived from holding the money, or using that lot to buy another lot, then the buyer will offer to bid for that lot in an auction. Both the utility of a win and a loss is required in determining how close a buyer should bid to his true valuation. If the loss of an auction is very costly, the buyer should bid the true valuation of the lot straight away, instead of trying to augment his utility by obtaining the lot at a lower price that he was willing to pay for it. The confidence that a buyer has in his valuation determines whether the valuation will be adjusted in the face of significantly higher or lower bids from other bidders present in the auction. A high confidence in the values means that the list of utility for each lot remains the same throughout an auction, while a lower confidence may mean that the auction itself is, to a certain extent, a price discovery mechanism for the bidder and his utility of obtaining the lot may vary in accordance with the current bid levels. In the same way, the seller also has lists of utilities and confidences, as well as a value for reputation which may result in a price discount or premium in an auction depending on factors that may be external to the lot, e.g. timeliness of delivery. Currently there are a few features listed in the class diagram that have not been fully implemented such as the buyer and seller utilities, confidence in their utility and their ratings at the auction website.

The class diagram uses standard Unified Modelling Language (UML) notation and shows different classes, their attributes and methods, and how they inter-relate. Private attributes and methods, which are known only to the class itself and to no other classes, are preceded with a (-) symbol, while, if public, preceded by a (+) symbol. There exists four main classes: the Auction Website, the Auction, the Buyer and the Seller. Classes connected by just a line indicate association, while a hollow diamond, aggregation, i.e. "Class A *has* Class B", with the hollow diamond-end connecting to the container class (A). A solid diamond, on the other hand, means composition, which is a stronger version of association, with the associated class often unable to exist if the class to which it is associated is destroyed. Numbers at two ends of the lines, indicate multiplicity, or the number of classes that participate in that association, e.g. "1..*" indicates an association that relates to "1 or more" classes. Thus, in Figure 6-2, an auction house/website runs one or more auctions which have parameters determining whether it is first-price or k-price, whether it has a minimum or maximum limit on the number of bids, and can either be a fixed time or variable time auction. Furthermore, each auction takes place by auctioning off a lot which is transferred from one owner (the seller) to another owner (the buyer), each of which has an associated reputation and wealth. Sellers can also set a reserve price on a lot, while each buyer has a valuation of each lot (consisting of an associated utility and confidence in that utility), which determines the amounts that the buyer should bid for those lots.

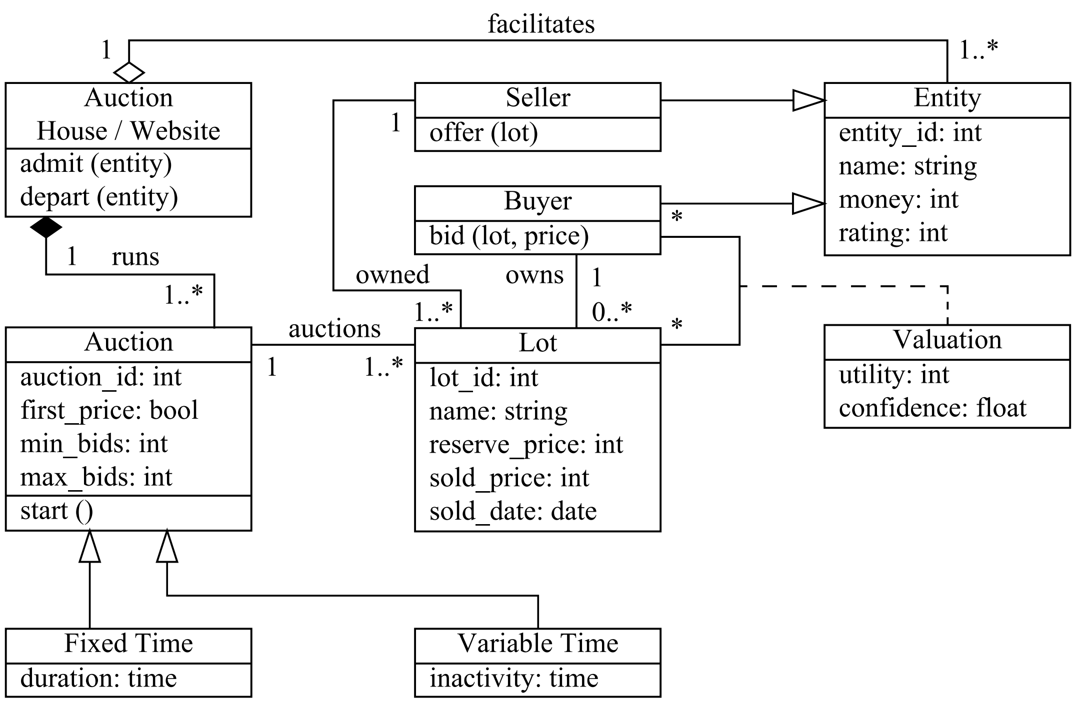

Figure 6-2: UML class diagram for the auction process simulator.

From the simulation experiments, the theoretical and simulation results are com-

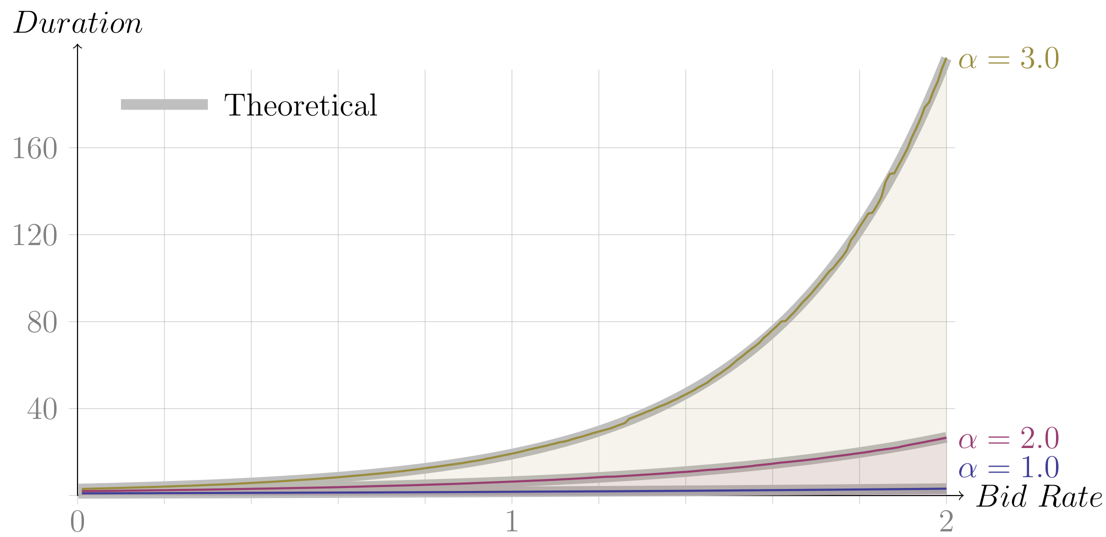

Figure 6-3: Average auction duration of variable-time first-price forward auction with fixed inactivity window.

pared, and the agreements are all extremely good. A sample of the graphs comparing the theoretical and simulation results is given below, where the thick curves are those obtained from theory and the thin curves are those obtained by simulation.

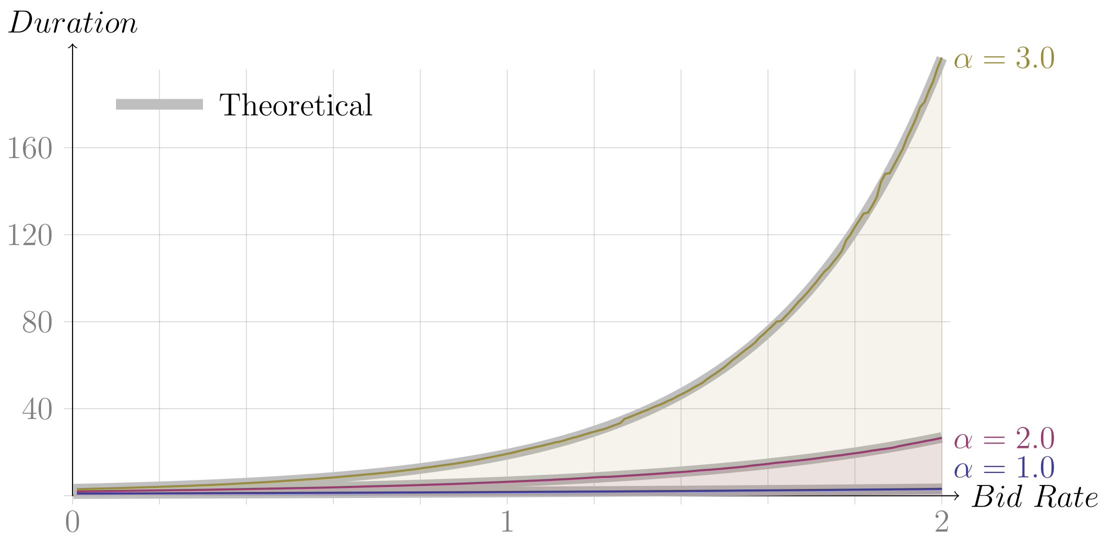

Figure 6-4: Average auction duration of variable-time first-price forward auction with fixed inactivity window.

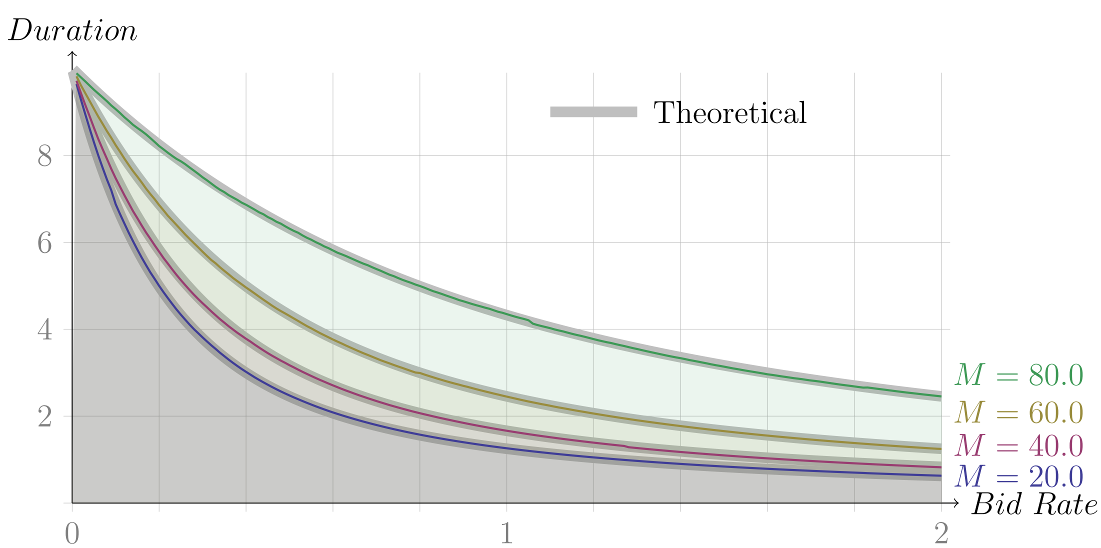

Figure 6-5: Average auction duration of fixed-time first-price forward auction with maximum threshold termination.

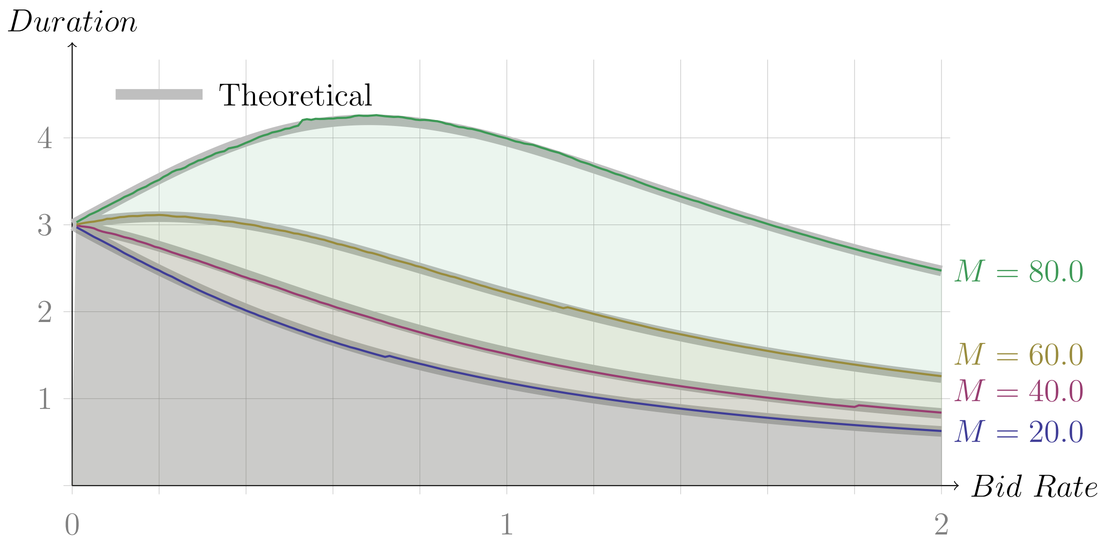

Figure 6-6: Average auction duration of variable-time first-price forward auction with fixed inactivity window and maximum threshold termination.

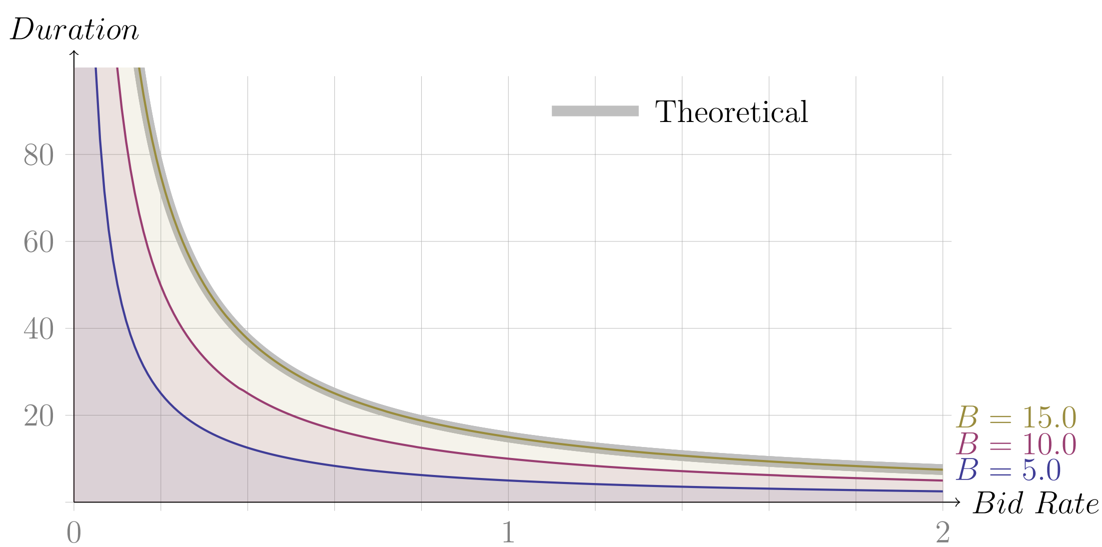

Figure 6-7: Average auction duration of first-price forward auction with bid enumeration termination.

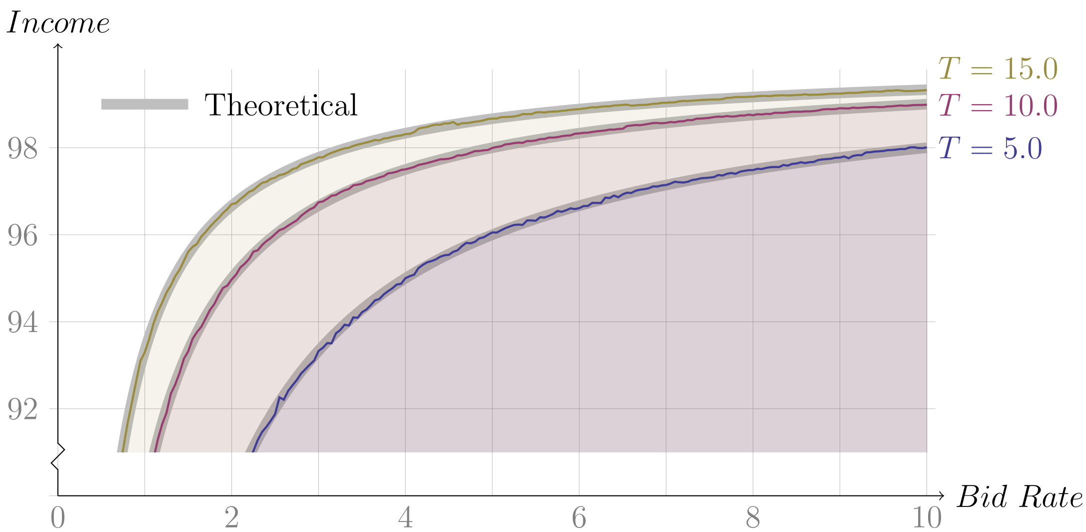

Figure 6-8: Average auction income of fixed-time first-price forward auction.

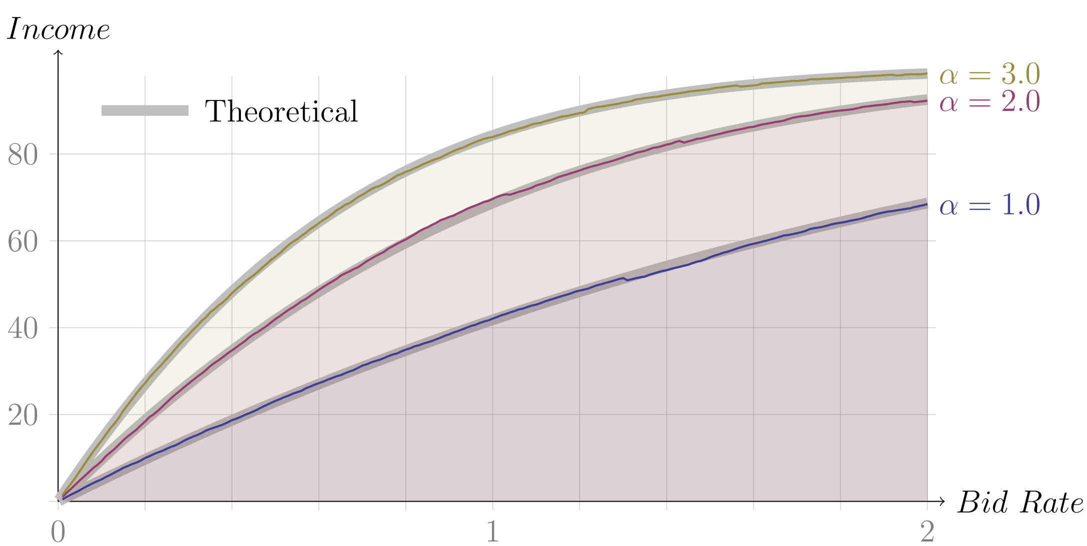

Figure 6-9: Average auction income of variable-time first-price forward auction with fixed inactivity window.

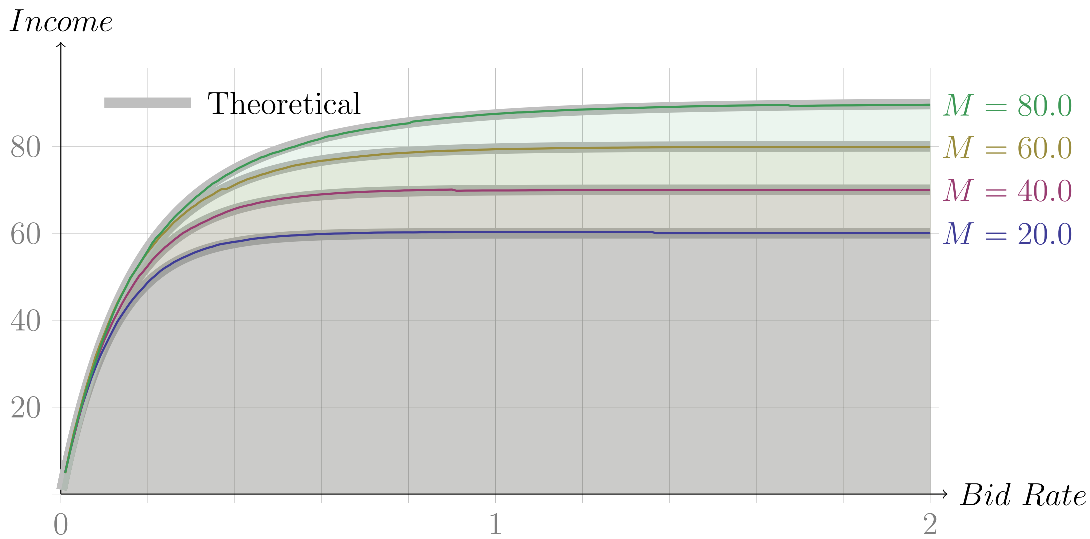

Figure 6-10: Average auction income of fixed-time first-price forward auction with maximum threshold termination.

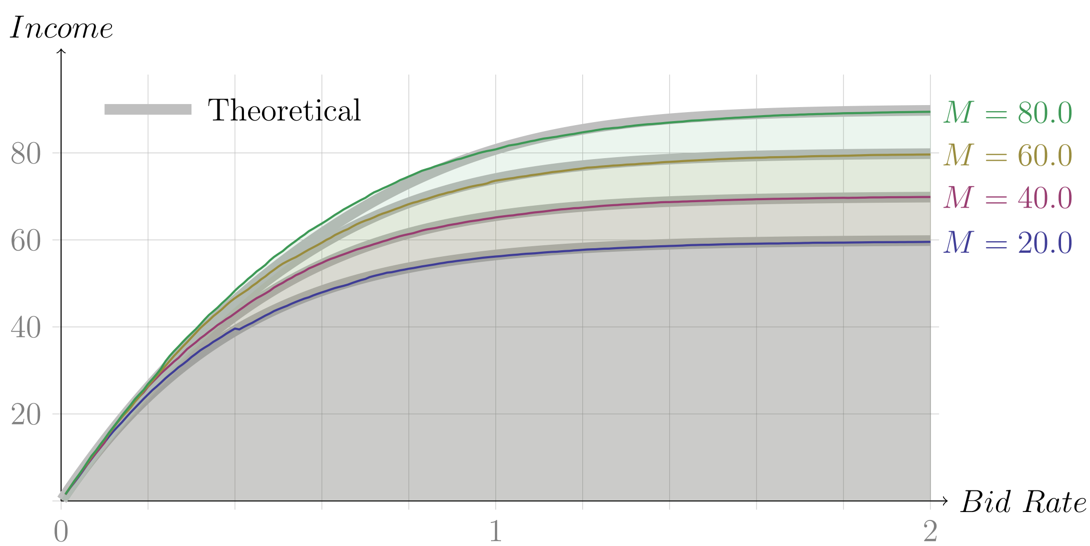

Figure 6-11: Average auction income of variable-time first-price forward auction with fixed inactivity window and maximum threshold termination.

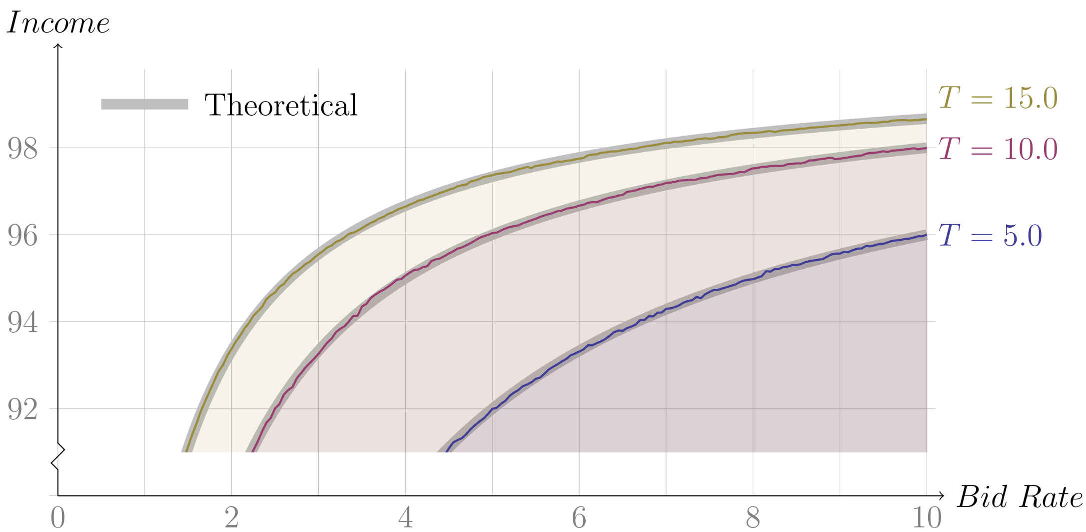

Figure 6-12: Average auction income of fixed-time Vickrey forward auction.

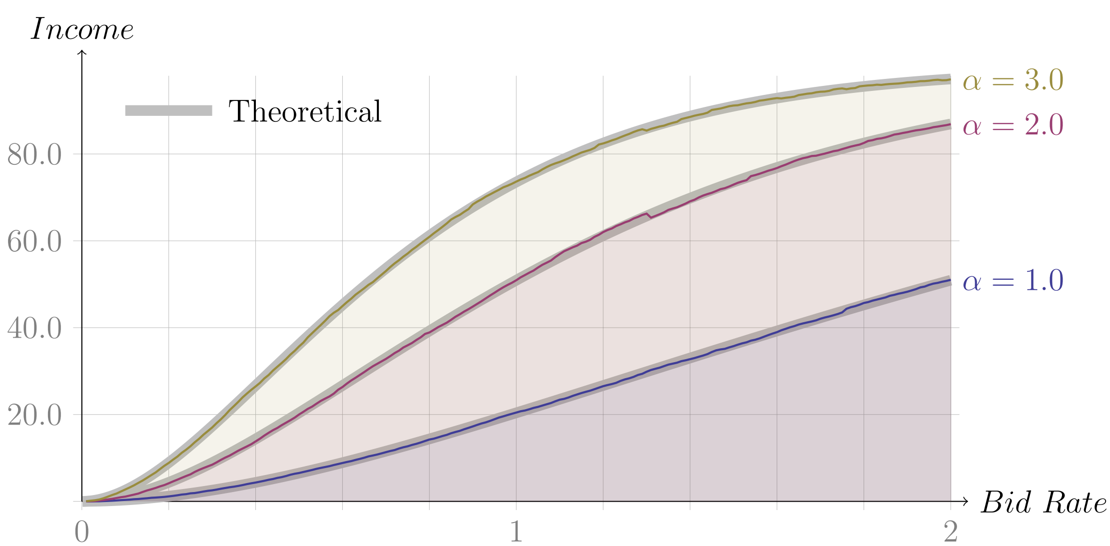

Figure 6-13: Average auction income of variable-time Vickrey forward auction with fixed inactivity window.

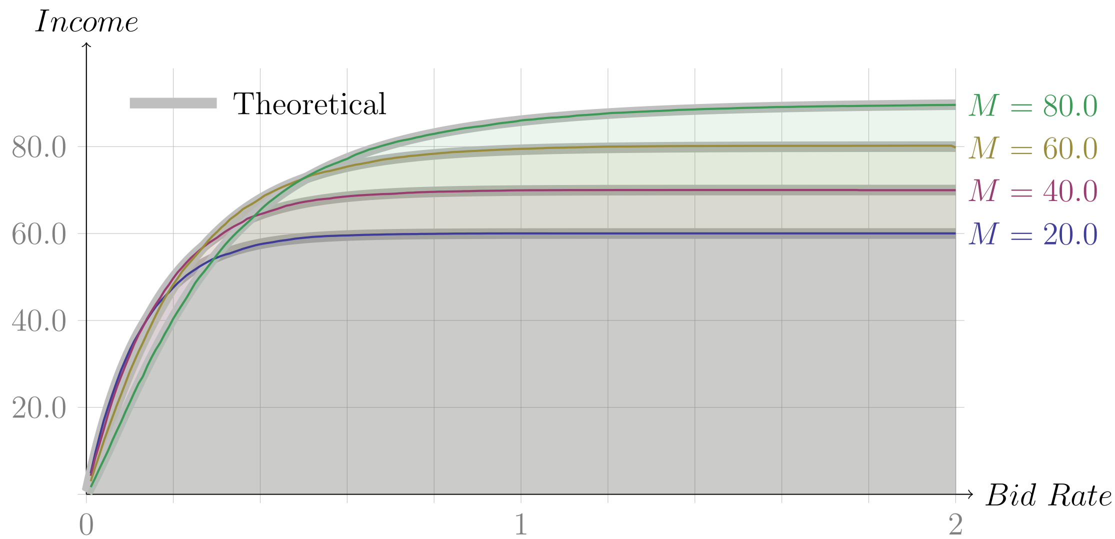

Figure 6-14: Average auction income of fixed-time Vickrey forward auction with maximum threshold termination.

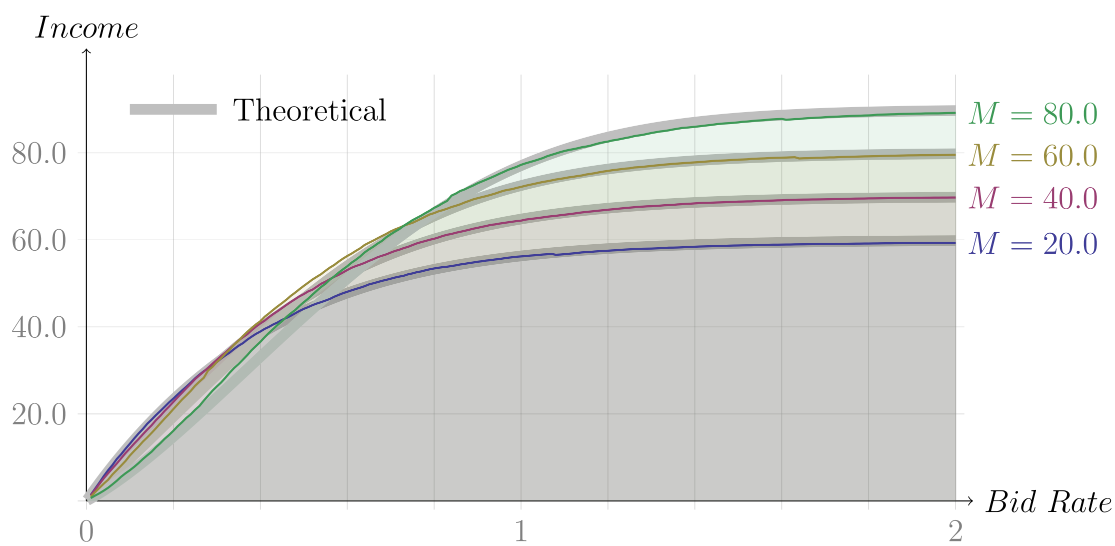

Figure 6-15: Average auction income of variable-time Vickrey forward auction with fixed inactivity window and maximum threshold termination.

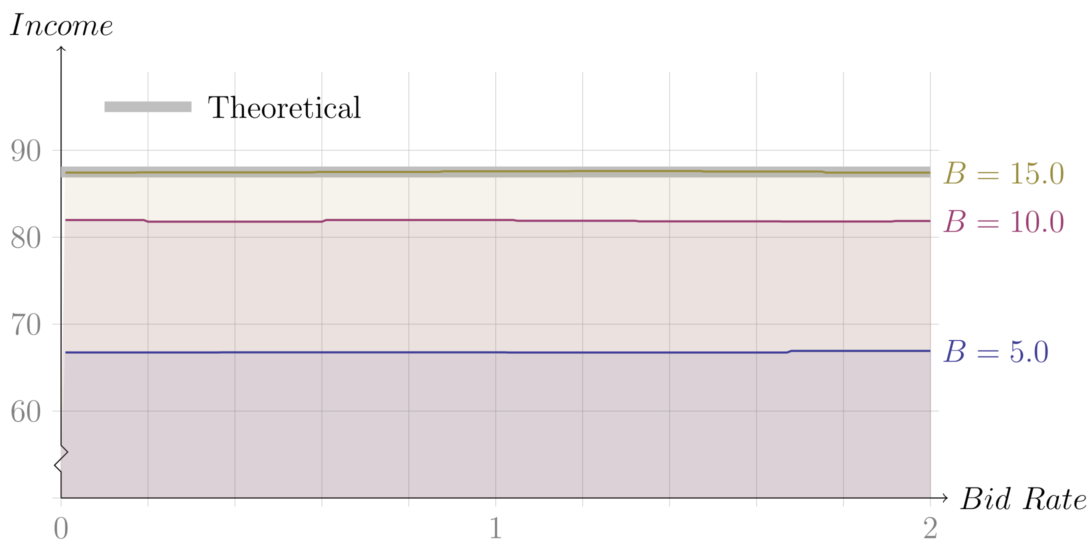

Figure 6-16: Average auction income of Vickrey forward auction with bid enumeration termination.

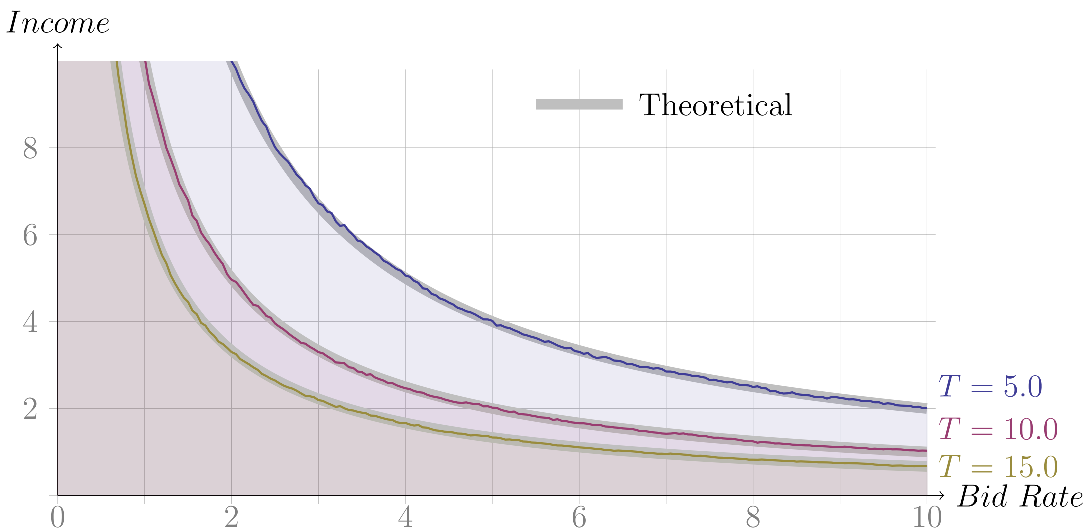

Figure 6-17: Average auction income of fixed-time last-price reverse auction.

6.2 Comparison with *eBay* Auction Data

Actual Internet auction data are obtained from *eBay* for comparison purposes. It is found that with many of the auctions, the number of bids is not high, often falling below ten. Among the most popular items are iPhone 4 with different colours, specifications and varying conditions. *eBay* auctions tend to be fixed time auctions, and we extracted the data from five *eBay* iPhone4 auctions that have at least 15 bids. These detailed auction data are given in Appendix A.

It must be emphasised that actual live data has a number of drawbacks. Firstly, unlike simulation experiments, these experimental observations are not reproducible—they can be somewhat haphazard due to the fact that the prevailing conditions governing the underlying processes are not experimentally repeatable. In fact, even in well-studied models such as those in queueing theory, it is generally unlikely to have the measurements taken from an actual queueing situation exactly matching the established theoretical results. Secondly, unlike simulation experiments, the experimental conditions are not under the control of the experimenter.

From these *eBay* data, we wish to examine the bid arrival pattern, the distribution of bid values, and the auction income.

We first examine the bid arrival pattern, and the relevant data are given in Ta-

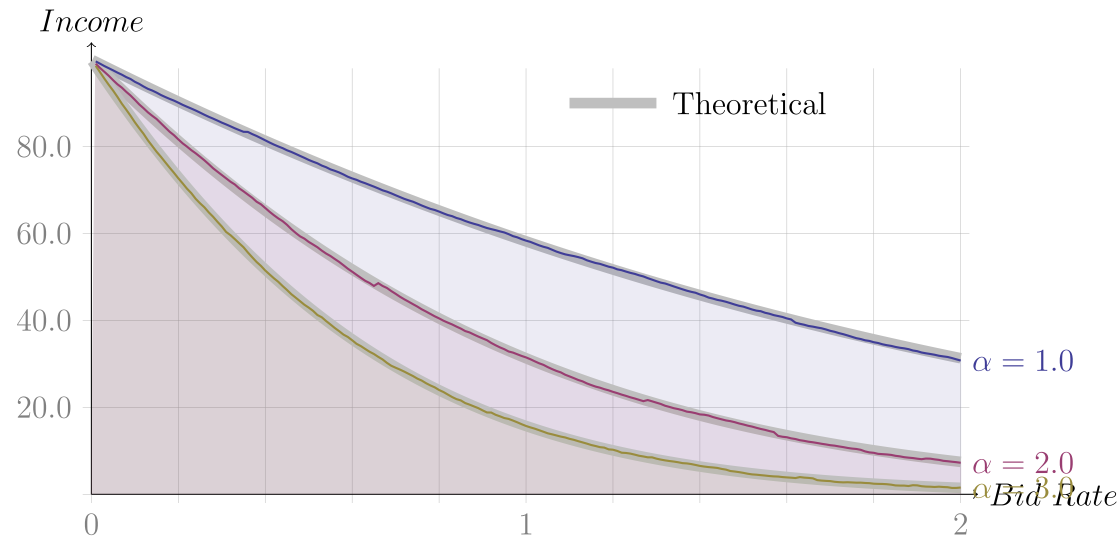

Figure 6-18: Average auction income of variable-time last-price reverse auction with fixed inactivity window.

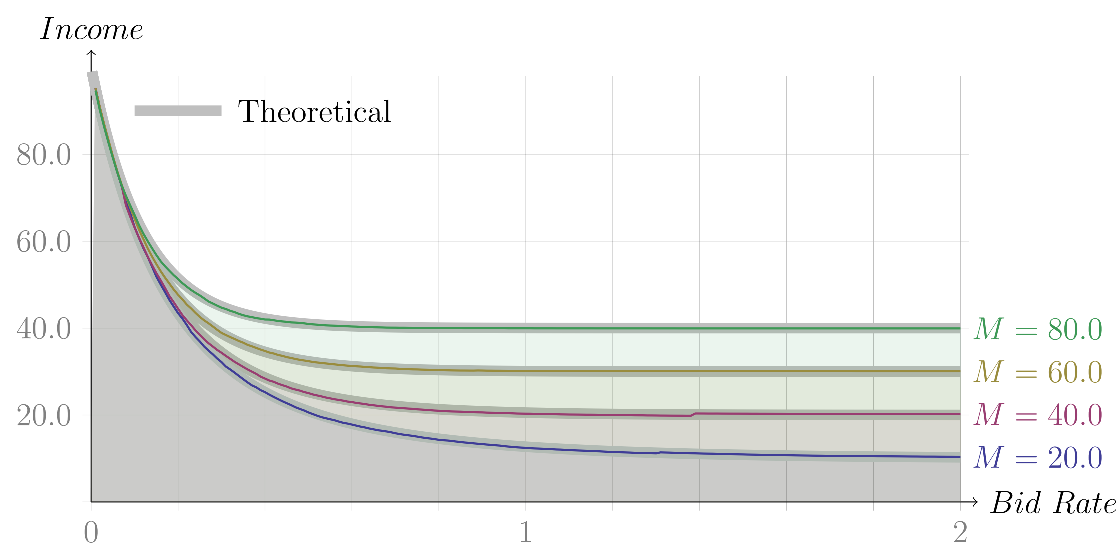

Figure 6-19: Average auction income of fixed-time last-price reverse auction with minimum threshold termination.

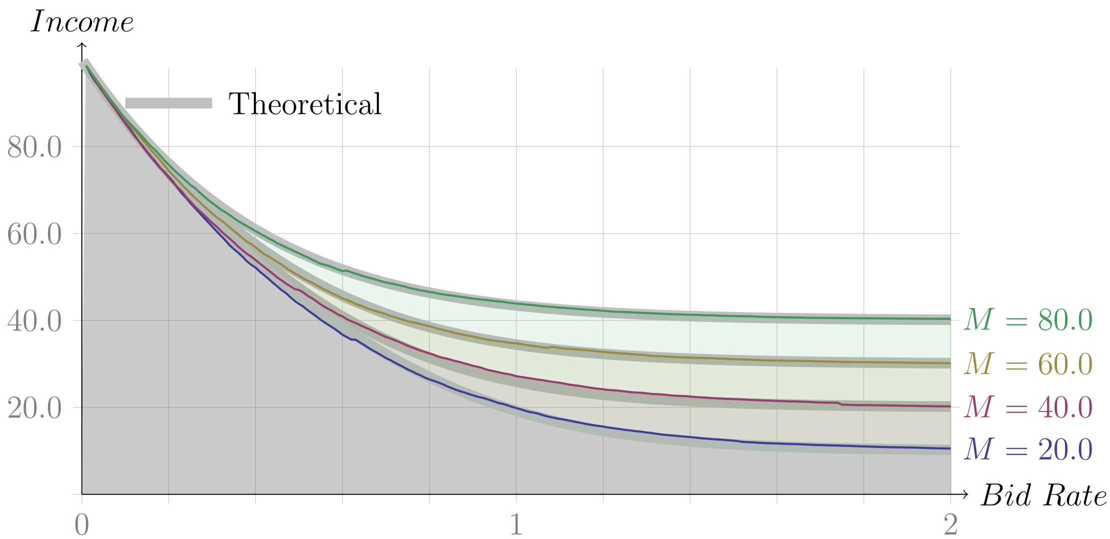

Figure 6-20: Average auction income of variable-time last-price reverse auction with fixed inactivity window and minimum threshold termination.

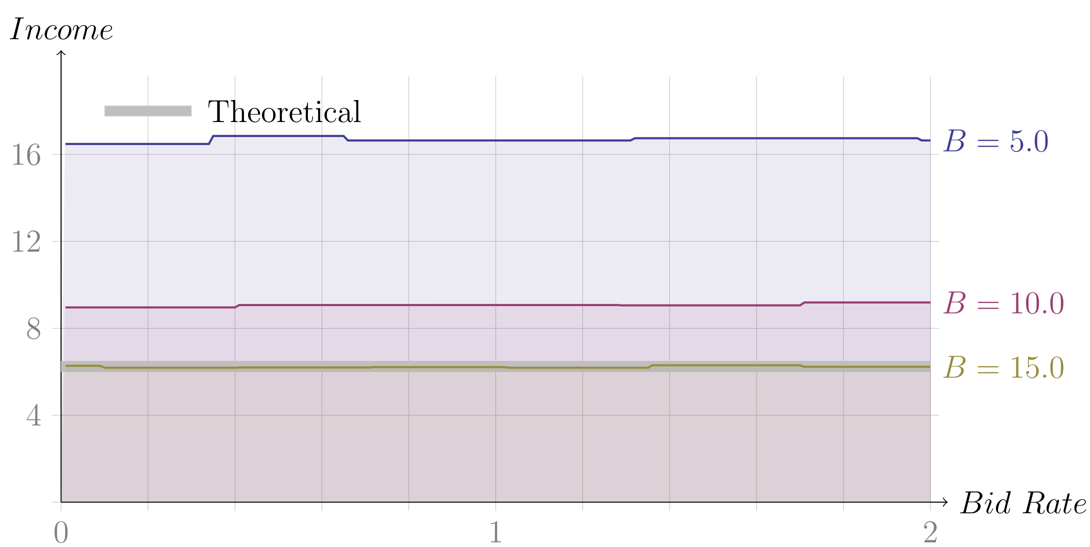

Figure 6-21: Average auction income of last-price reverse auction with bid enumeration termination.

ble 6.1. We note that for Poisson arrival, the inter-arrival time distribution has a coefficient of variation of one.

From Table 6.1, we see that for auction A1, the coefficient of variation of the inter-arrival time of 1.01 is very close to one indeed, with an error percentage of under 1%. For auction A3, the coefficient of variation of the inter-arrival time of 0.93 is also close to one, with an error percentage of around 7.5%. For auction A5, the coefficient of variation of the inter-arrival time of 1.2 is higher, but the error percentage of 16.7% is not excessive. The departure from the exponential inter-arrival time is more pronounced, however, in the cases of auctions A2 and A4. The average of the observed coefficients of variation of the inter-arrival time is 1.166, which deviates from the theoretical value by 16.6%, and the average error is 15.23%. However, it is interesting to note that, in the case of auction A4, when the arrival of the first three bids are not included, the inter-arrival time of the remaining 18 bids has a coefficient of variation of 1.032, which is rather close to the Poisson coefficient of variation. Moreover, in the case of auction A2, when the arrival of the first three bids are not included, the inter-arrival time of the remaining 14 bids has a coefficient of variation of 1.116, and also, in the case of auction A5, when the arrival of the three bids are not included, the inter-arrival time of the remaining 12 bids has a coefficient of variation of 1.017. It would appear particularly that, in the cases of auctions A4 and A5, while their overall bid arrival patterns are not strictly Poisson, they seem to converge to the Poisson pattern when some initial transient effects are disregarded. Column 6 of Table 6.1 shows the coefficients of variation after the transient effects of A2, A4, and A5 are adjusted with the inter-arrival times of the first three bids disregarded. The adjusted coefficients of variation have a mean of 1.022, which shows little deviation from the theoretical average Poisson coefficient of variation of one, while the corresponding average error is around 4.8%. Column 7 of Table 6.1 shows the coefficients of variation after the transient effects of all the five auctions are adjusted with the inter-arrival times of the first three bids disregarded. The adjusted coefficients of variation have a mean of 1.004, which is very close to one indeed, while the corresponding average error is around 6.7%. From these observations, it is possible

Table 6.1: Comparison of observed and theoretical bid patterns.

Auction	**Start Time**	**End Time**	**Bids**	**Observed coefficient of variation of the inter-arrival time (no transient effects adjustments)**	**Observed coefficient of variation of the inter-arrival time (with transient effects adjusted for A2, A4, A5)**	**Observed coefficient of variation of the inter-arrival time (with transient effects adjusted for A1, A2, A3, A4, A5)**	**Error% (No transient effects adjustments)**	**Error % (with transient effects adjusted for A2, A4, A5)**	**Error% (with transient effects adjusted for A1, A2, A3, A4, A5)**
A1	10 July 2011 18:20	13 July 2011 18:20	16	1.01	1.01	0.85	0.99%	0.99%	17.65%
A2	6 July 2011 18:50	13 July 2011 18:50	19 (Last 16)	1.28	1.12	1.12	21.88%	10.71%	10.71%
A3	8 July 2011 20:00	13 July 2011 20:00	17	0.93	0.93	1.00	7.53%	7.53%	0%
A4	12 July 2011 20:38	13 July 2011 20:38	21 (Last 18)	1.41	1.03	1.03	29.08%	2.91%	2.91%
A5	13 July 2011 3:05	14 July 2011 3:05	15 (Last 12)	1.20	1.02	1.02	16.67%	1.96%	1.96%
Average				1.166	1.022	1.004	15.23%	4.82%	6.65%

Table 6.2: Comparison of observed and theoretical bid value patterns.

Auction	**Start Time**	**End Time**	**Bids**	**Observed coefficient of variation of bid value**	**Error %**
A1	10 July 2011 18:20	13 July 2011 18:20	16	0.68	15.10%
A2	6 July 2011 18:50	13 July 2011 18:50	19	0.64	9.79%
A3	8 July 2011 20:00	13 July 2011 20:00	17	0.61	5.35%
A4	12 July 2011 20:38	13 July 2011 20:38	21	0.55	4.97%
A5	13 July 2011 3:05	14 July 2011 3:05	15	0.62	6.88%
Average				0.62	8.42%

to conclude that the Poisson arrival assumptions is a valid one for at least some of the actual auction processes, while it can be employed as a useful approximation for some others.

We next examine the bid value patterns, and the results are given in Table 6.2. We note that, for uniformly distributed bid values, the coefficient of variation is 0.58. We see that, with the exception of auction A1, all the other auctions have for the coefficient of variation of the bid value reasonably close to the theoretical value of 0.58, all having below 10% error. For auctions A3 and A4, their coefficients of variation of the bid value are quite close to the theoretical value of 0.58, having only around 5% error. For auction A1, its coefficient of variation of the bid value has a slightly higher error percentage of 15.1%, but is not excessive. The average of the observed coefficient of variation of the bid value is 0.62, which deviates from the theoretical value of 0.58 by 6.88%, and the average error is 8.42%. From these observations, it is possible to conclude that using uniformly distributed bid values generally provides a reasonable working approximation for the study of Internet auction processes.

We also compare the auction income for both the first-price forward auction and the Vickrey auction. The parameter L is estimated from Equation 3.1 with N taken

Table 6.3: First-price forward auction income.

Auction	**Start Time**	**End Time**	**Bids**	**Observed auction income**	**Theoretical auction income**	**Error %**
A1	10 July 2011 18:20	13 July 2011 18:20	16	575	572.8	0.4%
A2	6 July 2011 18:50	13 July 2011 18:50	19	420	418.8	0.3%
A3	8 July 2011 20:00	13 July 2011 20:00	17	475	473.4	0.3%
A4	12 July 2011 20:38	13 July 2011 20:38	21	540	538.8	0.2%
A5	13 July 2011 3:05	14 July 2011 3:05	15	525	522.7	0.4%

Table 6.4: Vickrey auction income.

Auction	**Start Time**	**End Time**	**Bids**	**Observed auction income**	**Theoretical auction income**	**Error %**
A1	10 July 2011 18:20	13 July 2011 18:20	16	565	534.6	5.4%
A2	6 July 2011 18:50	13 July 2011 18:50	19	410	395.6	3.5%
A3	8 July 2011 20:00	13 July 2011 20:00	17	470	443.7	5.6%
A4	12 July 2011 20:38	13 July 2011 20:38	21	530	511.8	3.4%
A5	13 July 2011 3:05	14 July 2011 3:05	15	515	485.3	5.8%

to be the number of bids. Since λT gives the average number of bids, the parameter λ is estimated by the number of bids divided by the auction duration. The results are given in Tables 6.3 and 6.4. We see that the agreement is generally good, with error percentage below 6%.

Table 6.5: Simulation using eBay parameters for first-price forward auctions.

Auction	**L**	λ	**T**	**Auction income (simulation)**	**Auction income (eBay)**	**Error %**
A1	610.97	0.222	72	572.75	575	0.39%
A2	442.11	0.113	168	418.82	420	0.28%
A3	502.94	0.142	120	473.42	475	0.33%
A4	565.71	0.875	24	538.77	540	0.23%
A5	560.00	0.625	24	522.66	525	0.44%

Table 6.6: Simulation using eBay parameters for Vickrey auctions.

Auction	**L**	λ	**T**	**Auction income (simulation)**	**Auction income (eBay)**	**Error %**
A1	610.97	0.222	72	534.52	565	5.39%
A2	442.11	0.113	168	395.53	410	3.53%
A3	502.94	0.142	120	443.91	470	5.55%
A4	565.71	0.875	24	511.83	530	3.43%
A5	560.00	0.625	24	485.33	515	5.76%

6.3 Simulations using eBay-determined Parameters

We also carry out simulations using the parameters as estimated from the *eBay* data above and the results are given in Table 6.5 and Table 6.6. We also see that the agreement is generally good.

Chapter 7

Sensitivity Analysis of the Auction Income With Respect to Different Independent Values Distributions

As we have seen, the Independent Private Values (IPV) model plays a fundamental role in the analyses of Internet auction performance. This model assumes privacy and independence, meaning that the private values of buyers are drawn from a common distribution, or in probabilistic terms, the series of values are independent and identically distributed. In this chapter, we shall study the sensitivity of the auction income with respect to different Independent Values Distributions and their appropriateness for Internet auctions. We shall also provide analytic approximations to the auction income for different Independent Values Distributions by introducing the damping factor which modifies the bid rate.

7.1 Importance of the Uniform Distribution for Independent Private Values

The uniform distribution for private values is commonly employed in auction studies (Katok and Kwasnica, 2008). It possesses the following desirable features that are widely used to characterise Internet auction behaviour.

1. *It is naturally capped by a finite value:* in most auctions, the private values do not go to infinity but is bounded by a certain upper limit. Since bids can often start at a very low value, the range of values permitted by uniformly private values may be presented by $(0, L)$, so that with the probability that a bid falling into a small value interval is given by $\Pr[x < Q_j < x + \Delta x] = \Delta x/L$.

2. *It represents that no information is available and no bias exists concerning the inclinations and behaviour of the underlying bidder group, as well as no interaction or correlation between bids:* any value is possible and they happen with equal probability. This property is also useful in expressing a wide cross-section of bidder valuation spreading over different geographical regions, disparate cultures, personal preferences and perceptions—typically found on the Internet—and with little or no communication or agreements among the bidders.

Key performance metrics are the average income per auction and the average duration of an auction. Since for hard close auctions, the auction duration is fixed, this quantity is of interest mainly for the soft close case, where the duration is a random variable. For each parameter setting, we carry out 100 auction experiments and the measurements collected are averaged. Figure 7-1 shows the average auction income for different values of λ for hard close auctions when $T = 10$, and $L = 100$. We see that the increase in bid rate up to about $\lambda = 1$ produces rather steep average auction income improvement. There seems to be a critical bid rate at around $\lambda = 1$, above which the improvement in income becomes less pronounced.

Figure 7-2 shows how auction income varies for soft close auctions with λ for $\alpha = 1, 2, 3$ and 4. We see that average auction income increases gradually for small values of λ, but accelerates for large values. As the bid rate increases, there is reduced chance of a no bid interval occurring, and the difference between $\alpha = 1$ and $\alpha = 4$ tends to widen as $\lambda \to 1$, but slowly narrows afterwards, especially between $\alpha = 2$, $\alpha = 3$ and $\alpha = 4$. Depending on the specified length of the no bid interval α, the auction duration for soft close auctions can show considerable divergence for high bid

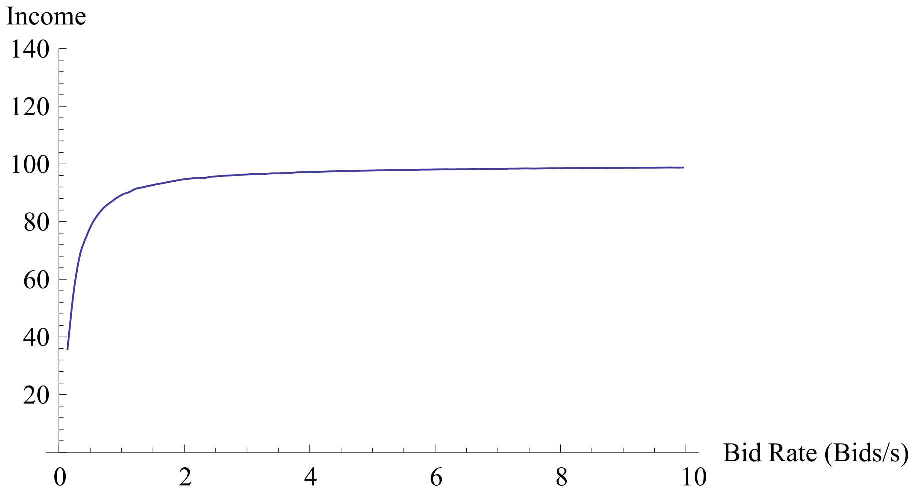

Figure 7-1: Auction income for uniformly distributed private values for hard close auctions.

rate, which is shown in Figure 7-3. We observe that for larger values of α (e.g. $\alpha = 4$), it can shoot up to very large values for high bid rates, which seems to be due the fact that under such circumstances, having no arriving bid in an interval of $\alpha = 4$ becomes increasingly unattainable, leading to almost an indefinite continuation of the auction.

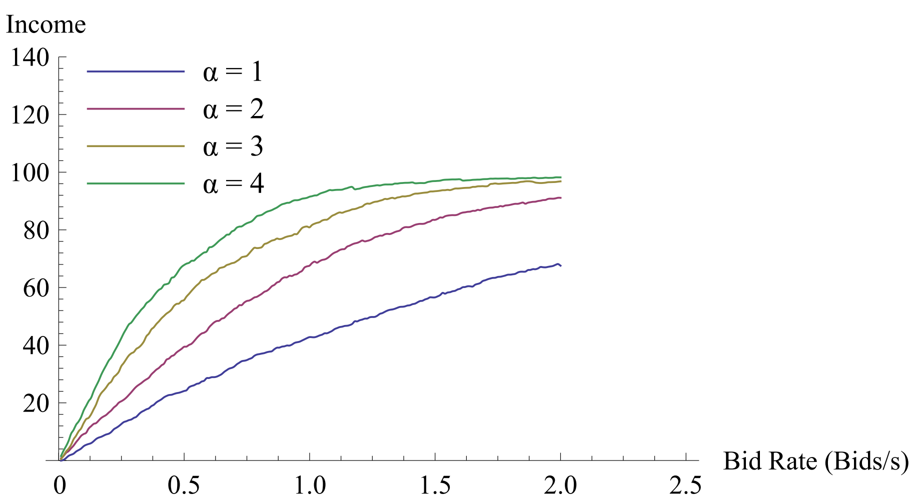

Figure 7-2: Auction income for uniformly distributed private values for soft close auctions.

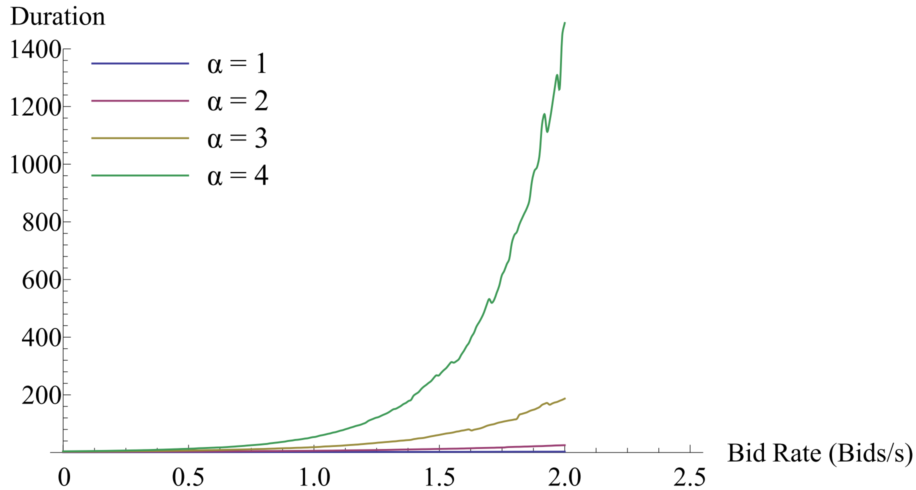

Figure 7-3: Auction duration for uniformly distributed private values for soft close auctions.

7.2 Exponentially distributed private values

The exponential distribution $\mu e^{-\mu v}$ is one of the most widely used distributions in stochastic modelling. Compared with the uniform distribution, it has an added natural advantage of representing private values since private values are not allowed to go negative, and the exponential random variable is always positive. Since for the exponential distribution, the smaller values has a higher probability of occurring than higher values, i.e. for $u < v$ and an arbitrary interval of length k,

$$\begin{aligned} \Pr\left[u \le Q_i \le u+k\right] &= \int_u^{u+k} \mu e^{-\mu t} dt \\ &> \\ \Pr\left[v \le Q_i \le v+k\right] &= \int_v^{v+k} \mu e^{-\mu t} dt, \end{aligned}$$

which is a consequence of the inequality $e^{-\mu u} > e^{-\mu v}$. Thus the exponential private values distribution represents a situation where people are looking for bargains from the auction, since the lower values are favoured. Figures 7-4 and 7-5 give the auction performance of the exponentially distributed private values; for effective comparison with the uniformly distributed private values, here we set $1/\mu = L/2 = 50$. The graph for the auction duration for soft close auctions is omitted as it is the same as

Figure 7-3, because the bid arrival process is independent of the private values. We see that the average auction income I for hard close auctions increases rather steeply at first but the increase slows down for higher bid rate. This is interesting as, unlike the uniform distribution, there is no cap on the private values. Again, unlike the uniform distribution, where the auction income is bounded by $L = 100$, the auction income here reaches to well over 200. For the soft close cases, it is particularly noteworthy to observe that, unlike the nonlinear behaviour for hard close auctions, the average auction income here increases approximately linearly with respect to the bid rate. In fact, for $\alpha = 2$, the average auction income appears to follow the relationship:

$$I = 100 \times \lambda. \tag{7.1}$$

Indeed, more generally, the relationship between income and bid rate appears to follow the equation:

$$I = \frac{\alpha\lambda}{\mu}. \tag{7.2}$$

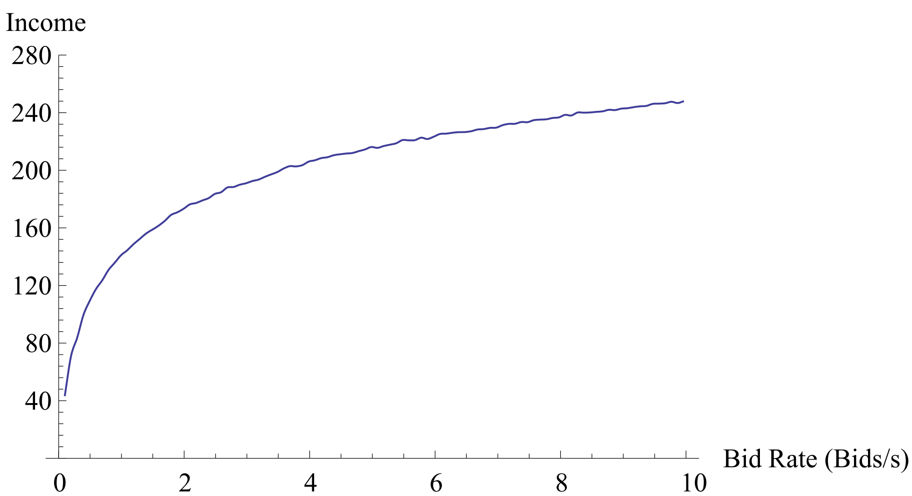

Figure 7-4: Auction income for exponentially distributed private values for hard close auctions.

As it stands, however, the ordinary exponential distribution in representing private values is not sufficiently realistic as it allows arbitrarily large values to be used, which

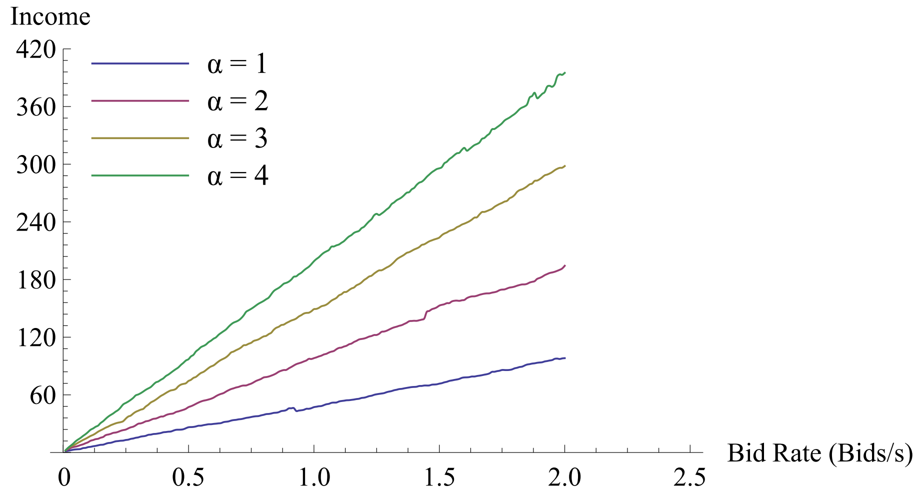

Figure 7-5: Auction income for exponentially distributed private values for soft close auctions.

is often not the case for practical auction processes. For example, for $\alpha = 4$, the average highest bid can go up to 400, which is over four times the auction income for the uniformly distributed private values case, even though both have the same mean private value. For a more realistic representation of private values, we make use of the truncated exponential distribution over the same interval as the uniform case, namely, the interval $(0, L)$, i.e.

$$f(v) \quad = \quad \frac{\mu e^{-\mu v}}{1 - e^{-\mu L}}, \text{ for } 0 < v < L, \tag{7.3}$$

and vanishes outside the interval $(0, L)$. Figures 7-6 and 7-7 give the auction performance for truncated exponentially distributed private values. We see that the performance behaviour of the ordinary exponential distribution and the truncated exponential distribution is rather different. However, there is a remarkable similarity in the auction income behaviour between the truncated exponential private values and the uniform private values. In the uniform case, the ceiling of 100 is approached faster than in the truncated exponential case, which as indicated above may be due to the fact that the bidders in this case is looking for bargains and there is a greater tendency (probability) to bid for the lower values than the higher ones, whereas in the uniform case, all the values happen equally likely.

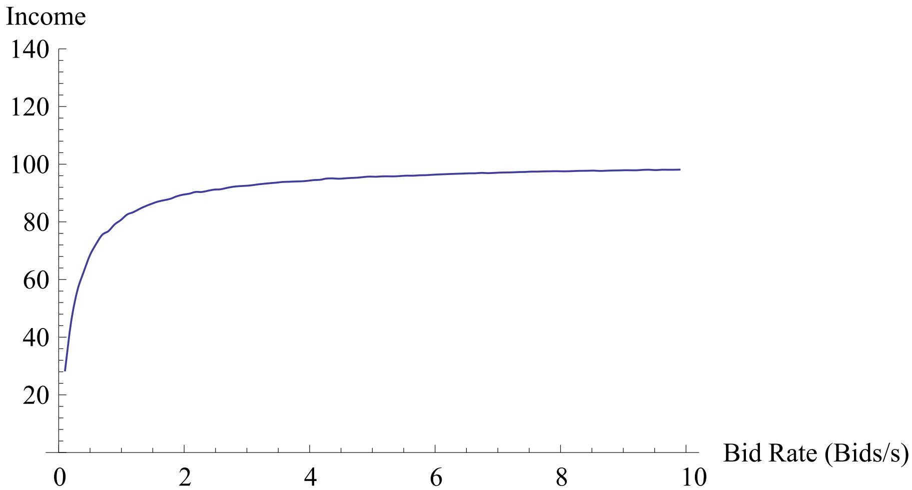

Figure 7-6: Auction income for truncated exponentially distributed private values for hard close auctions.

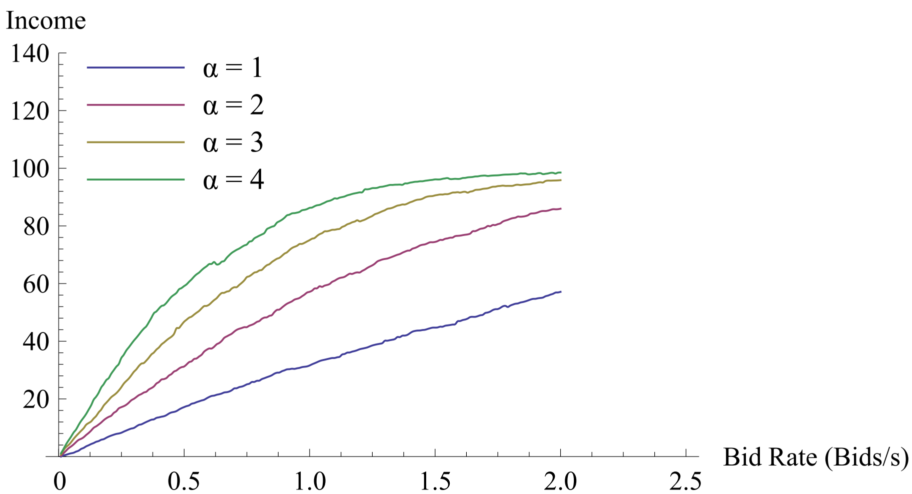

Figure 7-7: Auction income for truncated exponentially distributed private values for soft close auctions.

7.3 Normally distributed private values

The normal distribution is highly versatile and is representative of a variety of social and physical phenomena. Here, it represents some measure of implicit agreement concerning the value of the auction item among the bidder population, since there is a mode value which they roughly agree upon. In addition, there may even be some degree of indirect interaction or communication among the global community of bidders, perhaps through various forms of social networking. Here, we take the mean of the normal distribution η to correspond to the mean of the uniform distribution, which is $L/2 = 50$, and likewise we take the variance σ^2 to also correspond to the variance of the uniform distribution, which is $L^2/12 = 833.33$.

Figures 7-8 and 7-9 give the auction performance of the normally distributed private values. We see that, while the auction income definitely goes above 100, it is much more subdued compared with the exponential case for hard close auctions. Similar performance behaviour is also evident for soft close auctions. These are probably due to the strong clustering around the mean/mode. Unlike the exponential case, however, there is no simple linearity relationship observed, but rather a sub-linear relationship may be inferred.

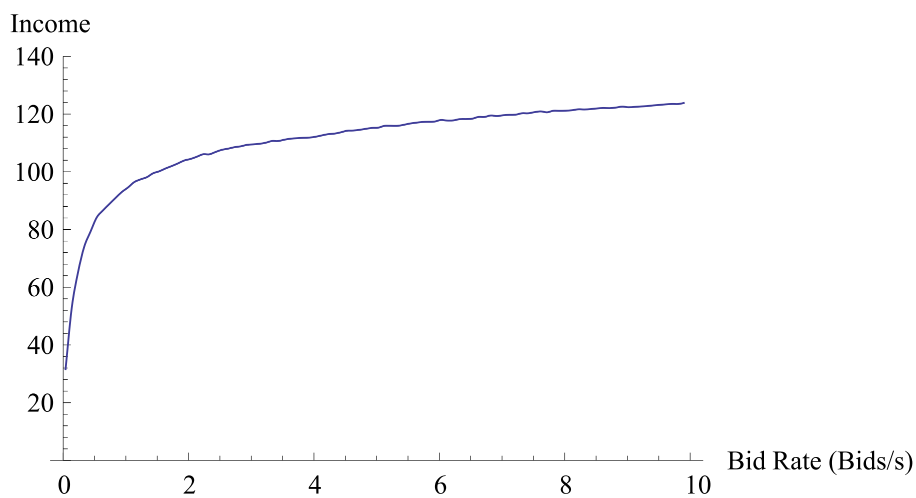

Figure 7-8: Auction income for normally distributed private values for hard close auctions.

However, the ordinary normal distribution has two properties which limits its usefulness for auction processes. First, it can go up to $+\infty$, which as indicated earlier for the exponential distribution, is often not realistic. Second, it can go down to $-\infty$, which means that negative values are unavoidable no matter how far we shift the mean to the right, and this again is not permitted in auction processes. As for the exponential case, we make use of the truncated normal distribution confined to the

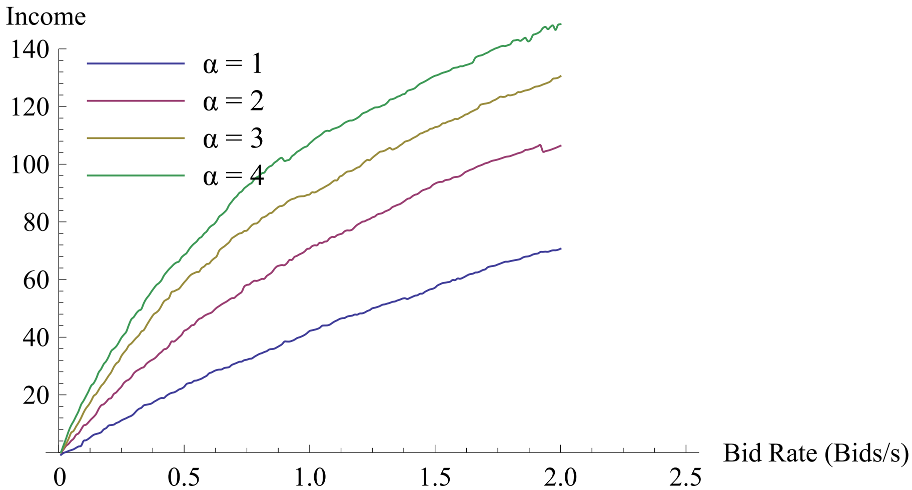

Figure 7-9: Auction income for normally distributed private values for soft close auctions.

interval $(0, L)$, i.e.

$$f(v) \quad = \quad \frac{\exp\left[-\frac{(v-\eta)^2}{2\sigma^2}\right]}{K\sqrt{2\pi\sigma^2}}, \text{ for } 0 < v < L, \tag{7.4}$$

but vanishes outside the interval (0, L), and K is the constant

$$K \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}}\int_0^L \exp\left[-\frac{(t-\eta)^2}{2\sigma^2}\right] dt. \tag{7.5}$$

Figures 7-10 and 7-11 give the auction performance for truncated normally distributed private values. Here, we also see that the performance behaviour of the ordinary normal distribution and the truncated normal distribution is rather different, and there is also a remarkable similarity in the auction income behaviour between the truncated normal private values and the uniform private values. Similar to the remarks for the truncated exponential case, in the truncated normal case, the ceiling of 100 is approached slower than in the uniform case which may be due to the clustering of values around the mean/mode that limits its scope in going up to very high private values.

We thus see that in the modelling of private values, the uniform distribution as well as the truncated versions of the normal and exponential distributions are more

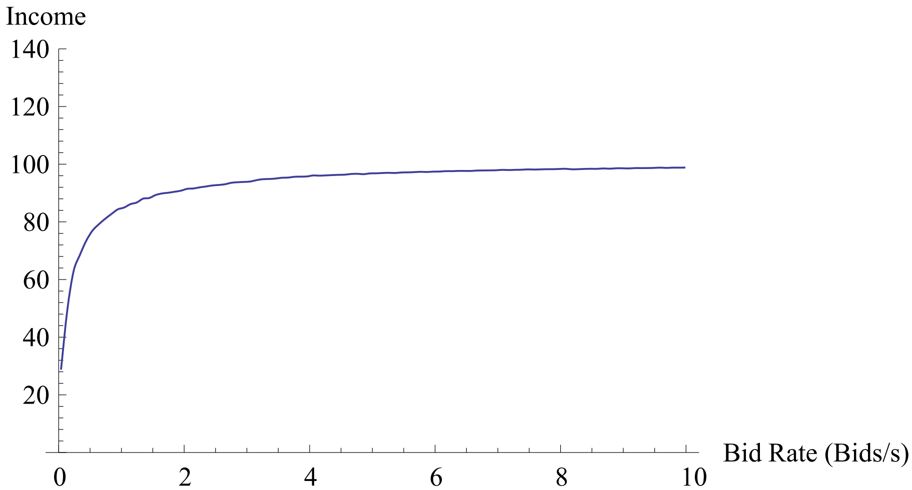

Figure 7-10: Auction income for truncated normally distributed private values for hard close auctions.

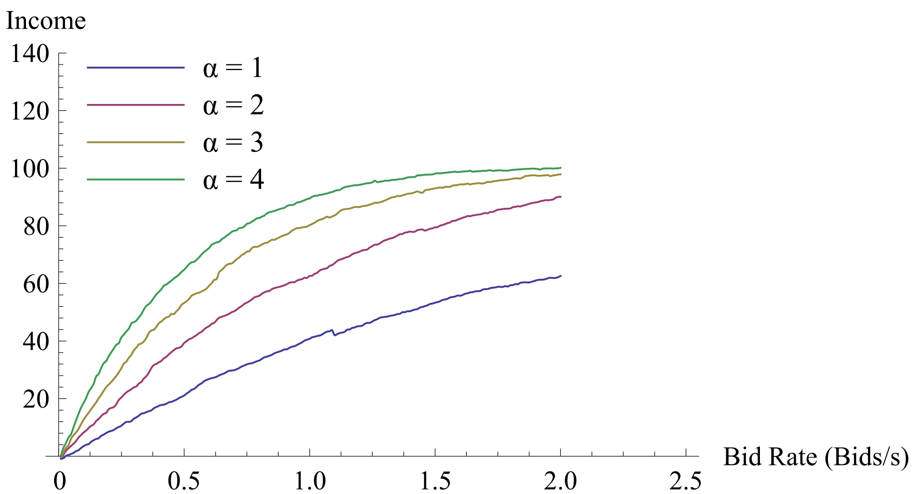

Figure 7-11: Auction income for truncated normally distributed private values for soft close auctions.

realistic and versatile. The average auction incomes for these three distributions exhibit similar behaviour in response to the bid level. It is interesting to compare how quickly the ceiling of L is approached in these cases. Table 7.1 shows the different bid rates in attaining 70%, 80%, and 90% of the ceiling L for the hard close auctions. From the sellers' point of view, the preferred private values distribution is the uniform, since it attains a high average income more quickly than the other two distributions. The least preferred is the truncated exponential distribution since it is the slowest among the three private values distribution to attain high average income, probably

Table 7.1: Comparison of the Rates of Convergence to the Ceiling L

Private Values Distribution	$0.7L$	$0.8L$	$0.9L$
Uniform	$\lambda = 0.3$	$\lambda = 0.5$	$\lambda = 1$
Truncated exponential	$\lambda = 0.6$	$\lambda = 1$	$\lambda = 2.5$
Truncated normal	$\lambda = 0.4$	$\lambda = 0.7$	$\lambda = 1.9$

due to the reluctance of the underlying bidder community in bidding reasonable values.

7.4 Sensitivity of the Auction Income to the Bid Rate and the Damping Factor

In view of the above observation, and the resemblance of the shape of the truncated exponential and truncated normal IPV auction incomes with that of the uniform IPV auction income, we infer that the behaviour of auctions with the truncated exponential and truncated normal IPV may be approximated by that of the uniform IPV, with the bid rate reduced by a damping factor $b < 1$. Thus modifying Equation 3.3 for the fixed time first-price auction, we have

$$E\left[Q_{(N)}\right] = \frac{L}{b\lambda T}\left(b\lambda T + e^{-b\lambda T} - 1\right). \tag{7.6}$$

For $T = 10$, $L = 100$, Figure 7-12 compares the above formula with the truncated exponential experimental values for $b = 1/2$, and we see that the agreement is quite close.

Modifying the fixed time Vickrey auction equation in the same way, we have

$$E\left[Q_{(N-1)}\right] = \frac{L}{b\lambda T}\left(b\lambda T + b\lambda T e^{-b\lambda T} + 2e^{-b\lambda T} - 2\right). \tag{7.7}$$

Figure 7-13 compares the above formula with the truncated exponential experimental values for $b = 1/2$, and we see that the agreement, while not as close as the first-price case, is still quite good.

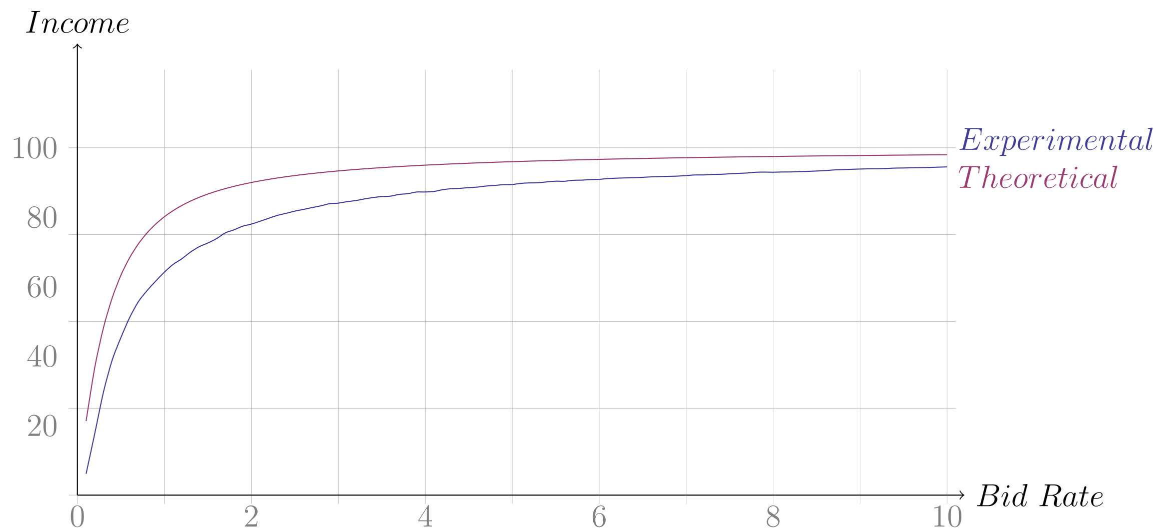

Figure 7-12: Analytic formula for the truncated exponentially distributed IPV for first-price auctions.

The damping factor applied to λ represents a fractional reduction in bid traffic, and hence a damping factor less than one indicates that, compared with the uniformly distributed IPV, the corresponding traffic gives a lesser auction income than the uniformly distributed IPV. The damping factor of $b = 1/2$ for the truncated exponential is a significant reduction, and as remarked earlier, it represents bidders who are looking for bargains and are less generous than those with the corresponding uniformly distributed private values. Indeed, it is useful to compare the generosity characteristics of the two distributions. Denoting by Q_{exp} the truncated exponential IPV bids, and $Q_{uniform}$ the uniform IPV bids, we have for $v < L$,

$$\Pr\left[Q_{exp} < v\right] = \frac{1 - e^{\mu v}}{1 - e^{-\mu L}} \tag{7.8}$$

$$\Pr\left[Q_{uniform} < v\right] = \frac{v}{L} \tag{7.9}$$

Since the truncated exponential IPV tend to have bids clustering more at the lower end compared with the uniform IPV, therefore the relative generosity of the uniform IPV may be indicated by the probability of a bid exceeding a certain value v

$$\Pr\left[Q_{exp} > v\right] < \Pr\left[Q_{uniform} > v\right], \tag{7.10}$$

which is equivalent to

$$\Pr\left[Q_{uniform} < v\right] < \Pr\left[Q_{exp} < v\right], \tag{7.11}$$

That is,

$$\frac{1 - e^{-\mu v}}{v} > \frac{1 - e^{-\mu v}}{L}. \tag{7.12}$$

Since $L > v$, thus condition (Equation 7.10) will be true if we can show that the function

$$f(x) \quad = \quad \frac{1 - e^{-\mu v}}{x}. \tag{7.13}$$

is a decreasing function, i.e. the derivative $f'(x) < 0$. Now

$$f'(x) \quad = \quad \frac{e^{-\mu x}(1 + \mu x) - 1}{x^2}. \tag{7.14}$$

Since $(1 + \mu x)/\exp(\mu x) < 1$, we see that $f'(x) < 0$, and so establishing condition (Equation 7.10), showing that adopting the truncated exponential IPV is more frugal than the uniform IPV.

As for the truncated normal distribution for the IPV, Figure 7-14 compares Equation 7.6 with the corresponding truncated normal experimental values for $b = 2/3$, and we also see that the agreement is quite close. Also, Figure 7-15 compares Equation 7.7 with the corresponding truncated normal experimental values for $b = 2/3$, and we observe that the agreement, while not as close as the first-price case, seems quite workable as an approximation.

Thus, we see that both for the truncated exponential and truncated normal IPVs, the auction income may be usefully approximated by Equations 7.6 and 7.7 respectively for the first-price and Vickrey auctions with appropriate damping factor b. In both of these cases, the damping factor is less than one, with that for the truncated exponential distribution smaller than that for the truncated normal distribution. Of

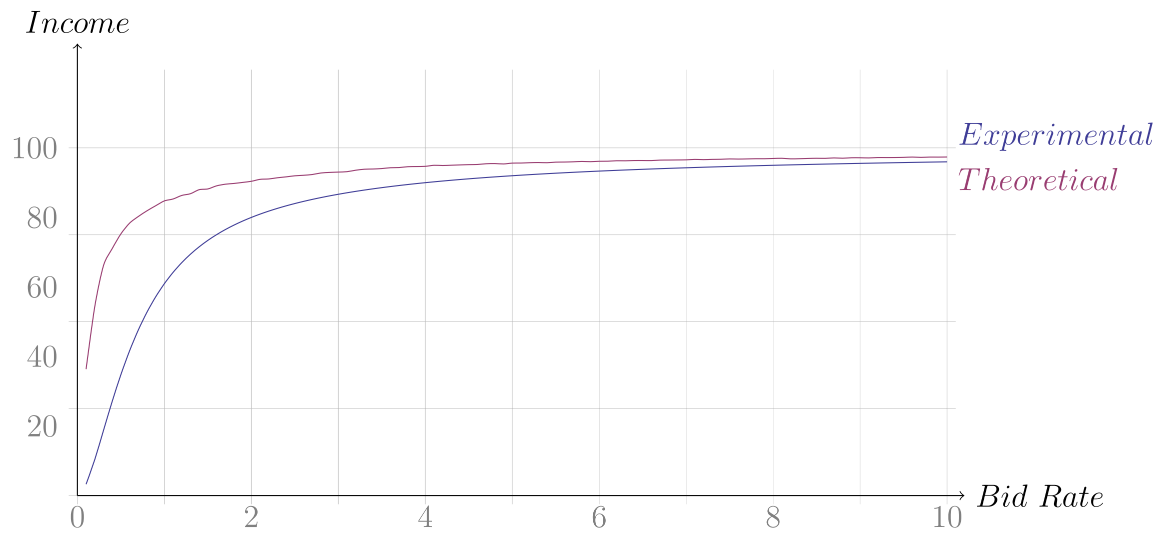

Figure 7-13: Analytic formula for the truncated exponentially distributed IPV for Vickrey auctions.

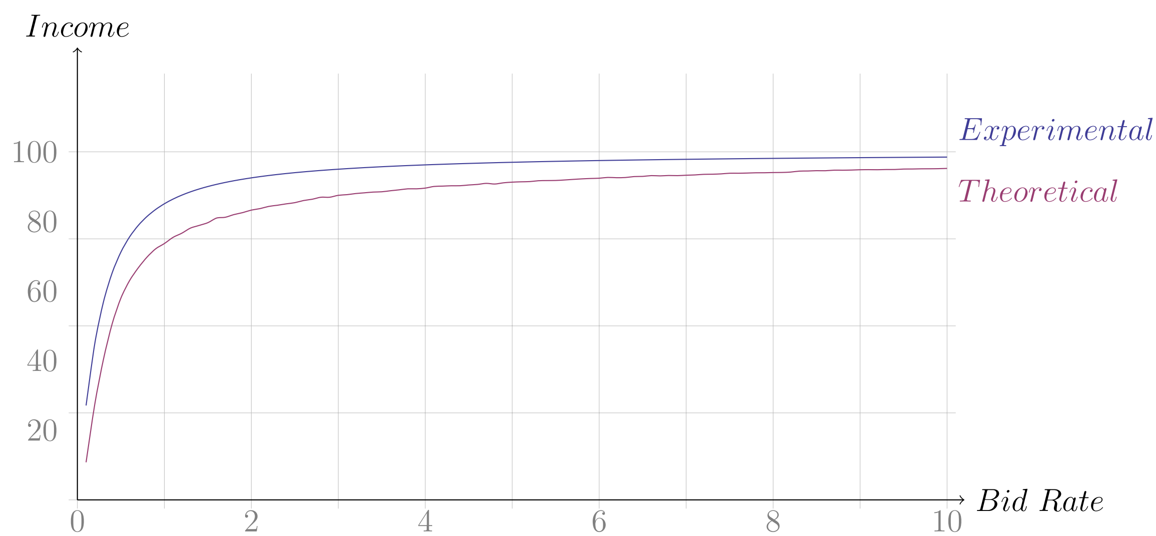

Figure 7-14: Analytic formula for the truncated normally distributed IPV for first-price auctions.

course, in the case of the uniform distribution, $b = 1$.

7.5 Summary

We have studied the sensitivity of the auction income to the form of the IPV distributions. To be realistic, we focus attention on those admitting a finite range of positive

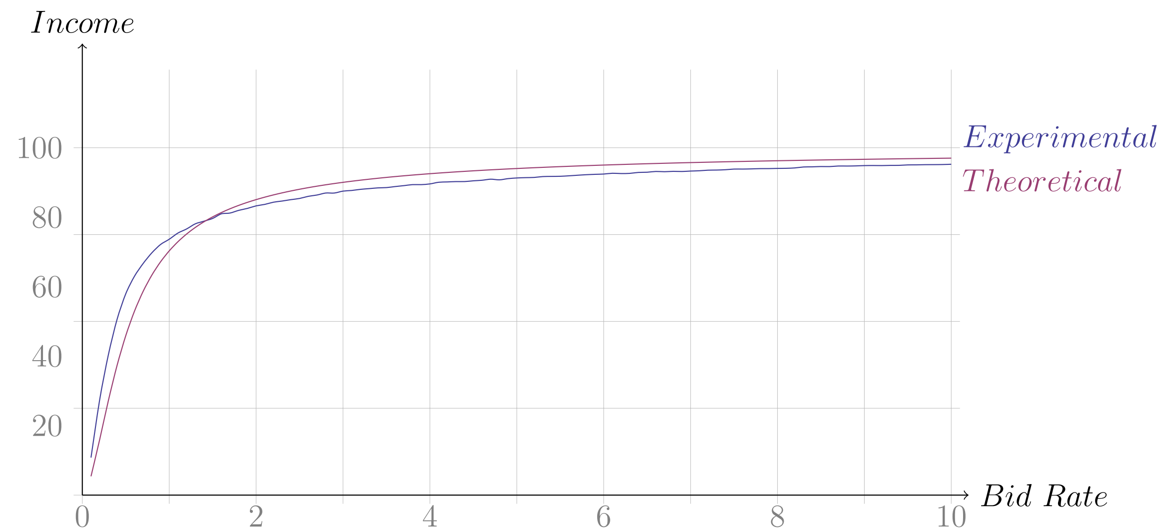

Figure 7-15: Analytic formula for the truncated normally distributed IPV for Vickrey auctions.

values, which include the uniform, truncated normal, and truncated exponential distributions. The uniform distribution may be used to represent bidders of whose IPV very little is known, with all values within the range occurring with equal probability. On the other hand, the truncated normal distribution may be used to represent the community of bidders in which there is some degree of agreement on the value of the item being auctioned. Since consumers are often looking for bargains in Internet auctions (as opposed to obtaining the good through conventional channels), many tend to bid at the lower end of the spectrum of values, and such social behaviour may be represented by the truncated exponential IPV distribution, where the lower value clusters have a greater probability of occurring. For these distributions, we have found that their performances exhibit a remarkable similarity in their pattern of increase. In all such IPV distributions, the average auction income tends to rise rather steeply at first and then gradually settles to approach the ceiling value. Using the damping factor which suitably reduces the bid rate, analytic formulae are obtained for the auction incomes for auctions which adopt the truncated exponential IPV and truncate normal IPV. We have found that the uniform IPV is the most generous with a damping factor of one. The truncated normal IPV is more frugal than the uniform IPV with a smaller damping factor, while the most frugal of the three is the truncated exponential IPV which has the smallest damping factor compared with the other IPV distributions.

Chapter 8

Conclusions and Future Work

8.1 Summary of Achievements

The present study provides the first systems model of Internet Auction Processes. It is able to give good prediction of Internet auction performance, and the closed-form analytic results are able to furnish fairly good agreements with observed measurements.

The value of such model may be usefully compared with those of Queueing Systems. While actual queueing systems in general admit considerable complexity such as transient behaviour, inter-dependent arrivals, rush hour characteristics, correlated service times, diverse scheduling rules etc., tractable and manageable system model such as the $M/G/1$ steady state queue (along with the well-known *Pollazcek-Khintchine* formula) nevertheless provides useful quantitative results as well as valuable insights and approximations into the performance and behavioural characteristics of queueing systems, even though the $M/G/1$ queue is recognised to be a gross simplification of reality. Similar to Kendall's notation for queueing systems, we can characterise Internet Auction Processes in terms of a set of properties.

The first is the arrival characteristics; e.g. as in queueing theory, we may use M to denote homogeneous Poisson (Markov) arrival, and $M(t)$ to denote non-homogeneous Poisson arrival.

The second is the distribution of the private values; e.g. we may use U to signify that the values are uniformly distributed, N to signify that the values are normally distributed, and M to signify that the values are exponentially distributed. Since private values are bounded, it is often more realistic to use the corresponding truncated distributions for N and M. Therefore the corresponding truncated distribution is assumed to be used whenever we indicate N or M.

The third is the relative ranking of the accepted bid; e.g. we may use FP to denote first-price auctions, VK to denote Vickrey auctions, and LP to denote lowest price auctions.

The fourth is the intended auction duration; e.g. we may use FT to denote fixed-time auctions, and VT to denote variable-time auctions.

The fifth is whether there is the option of early termination of an auction, e.g. on receiving a sufficiently high bid (such as Buy-It-Now or BIN) or a sufficiently low bid (Sell-It-Now or SIN); when such early termination option is not available, then we would simply omit this specification, meaning that the auction will always be allowed to run its normal course to completion. In the case where the auction is designed to terminate on receiving a pre-specified number of bids k, then we would simply use k to indicate this in the fifth component.

For example, a first-price auction with Poisson arrival, uniformly distributed private values, which uses a fixed auction duration but allows early termination at the point when a sufficiently large bid is received is denoted by $M/U/FP/FT/BIN$.

A comparison between some of the attributes of Internet Auction Systems and Queueing Systems is useful (see Table 8.1).

Internet Auction Systems	**Queueing Systems**
Independent Values Distribution	Service Time Distribution
Price Selection Criterion (e.g. First-Price)	Customer Selection Criterion (e.g. FCFS)
Number of Bids Accepted	Number of Servers
Auction Duration (exceeding which no bid is allowed)	Waiting Room Capacity (exceeding which no customer is allowed)
Premature Auction Termination	Service Preemption
Bid Arrival Pattern	Customer Arrival Pattern

Table 8.1: Comparison Between Internet Auction Systems and Queueing Systems.

We have obtained closed-form solutions to the following systems:

1. $M/U/FP/FT$
2. $M/U/FP/VT$
3. $M/U/FP/FT/BIN$
4. $M/U/FP/VT/BIN$
5. $M/U/FP/VT/k$
6. $M/U/VK/FT$
7. $M/U/VK/VT$
8. $M/U/VK/FT/BIN$
9. $M/U/VK/VT/BIN$
10. $M/U/VK/VT/k$
11. $M/U/LP/FT$
12. $M/U/LP/VT$
13. $M/U/LP/FT/SIN$
14. $M/U/LP/VT/SIN$
15. $M/U/LP/VT/k$

In addition, through the introduction of damping factors, we have also obtained analytic approximations to the following:

16. $M/M/FP/FT$
17. $M/M/VK/FT$
18. $M/N/FP/FT$

19. $M/N/VK/FT$

Moreover, through appropriately transforming the timescale, we have been able to generalise most of the above results from a homogeneous Poisson arrival pattern of M to a non-homogeneous arrival pattern of $M(t)$, which is able to model a wide variety of bid arrival characteristics in a versatile manner.

8.2 Optimal Auction Scenarios

From this thesis, I have learned auctions algorithms on the Internet have room for incorporating greater complexity. Some of the algorithms studied here are more complex than the existing Internet algorithms (e.g. Algorithms 3.3 and 3.4), and if users are willing to accept such complexity, these can be incorporated into Internet auctions.

The optimal auction scenario from the point-of-view of the seller is to run a fixed-time Vickrey auction, since Vickrey auctions encourage bidder participation, and consequently higher bid rates translate to higher incomes. Additionally, a high bid rate will narrow the price difference between the first-price and second price anyway. Choosing a fixed-time auction makes it easier to manage and as the seller, I can be assured of getting the money by a specific time rather than having to wait for a period of uncertain length in the case of a soft-close auction. The length of the auction will be so chosen that the expected income is higher than my target surplus.

The optimal auction scenario from the point-of-view of the buyer is to participate in an auction with Open BIN (either hard-close or soft-close) where M is lower than the price I am willing to pay. Doing so will mean that I can get the resource quickly without having to wait for the auction to finish. If such an auction is not available, I would choose a Vickrey auction, where I get the assurance that I won't overpay compared with other bidders.

8.3 Future Work

As for queueing systems, the varieties and types of Internet Auction processes admit wide variations, and the characteristics and performance of different such auction systems will clearly be worthy of further study. In future, some of the components of Internet Auction Systems may be extended to characterise different auction situations and algorithms. Unlike conventional auctions, arbitrarily complex auction algorithms may be designed and adopted for Internet Auctions.

While the probability distributional assumptions that we use have been commonly adopted in other studies, it will be useful in particular to extend the Independent Private Values characteristics to other distributions for modelling the behaviour of different types of bidding communities. For example, it will be interesting to generalise the above results to such widely-used distributions as the Erlang distribution and the Pareto distribution.

Indeed, any one of the five components of an Internet auction system may be usefully generalised and extended. Admittedly, however, exact results may not be easy to obtain, and approximations would have to be used. Such approximations may relate to the adoption of assumptions that are close but not completely matching the particular situations, or approximations that are related to the mathematical analysis of the model or both. Such approximate expressions, even though do not give a full and complete correspondence with the underlying reality, will nevertheless be able to provide useful insight into the operation of Internet auction processes and furnish mechanisms for the practical optimisation of the relevant auction parameters.

Since Internet auctions is becoming prevalent and ubiquitous, with significant parts of the commodities and resources markets based on auctions rather than the fixed-price model, the systematic quantitative study of Internet auction models will be increasingly useful and essential to impart a deeper understanding of the performance behaviour of such processes and mechanisms.

Appendix A

eBay Auction Data

Table A.1: Auction of "Apple iPhone 4 16GB- BLACK- UNLOCKED JAILBROKEN, NEW!!!" to "_***n" for USD575.00.

Bid Date	Bid Time	Bidder	Private Value	Time Between Bids (s)
7/13/2011	18:20			
7/13/2011	18:07	_***n	575.00	
7/13/2011	02:29	h***h	565.00	56268
7/12/2011	21:53	k***e	540.00	16559
7/12/2011	18:40	i***r	450.00	11602
7/12/2011	13:42	i***i	300.00	17867
7/12/2011	06:01	a***a	265.00	27649
7/12/2011	03:23	0***9	250.50	9465
7/11/2011	14:03	e***2	230.00	48021
7/11/2011	07:45	o***p	220.00	22651
7/11/2011	03:45	y***a	220.00	14439
7/11/2011	00:17	i***k	200.00	12451
7/11/2011	00:12	d***n	110.00	310
7/10/2011	20:28	r***y	75.00	13445
7/10/2011	20:09	m***s	50.00	1144
7/10/2011	18:55	l***t	189.23	4421
7/10/2011	18:45	n***c	3.00	635
7/10/2011	18:20			1468

Table A.2: Auction of “Apple iPhone 4 WHITE 16GB UNLOCKED JAILBROKEN BOXED 4.3” to “k***k” for GBP420.00.

Bid Date	**Bid Time**	**Bidder**	**Private Value**	**Time Between Bids (s)**
7/13/2011	18:50			
7/13/2011	15:09	k***k	420.00	
7/13/2011	11:36	e***t	345.00	12775
7/13/2011	00:55	m***r	410.00	38494
7/12/2011	18:16	9***9	272.00	23919
7/11/2011	18:07	i***i	262.00	86925
7/10/2011	16:30	a***o	240.00	92269
7/09/2011	03:26	s***u	210.00	133400
7/09/2011	03:04	1***2	325.00	1347
7/08/2011	02:21	a***b	200.00	88989
7/08/2011	00:49	e***i	195.00	5475
7/07/2011	19:24	t***h	183.00	19510
7/07/2011	13:48	o***b	175.00	20192
7/07/2011	13:33	0***1	165.00	885
7/07/2011	01:42	y***0	145.00	42674
7/06/2011	22:44	e***i	100.00	10671
7/06/2011	20:17	c***o	11 .00	8812
7/06/2011	19:50	i***h	50 .00	1631
7/06/2011	19:38	m***i	10 .00	710
7/06/2011	19:28	5***i	10 .00	627
7/06/2011	18:50			2240

Table A.3: Auction of “NEW! iPhone 4 16GB black JAILBROKEN!!! + FREE APPS” to “a***a” for USD475.00.

Bid Date	**Bid Time**	**Bidder**	**Private Value**	**Time Between Bids (s)**
7/13/2011	20:00			
7/13/2011	20:00	a***a	475.00	
7/13/2011	18:39	0***8	470.00	4850
7/13/2011	18:21	o***i	417.00	1093
7/13/2011	17:15	i***i	430.00	3943
7/13/2011	15:09	m***r	355.00	7545
7/13/2011	07:45	a***s	350.00	26668
7/13/2011	02:47	e***s	310.00	17871
7/12/2011	18:15	s***i	220.00	30729
7/12/2011	16:59	k***n	210.00	4539
7/12/2011	13:30	y***r	185.00	12531
7/12/2011	01:34	a***1	180.00	42967
7/11/2011	04:06	h***m	120.00	77308
7/10/2011	10:24	s***n	115.00	63700
7/10/2011	05:37	i***9	100.00	17215
7/10/2011	01:07	n***v	100.00	16208
7/09/2011	06:14	s***i	25 .00	67953
7/09/2011	00:37	p***n	75 .00	20222
7/08/2011	20:00			16655

Table A.4: Auction of “Apple iPhone 4 16GB new” to “c***o” for USD540.00.

Bid Date	Bid Time	Bidder	Private Value	Time Between Bids (s)
7/13/2011	20:38			
7/13/2011	18:31	c***o	540.00	
7/13/2011	15:53	b***a	500.00	9503
7/13/2011	13:25	u***1	450.00	8882
7/13/2011	10:58	i***k	530.00	8818
7/13/2011	08:59	i***n	410.00	7130
7/13/2011	08:41	n***e	405.00	1066
7/13/2011	07:59	1***a	380.00	2500
7/13/2011	07:26	n***d	360.00	1997
7/13/2011	07:09	a***a	400.00	1052
7/13/2011	07:07	t***i	300.00	81
7/13/2011	06:23	l***a	350.00	2649
7/13/2011	05:30	i***r	255.00	3169
7/13/2011	05:11	d***o	240.00	1128
7/13/2011	05:02	0***r	250.00	582
7/13/2011	04:06	0***e	200.00	3323
7/13/2011	03:22	3***e	192.00	2636
7/13/2011	03:08	n***6	90 .00	880
7/13/2011	03:02	r***n	170.00	335
7/13/2011	03:00	n***c	45 .00	153
7/13/2011	02:54	z***n	40.00	335
7/13/2011	02:53	r***x	20.00	72
7/12/2011	20:38			22490

Table A.5: Auction of "Apple iPhone 4 (Latest Model) - 16GB - White (AT&T) NEW" to "a***d" for USD525.00.

Bid Date	**Bid Time**	**Bidder**	**Private Value**	**Time Between Bids (s)**
7/14/2011	03:05			
7/14/2011	03:05	a***d	525.00	
7/14/2011	03:04	p***p	515.00	27
7/13/2011	22:29	j***m	496.00	16505
7/13/2011	18:56	i***i	485.01	12815
7/13/2011	17:44	a***c	350.00	4310
7/13/2011	14:40	s***f	320.00	11011
7/13/2011	13:22	a***m	450.00	4716
7/13/2011	07:37	g***1	290.00	20677
7/13/2011	04:24	k***a	260.00	11595
7/13/2011	04:17	e***e	300.99	426
7/13/2011	04:16	h***h	250.00	62
7/13/2011	03:51	y***e	70 .00	1510
7/13/2011	03:34	i***h	50 .00	1010
7/13/2011	03:14	8***m	27 .00	1158
7/13/2011	03:12	l***d	26 .00	118
7/13/2011	03:05			460

Appendix B

Generalisation to Non-Homogeneous Poisson Process

While many previous studies modelled the bid arrival process as a Poisson process with constant arrival rate λ, a more versatile model will be to allow λ to be a function of time, i.e.

$$\Pr[\text{an arrival occurring in } (t, t + \Delta t)] \approx \lambda(t)\Delta t. \tag{B.1}$$

In such a process, the number of bid arrivals in the time interval $(t, t + \Delta t)$ is still independent of what happens in other time intervals. This process is known by various names such as non-homogeneous Poisson process, non-stationary Poisson process, time-dependent Poisson process, and in our study, the term non-homogeneous Poisson process will be used. To generalise on the results of Chapter 3, we need to replace the number of bids N by $N(t)$, which signifies the number of bids arrived in the time interval $(0, t)$, and denoting by $R(t)$, the following integral

$$R(t) = \int_0^t \lambda(x)dx,$$

from Mieghem (2006), we have the non-homogeneous Poisson distribution

$$\Pr[N(t) = k] \quad = \quad \frac{e^{-R(t)} R(t)^k}{k!}, \tag{B.2}$$

the mean value of which is given by $R(t)$.

B.1 Algorithm I: Fixed Time Forward Auction

To determine the average income $E[Q_{N(T)}]$ for the present situation, we remove the condition on $N(t)$ in Equation 3.1 using the probabilities in Equation B.2 with $t = T$; i.e.

$$\begin{aligned}
E\left[Q_{N(T)}\right] &= \sum_{N(T)=1}^{\infty} \frac{N(T)L}{N(T)+1} \times \frac{e^{-R(T)} R(T)^{N(T)}}{N(T)!} \\
&= \sum_{N(T)=1}^{\infty} \left[L - \frac{L}{N(T)+1}\right] \times \frac{e^{-R(T)} R(T)^{N(T)}}{N(T)!} \\
&= \sum_{N(T)=1}^{\infty} L \times \frac{e^{-R(T)} R(T)^{N(T)}}{N(T)!} - \sum_{N(T)=1}^{\infty} \frac{L}{N(T)+1} \times \frac{e^{-R(T)} R(T)^{N(T)}}{N(T)!} \\
&= L\left(1 - e^{-R(T)}\right) - \frac{Le^{-R(T)}}{R(T)} \sum_{N(T)=1}^{\infty} \frac{R(T)^{N(T)+1}}{(N(T)+1)!} \\
&= L\left(1 - e^{-R(T)}\right) - \frac{Le^{-R(T)}}{R(T)} \left(e^{R(T)} - 1 - R(T)\right)
\end{aligned}$$

which simplifies to

$$E\left[Q_{N(T)}\right] \quad = \quad \frac{L}{R(T)} \left(R(T) + e^{-R(T)} - 1\right),$$

i.e.

$$E\left[Q_{N(T)}\right] \quad = \quad \frac{L\left\{\left(\int_0^T \lambda(x)dx\right) + \left[\exp\left(-\int_0^T \lambda(x)dx\right)\right] - 1\right\}}{\int_0^T \lambda(x)dx}, \tag{B.3}$$

Fixed time auctions are particularly susceptible to sniping where the closing time of an auction is known in advance. This may be modelled by

$$\lambda(t) \quad = \quad Ae^{\omega t}, \tag{B.4}$$

where $A > 0$, and, while ω may be positive or negative, the case $\omega > 0$ may be used to represent sniping situations since the bid arrival rate climbs steeply towards the end of the auction.

In this case, Equation B.3 becomes

$$E\left[Q_{N(T)}\right] \quad = \quad \frac{L\left\{\left(\int_0^T Ae^{\omega x}dx\right) + \left[\exp\left(-\int_0^T Ae^{\omega x}dx\right)\right] - 1\right\}}{\int_0^T Ae^{\omega x}dx},$$

This gives

$$E\left[Q_{N(T)}\right] \quad = \quad \frac{L}{A\left(e^{\omega T}-1\right)}\left[A\left(e^{\omega T}-1\right) + \omega\exp\left[\frac{A}{\omega}\left(e^{\omega T}-1\right)\right] - \omega\right]. \tag{B.5}$$

B.2 Algorithm II: Variable Time Forward Auctions with Fixed Inactivity Window

The generalisation of Algorithm II is complicated by the fact that, since the bid arrival process is time-dependent, the recursive arguments of Equation 3.6 cannot be applied. However, we shall make use of an important property of the non-homogeneous Poisson process (Mieghem, 2006), namely, that by transforming the time scale from the t-axis to a new u-axis using the transformation

$$u \quad = \quad \int_0^t \lambda(x)dx = R(t),$$

so that under the new time scale u, we have a homogeneous Poisson process with unit rate; i.e. if λ' signifies the new Poisson rate on the u-axis, we have $\lambda' = 1$. Thus, by carrying out the analysis like before under the new time scale, all the previous arguments will carry through, and then we can inverse transform the results back to the original time scale t.

Under the transformed time scale, the parameter α will become α', with

$$\alpha' \quad = \quad \int_0^{\alpha} \lambda(x)dx = R(\alpha), \tag{B.6}$$

and under this transformed time scale, the auction duration from Equation 3.7, bearing in mind that the rate of the process is now unity, is

$$\tau'_{\alpha'} \quad = \quad e^{\alpha'} - 1 = e^{R(\alpha)} - 1. \tag{B.7}$$

Hence, the required auction duration τ_α satisfies

$$R(\tau_\alpha) \quad = \quad e^{R(\alpha)} - 1. \tag{B.8}$$

Or in terms of the inverse function

$$\tau_\alpha \quad = \quad R^{-1}\left[e^{R(\alpha)} - 1\right]. \tag{B.9}$$

From Equation B.8 we may determine τ_α from the following equation

$$\int_0^{\tau_\alpha} \lambda(x)dx = \exp\left(\int_0^{\alpha} \lambda(x)dx\right) - 1. \tag{B.10}$$

which, in general, may be solved by numerical methods. For certain simple cases, closed form solutions may be obtained. For example, in the simple case $\lambda(t) = bt$, then we have from Equation B.10

$$\frac{b\tau_\alpha^2}{2} \quad = \quad e^{\frac{ba^2}{2}} - 1.$$

giving

$$\tau_\alpha \quad = \quad \sqrt{\frac{2e^{\frac{ba^2}{2}} - 2}{b}}.$$

Here, we have $\lambda(0) = 0$, which may sometimes be restrictive. For a more general linear form $\lambda(t) = a + bt$, the solution to τ_α is more complicated but still admits a closed-form solution. Here, we have

$$R(\tau_\alpha) \;=\; \int_0^{\tau_\alpha} (a + bx)dx = a\tau_\alpha + \frac{b\tau_\alpha^2}{2}.$$

To determine the inverse function R^{-1}, we let $y = R(\tau_\alpha)$, and solve for τ_α in terms of y. That is,

$$y \;=\; \alpha\tau_\alpha + \frac{b\tau_\alpha^2}{2},$$

which gives

$$b\tau_\alpha^2 + 2\alpha\tau_\alpha - 2y \;=\; 0.$$

Solving this using the quadratic formula, we have

$$\tau_\alpha \;=\; \frac{-2a \pm \sqrt{4a^2 + 8by}}{2b}.$$

Taking the positive square root and simplifying, we have

$$\tau_\alpha \;=\; \frac{-a + \sqrt{a^2 + 2by}}{b}.$$

That is,

$$R^{-1}(y) \;=\; \frac{-a + \sqrt{a^2 + 2by}}{b}.$$

From Equation B.9, we have

$$\tau_\alpha \;=\; R^{-1}\left[e^{R(\alpha)} - 1\right] = \frac{-a + \sqrt{a^2 + 2b\left[e^{R(\alpha)} - 1\right]}}{b}.$$

That is, we obtain

$$\tau_\alpha = R^{-1}\left[e^{R(\alpha)} - 1\right] = \frac{-a + \sqrt{a^2 + 2b\left[e^{a\alpha + b\alpha^2/2} - 1\right]}}{b}. \quad \text{(B.11)}$$

It is useful to note that as $b \to 0$, $\lambda \to a$, the limit of the right hand side of Equation B.11 tends to $\left(e^{a\alpha} - 1\right)/a$, which can be seen by application of L'Hôpital's rule, and this result is in agreement with Equation 3.7.

For

$$\lambda(t) = Ae^{\omega t},$$

from B.10, we have

$$\int_0^{\tau_\alpha} Ae^{\omega x} dx = \exp\left(\int_0^{\alpha} Ae^{\omega x} dx\right) - 1, \quad \text{(B.12)}$$

i.e.

$$\frac{Ae^{\omega\tau_\alpha}}{\omega} = \left[\exp\left(\int_0^{\alpha} \frac{Ae^{\omega\alpha}}{\omega}\right)\right] - 1 + \frac{A}{\omega}$$

or

$$e^{\omega\tau_\alpha} = \frac{\omega}{A}\left[\exp\left(\frac{A}{\omega}\left(e^{\omega\alpha} - 1\right)\right)\right] - \frac{\omega}{A} + 1$$

giving

$$\tau_\alpha = \frac{1}{\omega}\log_e\left(\frac{\omega}{A}\left[\exp\left(\frac{A}{\omega}\left(e^{\omega\alpha} - 1\right)\right)\right] - \frac{\omega}{A} + 1\right).$$

We also note that as $\omega \to 0$, $\lambda \to A$, and τ_α tends to $\left(e^{A\alpha} - 1\right)/A$, which can be seen most readily from Equation B.12, and this result is in agreement with Equation 3.7.

Since bidding sometimes takes place more frequently at certain times of day but less so at other times, a periodic $\lambda(t)$ may be usefully employed to model this situation;

here we may take

$$\lambda(t) \quad = \quad A_1 \cos(\omega t - \phi) + A_0. \tag{B.13}$$

From B.10, we have

$$\int_0^{\tau_\alpha} (A_1 \cos(\omega x - \phi) + A_0)\, dx \quad = \quad \exp\left(\int_0^{\alpha} (A_1 \cos(\omega x - \phi) + A_0)\, dx\right) - 1,$$

i.e.

$$\frac{A_1 \left(\sin\left(\omega\tau_\alpha - \phi\right) + \sin\phi\right)}{\omega} + A_0\tau_\alpha \quad = \quad e^{(A_1/\omega)(\sin(\omega\alpha - \phi) + \sin\phi) + A_0\alpha} - 1,$$

from which τ_α may be solved numerically. In the special case $\phi = 0$, the above simplifies to

$$\frac{A_1 \sin \omega\tau_\alpha}{\omega} + A_0\tau_\alpha \quad = \quad e^{(A_1 \sin \omega\alpha/\omega) + A_0\alpha} - 1, \tag{B.14}$$

For example, if $A_0 = 2$, $A_1 = 1$, $\omega = 1$, and $\alpha = \pi$, Equation B.14 becomes

$$2\tau_\alpha + \sin\tau_\alpha \quad = \quad e^{2\pi} - 1$$

Solving for τ_α numerically gives $\tau_\alpha = 267.5$. It is interesting to note that when $A_1 = 0$, then $\lambda(t) = A_0$ and Equation B.14 becomes

$$A_0\tau_\alpha \quad = \quad e^{A_0\alpha} - 1,$$

which is in agreement with Equation 3.7.

In general, the inverse function would be difficult to obtain, but if $\lambda(t)$ is a monotonic function, then useful bounds may be obtained. For example, if $\lambda(t)$ is monotonically increasing with $\lambda(0) \leq \lambda(t)$, then since we know from previous analysis of Algorithm II that the auction duration is an increasing function of λ, thus the auction duration, from Equation 3.7, will be bounded below by

$$\tau_\alpha \;\geq\; \frac{e^{\lambda(0)\alpha}-1}{\lambda(0)}.$$

Likewise if $\lambda(t)$ is monotonically decreasing with $\lambda(0) \geq \lambda(t)$, then the auction duration, from Equation 3.7, will be bounded above by

$$\tau_\alpha \;\leq\; \frac{e^{\lambda(0)\alpha}-1}{\lambda(0)}.$$

As for the auction income, we note that in the homogeneous Poisson case, the income depends on α only through the probabilities (Equation 3.8). Thus, unlike the auction duration where constants also need to be transformed (e.g. a duration of 1second in the t-axis will not necessarily be 1 second in the u-axis), here our conversion need to only concentrate on α.

Thus, viewing the process from the u-axis, from Equation 3.11, we have

$$E\left[Q_{(N)}\right] \;=\; \frac{L}{e^{\alpha'}-1}\left[e^{\alpha'}-\alpha'-1\right]. \tag{B.15}$$

That is,

$$E\left[Q_{(N)}\right] \;=\; \frac{L}{e^{R(\alpha)}-1}\left[e^{R(\alpha)}-R(\alpha)-1\right], \tag{B.16}$$

or more explicitly

$$E\left[Q_{(N)}\right] \;=\; \frac{L\left[\exp\left(\int_0^\alpha \lambda(x)dx\right)-\int_0^\alpha \lambda(x)dx-1\right]}{\exp\left(\int_0^\alpha \lambda(x)dx\right)-1}.$$

It may be useful to verify the approach leading to Equation B.15 from another angle. Combining Equation 3.7 and 3.11 gives for the homogeneous Poisson case,

$$E\left[Q_{(N)}\right] \;=\; L-\frac{L\alpha}{\tau_\alpha}.$$

Since L is not a time quantity, it need not be transformed. Thus the only factors that need to be transformed here are α and τ_α, giving for the non-homogeneous Poisson case

$$E\left[Q_{(N)}\right] = L - \frac{L\alpha'}{\tau'_{\alpha'}}.$$

From Equations B.6 and B.7 this gives

$$E\left[Q_{(N)}\right] = L - \frac{LR(\alpha)}{e^{R(\alpha)}-1} = L\left[1 - \frac{R(\alpha)}{e^{R(\alpha)}-1}\right].$$

which again yields Equation B.16. This indicates that the above approach that concentrates only on the conversion of α and the arrival rate will yield correct results. That is, since the income is measured in monetary terms, it should be the same irrespective of which time scale is used, and consequently only time-related parameters, such as the length of the inactivity window and the arrival rate, need to be adjusted.

In the case when $\lambda(t) = a + bt$, we have

$$E\left[Q_{(N)}\right] = \frac{L\left[e^{a\alpha+b\alpha^2/2} - a\alpha - b\alpha^2/2 - 1\right]}{e^{(a\alpha+b\alpha^2/2)} - 1}$$

We see that as $b \to 0$, $\lambda \to \alpha$, and the above tends to

$$E\left[Q_{(N)}\right] = \frac{L}{e^{a\alpha}-1}\left[e^{a\alpha} - a\alpha - 1\right],$$

which is in agreement with Equation 3.11. In the case when $\lambda(t)$ is given by Equation B.4, we have

$$\begin{aligned} E\left[Q_{(N)}\right] &= \frac{L\left[\exp\left(\int_0^\alpha Ae^{\omega x}dx\right) - \int_0^\alpha Ae^{\omega x}dx - 1\right]}{\int_0^\alpha Ae^{\omega x}dx - 1} \\ &= \frac{\omega L \times \exp\left[\frac{A}{\omega}\left(e^{\omega\alpha}-1\right) - \frac{A}{\omega}\left(e^{\omega\alpha}-1\right) - 1\right]}{A\left(e^{\omega\alpha}-1\right)}. \end{aligned}$$

We see that as $\omega \to 0$, $\lambda \to A$, and the above tends to

$$E\left[Q_{(N)}\right] = \frac{L}{e^{A\alpha}-1}\left[e^{A\alpha} - A\alpha - 1\right],$$

which is in agreement with Equation 3.11. In the case when $\lambda(t)$ is periodic as given by Equation B.13, we have from Equation B.16

$$E\left[Q_{(N)}\right] = \frac{Le^{(A_1/\omega)[\sin(\omega\alpha-\phi)+\sin\phi]+A_0\alpha} - (A_1/\omega)\left[\sin(\omega\alpha-\phi)+\sin\phi\right] - A_0\alpha - 1}{e^{(A_1/\omega)[\sin(\omega\alpha-\phi)+\sin\phi]+A_0\alpha} - 1}$$

We see that as $A_1 = 0$, then $\lambda = A_0$, and the above tends to

$$E\left[Q_{(N)}\right] = \frac{L}{e^{A_0\alpha} - 1}\left[e^{A_0\alpha} - A_0\alpha - 1\right],$$

which, again, is in agreement with Equation 3.11.

B.3 Algorithm III: Fixed Time Forward Auctions with Fixed Inactivity Window and Maximum Threshold Termination

Consider the Poisson stream. From (Van Micghem, 2006), the probability density function of the inter-arrival time $f(t)$ of this stream is

$$f(t) = \Lambda(t)\exp\left(-\int_0^t \Lambda(x)dx\right) = p\lambda(t)e^{-pR(t)}. \tag{B.17}$$

The average auction duration by combining the averages of (i) having an arrival from $\Lambda(t)$ in the interval $(0, T)$, and (ii) having no arrival from $\Lambda(t)$ in the interval $(0, T)$, and making use of Equation B.2 for the $\Lambda(t)$ stream, is

$$E\left[T_M\right] = \int_0^T tf(t)dt + T\exp\left(-\int_0^T \Lambda(x)dx\right). \tag{B.18}$$

That is,

$$E\left[T_M\right] = p\int_0^T t\lambda(t)e^{-pR(t)}dt + Te^{-pR(T)}. \tag{B.19}$$

As for the auction income, we consider the process on the u-axis, so that from

Equation 3.15, and noting that $\Lambda(t)$ becomes p on the u-axis (since $\lambda(t)$ is transformed to unit arrival rate),

$$E\left[Q_M\right] = \frac{(M+L)\left(1-e^{-pT'}\right)}{2} + \frac{Me^{-pT'}}{(1-p)T'}\left((1-p)T' + e^{-(1-p)T'} - 1\right) \tag{B.20}$$

Since $T' = R(T)$, this becomes

$$E\left[Q_M\right] = \frac{(M+L)\left(1-e^{-pR(T)}\right)}{2} + \frac{Me^{-pR(T)}}{(1-p)R(T)}\left((1-p)R(T) + e^{-(1-p)R(T)} - 1\right) \tag{B.21}$$

B.4 Algorithm IV: Variable Time Forward Auctions with Fixed Inactivity Window and Maximum Threshold Termination

From Chapter 3, and carrying out the above time transformation, we have for the average auction duration in the t-axis

$$E\left[\tau_M\right] = R^{-1}\left\{\frac{1-e^{-\alpha'}}{p+p'e^{-\alpha'}}\right\}. \tag{B.22}$$

For the average auction income we have, following the same arguments as in Chapter 3,

$$E\left[Q'_M\right] = \frac{1}{p_1+p_2}\left\{p_1 M\left[1-\frac{p'\alpha'}{e^{p'\alpha'}-1}\right] + \frac{p_2(M+L)}{2}\right\}. \tag{B.23}$$

B.5 Algorithm V: Auctions on Attaining a Given Number of Bids

As indicated in Equation B.2, the mean number of bids for the non-homogeneous Poisson process is given by $R(t)$. Thus the average time τ_B required to attain B bids is given by

$$R(\tau_B) = \int_0^{\tau_B} \lambda(x)dx = B. \tag{B.24}$$

Thus,

$$\tau_B = R^{-1}(B). \tag{B.25}$$

For $\lambda(t) = a + bt$, we have

$$R(\tau_B) = \int_0^{\tau_B} (a + bx)dx = a\tau_B + \frac{b\tau_B^2}{2} = B. \tag{B.26}$$

This gives

$$\tau_B = \frac{-a + \sqrt{a^2 + 2bB}}{b} \tag{B.27}$$

If $b \to 0$, $\lambda(t) \to a$, and the above tends to B/a, which is in agreement with the homogeneous Poisson case.

For

$$\lambda(t) = Ae^{\omega t} \tag{B.28}$$

we have

$$\int_0^{\tau_B} Ae^{\omega x}dx = \frac{Ae^{\omega\tau_B}}{\omega} - \frac{A}{\omega} = B, \tag{B.29}$$

which gives

$$\tau_B = \frac{1}{\omega}\log_e\left(\frac{\omega B}{A} + 1\right). \tag{B.30}$$

Since the number of bids is the same, the auction income for this case is the same as for the corresponding homogeneous Poisson case.

B.6 Algorithm VI: Fixed Time Vickrey Auctions

To determine the average income $E\left[Q_{(N(T))-1}\right]$ for the present situation, we remove the condition on $N(t)$ in Equation 3.8 using the probabilities given in Equation B.2 with $t = T$; i.e.

$$\begin{aligned}
E\left[Q_{(N(T))-1}\right] &= \sum_{N(T)=1}^{\infty} \frac{(N(T)-1)\,L}{N(T)+1} \times \frac{e^{-R(T)}R(T)^{N(T)}}{N(T)!} \\
&= \sum_{N(T)=1}^{\infty} \left[L - \frac{2L}{N(T)+1}\right] \times \frac{e^{-R(T)}R(T)^{N(T)}}{N(T)!} \\
&= \sum_{N(T)=1}^{\infty} L \times \frac{e^{-R(T)}R(T)^{N(T)}}{N(T)!} - \sum_{N(T)=1}^{\infty} \frac{2L}{N(T)+1} \times \frac{e^{-R(T)}R(T)^{N(T)}}{N(T)!} \\
&= L\left(1 - e^{-R(T)}\right) - \frac{2Le^{-R(T)}}{R(T)} \sum_{N(T)=1}^{\infty} \frac{R(T)^{N(T)+1}}{(N(T)+1)!} \\
&= L\left(1 - e^{-R(T)}\right) - \frac{2Le^{-R(T)}}{R(T)} \left(e^{R(T)} - 1 - R(T)\right)
\end{aligned}$$

which simplifies to

$$E\left[Q_{(N(T))-1}\right] = \frac{L}{R(T)}\left(R(T) + R(T)e^{-R(T)} + 2e^{-R(T)} - 2\right), \tag{B.31}$$

i.e.

$$E\left[Q_{(N(T))-1}\right] = \frac{L\left(\int_0^T \lambda(x)dx + \left[\int_0^T \lambda(x)dx \times \exp\left(-\int_0^T \lambda(x)dx\right)\right] + \left[2\exp\left(-\int_0^T \lambda(x)dx\right)\right] - 2\right)}{\int_0^T \lambda(x)dx} \tag{B.32}$$

B.7 Algorithm VII: Variable Time Vickrey Auctions with Fixed Inactivity Window

The average auction duration is given in Section B.2. For the determination of average auction income, as before, we carry out the time transformation in Equation B.6 so that the parameter α in the t-axis will become α' in the u-axis, and from Equation 4.6, we have

$$E\left[Q_{(N(T)-1)}\right] \quad = \quad \frac{2L}{e^{\alpha'}-1}\left[\sinh\alpha' - \alpha'\right], \tag{B.33}$$

i.e.

$$E\left[Q_{(N(T)-1)}\right] \quad = \quad \frac{2L}{e^{R(\alpha)}-1}\left[\sinh R(\alpha) - R(\alpha)\right]. \tag{B.34}$$

This gives

$$E\left[Q_{(N(T)-1)}\right] \quad = \quad \frac{2L\left[\sinh\int_0^\alpha \lambda(x)dx - \int_0^\alpha \lambda(x)dx\right]}{e^{\int_0^\alpha \lambda(x)dx}-1}. \tag{B.35}$$

B.8 Algorithm VIII: Fixed Time Vickrey Auctions with Maximum Threshold Termination

The average auction duration is given in Section B.3. For the determination of average auction income, as before, we carry out the time transformation in Equation B.6 and using the unit arrival rate in the u-axis, we have from Equation 4.9,

$$E\left[Q_{Mv}\right] \quad = \quad \frac{(M+L)\left(1-e^{-pT'}\right)}{2} + \frac{Me^{-pT'}}{p'T'}\left(p'T' + p'T'e^{-p'T'} + 2e^{-p'T'} - 2\right), \tag{B.36}$$

where

$$T' \quad = \quad \int_0^T \lambda(x)dx. \tag{B.37}$$

B.9 Algorithm IX: Variable Time Vickrey Auctions with Fixed Inactivity Window and Maximum Threshold Termination

Here, we carry out the same transformation, and following similar arguments as previously, we have for the average auction income,

$$E\left[Q'_M\right] = \frac{1}{p_1+p_2}\left[p_1\left(\sinh\left(p'\alpha'\right)+p'\alpha'\right)+p_2\frac{M+L}{2}\right]. \tag{B.38}$$

The average auction duration here is the same as that given in Algorithm IV.

B.10 Algorithm X: Vickrey Auctions on Attaining a Given Number of Bids

Here, the average magnitude of the accepted bid is given by Equation 4.13, while the average auction duration is given by Equation B.25.

B.11 Algorithm XI: Fixed Time Reverse Auctions

Here, the auction duration is fixed and equals T. Following the techniques of previous analysis, we have, for the average auction income

$$\begin{aligned} E\left[Q_{(1)}\right] &= \sum_{N=0}^{\infty}\frac{L}{N(T)+1}\times\frac{e^{-R(T)}R(T)^{N(T)}}{N(T)!} \\ &= \frac{Le^{-R(T)}}{R(T)}\left(e^{R(T)}-1\right). \end{aligned}$$

That is,

$$E\left[Q_{(1)}\right] = \frac{L}{R(T)}\left(1-e^{-R(T)}\right) \tag{B.39}$$

B.12 Algorithm XII: Variable Time Reverse Auctions with Fixed Inactivity Window

Similar to previous approach, from Chapter 4, we have for the auction income

$$E\left[Q_{(1)}\right] = \frac{LR(\alpha)}{e^{R(\alpha)} - 1}. \tag{B.40}$$

The average auction duration is similar to that obtained for Algorithm II.

B.13 Algorithm XIII: Fixed Time Reverse Auctions with Minimum Threshold Termination

The average auction duration is similar to that obtained for Algorithm III. Carrying out the same transformation as before, the average magnitude of the accepted bid $E\left[Q_m\right]$, from Chapter 4, is

$$E\left[Q_m\right] = L - \left(\frac{(2L-m)\left(1-e^{-pT'}\right)}{2} + \frac{(L-m)e^{-pT'}}{p'T'}\left[p'T' + e^{-p'T'} - 1\right]\right) \tag{B.41}$$

B.14 Algorithm XIV: Variable Time Reverse Auctions with Fixed Inactivity Window and Minimum Threshold Termination

Following the same approach as previously, the average magnitude of the accepted bid $E\left[Q_m\right]$ is, from Chapter 4,

$$E\left[Q_m\right] = L - \left(\frac{e^{-\alpha'}\left[(L-m)\left(1-\frac{p'\alpha'}{e^{-p'\alpha'}-1}\right)\right] + \frac{2L-m}{2}\left(1-e^{-p\alpha'}\right)\left(p+p'e^{-\alpha'}\right)}{e^{-\alpha'} + \left(1-e^{-p\alpha'}\right)\left(p+p'e^{-\alpha'}\right)}\right) \tag{B.42}$$

where $p = \frac{m}{L}$. The average auction duration is similar to that obtained for Algorithm IV.

B.15 Algorithm XV: Reverse Auctions on Attaining a Given Number of Bids

Here, as before, the average auction duration required to attain B bids is the same as Algorithm V, while the auction income is the same as for the corresponding non-homogeneous Poisson case.

B.16 Empirical Representation of eBay arrivals using Non-Homogeneous Poisson Process

From the *eBay* auction data in Appendix A, some degree of non?uniformity seems to exist in the bid arrival pattern within individual auctions. Thus, the non?homogeneous ?Poisson process may be usefully employed to represent some of these situations.

?For example, for Auction A2, we can represent the arrival pattern as (using day as the ?time unit and counting the number of bid arrivals in each 24?hour period): ?$\lambda(1) = 8; \lambda(2) = 3?; \lambda(3) = 2; \lambda?(4) = 1; \lambda?(5) = 1; \lambda?(6) = 1; \lambda?(7) = 4,?$ the rates of which show considerable variation and range from 1 to 8. Also, for Auction ?A3, we can represent the arrival pattern as $\lambda(1) = 2; \lambda(2) = 3; \lambda(3) = 1; \lambda(4) = 4; \lambda(5) = 8$, which shows a comparable degree of variability over each 24?hour period as Auction A2.

Thus, modelling some Internet auctions using the non?homogeneous Poisson process will offer a greater degree of versatility and flexibility, especially those with more lengthy durations.

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