Interaction and Depth against Nondeterminism in Proof Search

Ozan Kahramanoğulları[⋆†]

[⋆]Imperial College London, Department of Computing
ozank@doc.ic.ac.uk

Abstract

Deep inference is a proof theoretical methodology that generalizes the traditional notion of inference of the sequent calculus. Deep inference provides more freedom in design of deductive systems and a rich combinatoric analysis of proofs. In particular, construction of exponentially shorter analytic proofs becomes possible, but with the cost of a greater nondeterminism than in the sequent calculus. In this paper, we report on our on going work on proof search with deep inference deductive systems. We argue that, by exploiting an interaction and depth scheme in the logical expressions, the nondeterminism in proof search can be reduced without losing the shorter proofs and breaking proof theoretical properties.

Introduction

Deep inference (Guglielmi, 2007) is a proof theoretical methodology that generalizes the traditional notion of inference of the sequent calculus. In contrast to the sequent calculus, the deductive systems with deep inference do not rely on the notion of main connective and permit the application of the inference rules at any depth inside logical expressions. Deep inference provides a rich combinatoric analysis of the logic begin studied (see, e.g.,(Brünnler, 2003; Straßburger, 2003)) and makes it possible to design deductive systems that are provably not designable in a standard sequent calculus [1].

Availability of deep inference provides shorter proofs than in the sequent calculus. For example, for the case of classical logic, there is a class of theorems, called the Statman's tautologies, for which the size of proofs in the sequent calculus grows exponentially over the size of the theorems. However, over the same class, there are deep inference proofs that grow polynomially (Guglielmi, 2004). This is because applicability of the inference rules at any depth inside logical expressions makes it possible to start the construction of a proof by manipulating and annihilating subformulae without any prior branching. However, because inference rules can be applied in many more ways, this results in a greater nondeterminism which increases the breadth of the search space rather quickly.

In this paper, we report on our on going work on reducing nondeterminism while searching for proofs in deep inference deductive systems, building on the ideas presented in (Kahramanoğulları, 2006). We argue that, by exploiting an interaction and depth scheme in the logical expressions, the nondeterminism in proof search can be reduced without losing the shorter proofs and without breaking proof theoretical properties. We present these ideas on a classical logic system and argue that they generalize to other deep inference systems for other logics.

Interaction and Depth against Nondeterminism

There are infinitely many atomic formulae, denoted by a, b, c... The negation of a formula is denoted by $\bar{\cdot}$. A *context*, denoted as in $S\{\ \}$, is a formula with a hole that does not appear in the scope of negation. The formula R is a *subformula* of $S\{R\}$ and $S\{\ \}$ is its *context*. The system KSg for classical logic (Brünnler, 2003) is the system consisting of the rules

$$\mathsf{ai}\!\downarrow \frac{S\{\mathtt{tt}\}}{S(a \vee \bar{a})}\ , \qquad \mathsf{s}\,\frac{S((R \vee U) \wedge T)}{S((R \wedge T) \vee U)}\ , \qquad \mathsf{w}\!\downarrow \frac{S\{\mathtt{ff}\}}{S\{R\}}\ , \qquad \mathsf{c}\!\downarrow \frac{S(R \vee R)}{S\{R\}}\ .$$

The rules of the system KSg are called *atomic interaction*, *switch*, *weakening*, and *contratction*, respectively. $\wedge$ and $\vee$ are associative and commutative. $\mathtt{ff}$ is the unit for the$\vee$ and $\mathtt{tt}$ is the unit for the $\wedge$. We also impose the equalities $(\mathtt{tt} \vee \mathtt{tt}) = \mathtt{tt}$ and $(\mathtt{ff} \wedge \mathtt{ff}) = \mathtt{ff}$. A derivation Δ is a finite chain of instances of the inference rules. The topmost formula in a derivation, if present, is called the *premise* of the derivation, and the bottommost formula is called its *conclusion*. Deep inference systems follow a common scheme where the context management of the commutative operators is performed by the switch rule. All the deep inference systems for linear logic (Straßburger, 2003), classical logic (Brünnler, 2003), systems BV (Guglielmi, 2007), and NEL (Straßburger, 2003), and modal logics follow this scheme.

We consider the subformulae which are in a disjunction relation as interacting formulae, whereas those formula in a conjunction relation as non-interacting formula. For example, when we consider the formula $a \vee b \vee (\bar{a} \wedge \bar{b})$, a is interacting with b, $\bar{a}$, and $\bar{b}$, whereas $\bar{a}$ is interacting with a and b, but it is not interacting with $\bar{b}$. We redesign the switch rule, in such a way that this rule can be applied only in those ways which promote the interaction between dual atoms: The notation

[1]For more information on deep inference, see http://alessio.guglielmi.name/res/cos/index.html.

$\mathsf{at}\, R$ denotes the set of atoms appearing in formula R. The rule *lazy interaction switch* lis is the rule obtained by imposing the following restriction on the rule switch; formula U is not a disjunction and $\mathsf{at}\, R \cap \mathsf{at}\, \bar{U} \neq \emptyset$.

In (Kahramanoğulları, 2006), we have shown that switch rule can be replaced with the rule lazy interaction switch in system KSg without losing completeness. The completeness proof uses a technique closely related to cut-elimination. Thus, this modification in system KSg, results in reduction of the breadth of the search space without losing shorter proofs and proof theoretic cleanliness. However, although checking the condition of the rule lis can be performed in linear time, in proof search applying this check on all the subformulae can become computationally expensive. In order to see this on an example, consider the following formula: $a \vee b \vee (\bar{a} \wedge \bar{b} \wedge (c \vee d \vee (\bar{c} \wedge \bar{d} \wedge (e \vee f \vee (\bar{e} \wedge \bar{f})))))$. To this formula, switch rule can be applied in 42 different ways, and the rule lis can be applied in 14 different ways. Thus, the condition of the rule lis must be performed for all the 42 potential rule instances.

When we consider the interaction scheme mentioned above as a graph, where interacting atoms are connected by nodes, we make the following observation: By applying the inference rules to subformulae that consist of atoms in close proximity with respect to number of connecting arcs, we are able to construct shorter proofs. When we carry this observation to syntactic level, this notion of being close proximity corresponds to the subformulae at the deepest positions. Thus, by introducing a plausible notion of "deepest" inference that is complete, we can reduce the nondeterminism further in a way that is orthogonal and complementary to that of the rule lis.

We have the following definition: An instance of the rule switch is an instance of the rule *deepest switch*, denoted by ds, if the formula R is not a conjunction. Let system KSgd be the system obtained by replacing the rule s in system KSg with the rule ds.

Theorem 0.1 *Systems* KSg *and* KSgd *are equivalent, that is, they prove the same formulae.*

The proof of this theorem uses the decomposition of the KSg proofs into phases where different rules are applied, and then permutation of the instances of the rule s, that are not instances of the rule ds, over the other rule instances.

An interesting question here is if it is possible to combine the ideas from the rule ds and lis: An instance of the rule switch is an instance of the rule *deepest lazy interaction switch*, denoted by dis, if the formula R is not a conjunction, formula U is not a disjunction, and $\mathsf{at}\, R \cap \mathsf{at}\, \bar{U} \neq \emptyset$. Let system KSgid be the system obtained by replacing the rule s in system KSg with the rule dis.

Theorem 0.2 *Systems* KSg *and* KSgd *are equivalent, that is, they prove the same formulae.*

The rule dis does not only reduce the nondeterminism in proof search further, but also reduces the cost of performing the check of the condition, because the notion of deepest switch forces the substructures R to be smaller subformulae.

We also conjecture that by means of a procedure for cut elimination, we can also eliminate the associativity and commutativity of conjunction, in a way which reduces nondeterminism in proof search without loosing shorter proofs.

Discussion

In (Guglielmi, 2004), Guglielmi has shown that for a class of classical tautologies called Statman's tautologies, deep inference provides an exponential speed up in contrast to the sequent calculus proofs. The restrictions imposed by the rules above preserve the shortest proofs of (Guglielmi, 2004). We have been experimenting with the Maude language for implementing deep inference deductive systems as term rewriting systems, including those discussed above. An extensive comparison of these implementations and proof complexity analysis are included in on going work. Other deep inference systems and a deep inference system for the logic of bunched implications (O'Hearn and Pym, 1999) are potential applications of the ideas above for future work.

References

Kai Brünnler. *Deep Inference and Symmetry in Classical Proofs*. PhD thesis, TU Dresden, 2003.

Alessio Guglielmi. A system of interaction and structure. *ACM Transactions on Computational Logic*, 8(1):1–64, 2007.

Alessio Guglielmi. Polynomial size deep-inference proofs instead of exponential size shallow-inference proofs. Available on the web at http://cs.bath.ac.uk/ag/p/AG12.pdf, 2004.

Ozan Kahramanoğulları. Reducing nondeterminism in the calculus of structures. In Miki Hermann and Andrei Voronkov, editors, *LPAR 2006, Phnom Penh, Cambodia*, volume 4246 of *LNCS*, pages 272–286. Springer, 2006.

P.W. O'Hearn and D.J. Pym. The logic of bunched implications. *Bulletin of Symbolic Logic*, 5(2):215–244, 1999.

Lutz Straßburger. *Linear Logic and Noncommutativity in the Calculus of Structures*. PhD thesis, TU Dresden, 2003.