Matrix Geometric and Spectral expansion Meh-thods

Harf Zatschler

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1 Introduction: Two methods, same thing (meh)

Usually, whenever the steady-state state occupation probabilities of long queues with some regular structure is sought, exactly one of spectral expansion [1] or matrix geometric [4] methods is used. Both methods have their advantages and disadvantages, and the choice of which of the two is used, usually depends on which a particular researcher has encountered first. There is overlap between the the two methods, with the feature that distinguishes them being the manner in which the solution space of the *repeating region* is represented. Within this region, depicted in Figure 1 for a simple, finite queue, adjacent state occupation probabilities satisfy a homogeneous matrix recurrence relation.

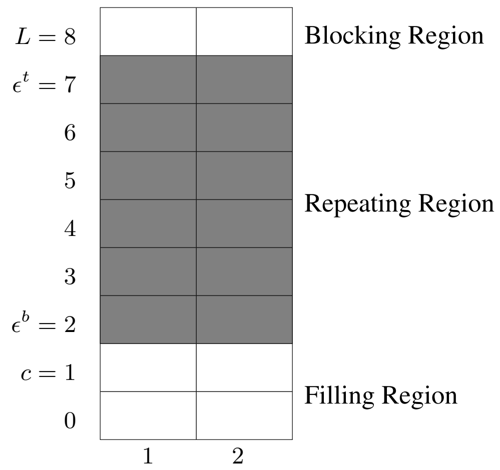

Figure 1: Queue regions in a finite queue. Infinite queues would not have a blocking region.

As an example, we consider a 2-state markov modulated queue (modulation matrix $\underline{\mathbf{Q}} = ((-q_{1,2}, q_{1,2}), (q_{2,1}, -q_{2,1}))$, with Poisson arrivals $\underline{\mathbf{\Lambda}} = ((\lambda_1, 0), (0, \lambda_2))$

and Poisson service times $\underline{\mathbf{M}} = ((\mu_1, 0), (0, \mu_2))$. These transitions at a typical queue length j within the repeating region is depicted in Figure 2.

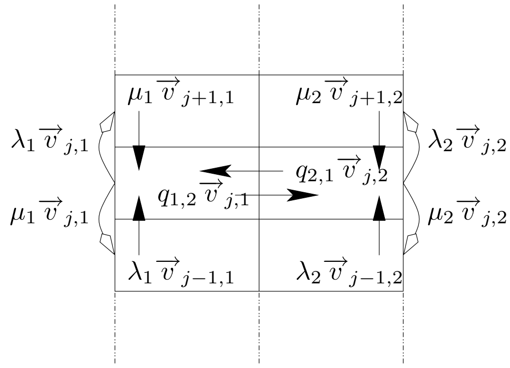

Figure 2: Sample transisions in a modulated queue.

Let $\overrightarrow{\mathbf{v}}_j$ be the (unknown) vector of occupation probabilities of the modulation states at queue length j, then

$$\overrightarrow{\mathbf{v}}_j(\underline{\mathbf{Q}} - \underline{\mathbf{\Lambda}} - \underline{\mathbf{M}}) - \overrightarrow{\mathbf{v}}_{j-1}\underline{\mathbf{\Lambda}} - \overrightarrow{\mathbf{v}}_{j+1}\underline{\mathbf{M}} = \underline{\mathbf{0}} \tag{1}$$

is the second order, homogeneous matrix recurrence that must be satisfied.

More complex queues, especially those with geometrically distributed batches [6], lead to higher order matrix recurrences. We label the individual matrix terms in (1) as $\underline{\mathbf{Q}}_0$, $\underline{\mathbf{Q}}_1$ and $\underline{\mathbf{Q}}_2$, so that here

$$\overrightarrow{\mathbf{v}}_{j-1}\underline{\mathbf{Q}}_0 + \overrightarrow{\mathbf{v}}_j\underline{\mathbf{Q}}_1 + \overrightarrow{\mathbf{v}}_{j+1}\underline{\mathbf{Q}}_2 = \underline{\mathbf{0}}$$

or,

$$\sum_{k=0}^{M} \overrightarrow{\mathbf{v}}_{j+k}\underline{\mathbf{Q}}_k = \underline{\mathbf{0}} \tag{2}$$

2 Spectral expansion solution

The spectral expansion solution to a homogeneous matrix equation of the form (2) is the sum of geometric series provided by the eigenvalues ξ_i of the characteristic matrix

$$\underline{\mathbf{Q}(\xi)} = \sum_{k=0}^{M} \xi^k \underline{\mathbf{Q_k}} \tag{3}$$

found by setting $\det\left|\underline{\mathbf{Q}(\xi)}\right| = 0$. Each of the eigenvalues, ξ_i is then used to determine its corresponding eigenvector $\overrightarrow{\psi}_i$ from $\overrightarrow{\psi}_i\underline{\mathbf{Q}(\xi_\mathbf{i})} = 0$.

Let there be n^* eigenvalue/eigenvector pairs. Each of these defines a basis function component, and by summing the ensemble with each component scaled appropriately, we can satisfy the boundary conditions defined by the balance equations of the levels within the filling and blocking regions as well as the normalization constraint.

At queue lengths j falling under the repeating region balance equation, we have:

$$\overrightarrow{\mathbf{v}}_j = \sum_{i=1}^{n^*} \alpha_i \xi_i^j \overrightarrow{\boldsymbol{\psi}}_i \quad \epsilon^b \leq j \leq \epsilon^t \tag{4}$$

The coefficients α_i are free variables to be constrained by the boundary conditions imposed by the rest of the queue (the filling region, and in the case of a finite queue, the blocking region.) These boundary conditions are imposed by the inclusion of balance equations which include both explicit $\overrightarrow{\mathbf{v}}_j$ vectors outside the repeating region, and vectors defined by the eigensystem (4).

The choice of raising the eigenvalue to the power j, the queue level, is somewhat arbitrary: the scaling due to any constant offset, $j + X$, could be absorbed into the associated α value.

When the queue is unbounded, any eigenvalues of magnitude greater than or equal to 1 (*i.e.* lying on or outside the unit disk in the Argand diagram) must take zero coefficients, as their infinite sum does not converge, and hence cannot be normalized. Eigenvectors with zero eigenvalue can also not contribute to the solution as, in a geometric series, a factor of zero raised to the power zero is equal to zero.

3 Matrix Geometric solution methods

Unlike spectral expansion, which represents the probabilities within the repeating region ($\epsilon^b \leq j \leq \epsilon^t$) using eigenvalues and vectors, matrix geometric methods [3] utilize a matrix $\underline{\mathbf{F}}$ to describe changes in probability between two adjacent levels *i.e.*

$$\overrightarrow{\mathbf{v}}_{j+1} = \overrightarrow{\mathbf{v}}_j \underline{\mathbf{F}} \Leftrightarrow \overrightarrow{\mathbf{v}}_{j+i} = \overrightarrow{\mathbf{v}}_j \underline{\mathbf{F}}^i \qquad \epsilon^b \leq j \leq j+i \leq \epsilon^t$$

or, by anchoring the expression at the bottom of the repeating region,

$$\overrightarrow{\mathbf{v}}_j = \overrightarrow{\mathbf{f}}\,\underline{\mathbf{F}}^{j-\epsilon^b} \qquad \epsilon^b \leq j \leq \epsilon^t$$

Here, $\overrightarrow{\mathbf{f}}$ takes the place of the α's in SE to be constrained by the boundary conditions imposed by the rest of the queue *i.e.* the processing region, and in the case of a finite queue, the blocking region. The power we raise the matrix $\underline{\mathbf{F}}$ to could be offset by any integer as any change here can be absorbed in an opposite change in the value of $\overrightarrow{\mathbf{f}}$. We have chosen to use $\underline{\mathbf{F}}^{j-\epsilon^b}$ because this both minimizes the powers that $\underline{\mathbf{F}}$ has to be raised to in calculations and ensures that $\overrightarrow{\mathbf{v}}_{\epsilon^b} = \overrightarrow{\mathbf{f}}$ so that

$$\overrightarrow{\mathbf{v}}_j = \overrightarrow{\mathbf{v}}_{\epsilon^b}\underline{\mathbf{F}}^{j-\epsilon^b} \qquad \epsilon^b \leq j \leq \epsilon^t \tag{5}$$

By substituting (5) into the Matrix recurrence (2) we see that a solution using this representation can only exist if $\underline{\mathbf{F}}$ satisfies the matrix polynomial

$$\sum_{k=0}^{M} \underline{\mathbf{F}}^k \underline{\mathbf{Q}}_k = \underline{\mathbf{0}} \tag{6}$$

Efficient, modern methods of solving for $\underline{\mathbf{F}}$ are presented in [4] and [2].

3.1 Correspondence in eigenmodes

Let $\hat{\underline{\mathbf{F}}}$ be a non-singular matrix found using one of the above methods that satisfies the matrix polynomial (6). Having full rank N, the matrix $\hat{\underline{\mathbf{F}}}$ has N non-zero eigenvalues ζ_n and corresponding eigenvectors $\overrightarrow{\boldsymbol{\chi}}_n$.

We can uniquely express the vector $\overrightarrow{\mathbf{v}}_{\epsilon^b}$ as a linear combination of these eigenvectors.

$$\overrightarrow{\mathbf{v}}_{\epsilon^b} = \sum_{n=1}^{N} \gamma_n \overrightarrow{\boldsymbol{\chi}}_n$$

then from equation (5) we have

$$\overrightarrow{\mathbf{v}}_j = \sum_{n=1}^{N} \gamma_n \zeta_n^{j-\epsilon^b} \overrightarrow{\boldsymbol{\chi}}_n \qquad \epsilon^b \leq j \leq \epsilon^t$$

We now compare this expression to the one derived for spectral expansion

$$\overrightarrow{\mathbf{v}}_j = \sum_{i=1}^{n^*} \alpha_i \xi_i^j \overrightarrow{\boldsymbol{\psi}}_i \qquad \epsilon^b \leq j \leq \epsilon^t$$

If the Matrix Geometric method gives a correct solution, it must be equal to that derived when using generalized spectral expansion.

$$\sum_{n=1}^{N} \gamma_n \zeta_n^{j-\epsilon^b} \overrightarrow{\boldsymbol{\chi}}_n = \sum_{i=1}^{n^*} \alpha_i \xi_i^j \overrightarrow{\boldsymbol{\psi}}_i \qquad \epsilon^b \leq j \leq \epsilon^t \tag{7}$$

Correspondence in eigenvalues

We now claim that any non-zero eigenvalue ζ_n of $\hat{\underline{\mathbf{F}}}$ (found using matrix geometric methods) is equal to an eigenvalue ξ_m found during generalized spectral expansion – *i.e.* that the sets of non-zero eigenvectors $\zeta = \{\zeta_n\}$ and $\xi = \{\xi_m\}$ satisfy $\zeta \subseteq \xi$.

Proof

Assume that ζ is not a subset of ξ. Therefore there exists a non-zero eigenvalue $\zeta_m \in \zeta$ such that $\zeta_m \notin \xi$. The equality (7) holds for any choice of the free variables γ_n, so that we chose $\gamma_n = \delta_{n,m}$ and try to determine the values of the free variables, α_i that the generalized spectral expansion method uses to represent this particular solution.

$$\begin{aligned}
\sum_{n=1}^{N} \delta_{n,m}\zeta_n^{j-\epsilon^b}\overrightarrow{\boldsymbol{\chi}}_n &= \sum_{i=1}^{n^*} \alpha_i \xi_i^j \overrightarrow{\boldsymbol{\psi}}_i \qquad && \epsilon^b \le j \le \epsilon^t \\
\zeta_m^{j-\epsilon^b}\overrightarrow{\boldsymbol{\chi}}_m &= \sum_{i=1}^{n^*} \alpha_i \xi_i^j \overrightarrow{\boldsymbol{\psi}}_i && \epsilon^b \le j \le \epsilon^t \\
\overrightarrow{\boldsymbol{\chi}}_m &= \sum_{i=1}^{n^*} \alpha_i \frac{\xi_i^j}{\zeta_m^{j-\epsilon^b}} \overrightarrow{\boldsymbol{\psi}}_i && \epsilon^b \le j \le \epsilon^t \\
\overrightarrow{\boldsymbol{\chi}}_m &= \sum_{i=1}^{n^*} \frac{\alpha_i}{\zeta_m^{\epsilon^b}} \left(\frac{\xi_i}{\zeta_m}\right)^j \overrightarrow{\boldsymbol{\psi}}_i && \epsilon^b \le j \le \epsilon^t
\end{aligned}$$

The LHS of this expression is constant, whereas the right hand side is a function of j, except in the following two cases.

Case 1. $\alpha_i = 0$ for all i. From this it follows that $\overrightarrow{\boldsymbol{\chi}}_m = \overrightarrow{\mathbf{0}}$. This contradicts the assumption that ζ_m is non-zero.

Case 2. Either $\frac{\xi_i}{\zeta_m} = 1$ or $\alpha_i = 0$ for all $1 \le i \le n^*$. It follows that for at least one i, we have $\xi_i = \zeta_m$. This contradicts the assumption that $\zeta_m \notin \xi$.

Therefore, the claim is proved by contradiction. ■

Correspondence in eigenvectors

To show that the set of (non-zero) eigen*vectors* satisfy the property that any member $\overrightarrow{\boldsymbol{\chi}}_m$ of $\overrightarrow{\boldsymbol{\chi}}$ is lies within a subspace[1] spanned by members $\overrightarrow{\boldsymbol{\psi}}_n$ of $\overrightarrow{\boldsymbol{\psi}}$, with the same associated eigenvalue. We again use $\gamma_n = \delta_{n,m}$ in the expression (7).

$$\begin{aligned}
\sum_{n=1}^{N} \delta_{n,m}\zeta_n^{j-\epsilon^b}\overrightarrow{\boldsymbol{\chi}}_n &= \sum_{i=1}^{n^*} \alpha_i \xi_i^j \overrightarrow{\boldsymbol{\psi}}_i \qquad && \epsilon^b \le j \le \epsilon^t \\
\overrightarrow{\boldsymbol{\chi}}_m &= \sum_{i=1}^{n^*} \frac{\alpha_i}{\zeta_m^{\epsilon^b}} \left(\frac{\xi_i}{\zeta_m}\right)^j \overrightarrow{\boldsymbol{\psi}}_i && \epsilon^b \le j \le \epsilon^t
\end{aligned}$$

There can not be any j-dependence on the RHS. This forces $\alpha_i = 0$, whenever $\xi_i \neq \zeta_m$. From before, we know that there is at least one i such that $\xi_i = \zeta_m$. Let

[1] Usually, this subspace only has dimension 1

there be a total of k equal eigenvalues, where $N \geq k \geq 1$. After re-labelling all terms for which we did not set the coefficient $\alpha_i = 0$, we have

$$\overrightarrow{\boldsymbol{\chi}}_m = \sum_{i=1}^{k} \frac{\alpha_i}{\zeta_m^{\epsilon^b}} \left(\frac{\xi_i}{\zeta_m} \right)^j \overrightarrow{\boldsymbol{\psi}}_i \qquad \epsilon^b \leq j \leq \epsilon^t$$

$$\overrightarrow{\boldsymbol{\chi}}_m = \sum_{i=1}^{k} \frac{\alpha_i}{\zeta_m^{\epsilon^b}} 1^j \overrightarrow{\boldsymbol{\psi}}_i \qquad \epsilon^b \leq j \leq \epsilon^t$$

It follows that $\overrightarrow{\boldsymbol{\chi}}_m$ is a vector that lies within the subspace spanned by $\left\{ \overrightarrow{\boldsymbol{\psi}}_i \right\}_{1 \leq i \leq k}$, where the spanning vectors are all eigenvectors of the SE method with the same eigenvalue, ζ_m. ■

3.2 Backward series

So far we have identified a subset of size N of the eigenvalues and eigenvectors contained within the complete solution. To gain access to additional eigenmodes, practitioners of the matrix geometric methods use the backward series

$$\sum_{k=0}^{M} \overrightarrow{\mathbf{v}}_{j+k} \underline{\mathbf{Q}}_{M-k} = \underline{\mathbf{0}}$$

This matrix recurrence has the same structure, but with the $\underline{\mathbf{Q}}$'s running the other way, as the original forward series given in (2) used to fing $\underline{\mathbf{F}}$ and we can use the same algorithms to find a numerical approximation $\hat{\underline{\mathbf{B}}}$ to $\underline{\mathbf{B}}$ with $\overrightarrow{\mathbf{v}}_{j+1} = \overrightarrow{\mathbf{v}}_j \underline{\mathbf{B}}$, a solution to

$$\sum_{k=0}^{M} \underline{\mathbf{B}}^k \underline{\mathbf{Q}}_{M-k} = \underline{\mathbf{0}}$$

When using the forward and backward series, the repeating region is represented as follows

$$\overrightarrow{\mathbf{v}}_j = \overrightarrow{\mathbf{v}}_{\epsilon^b} \underline{\mathbf{F}}^{j-\epsilon^b} + \overrightarrow{\mathbf{v}}_{\epsilon^t} \underline{\mathbf{B}}^{\epsilon^t - j}$$

This representation is only well-defined when the queue is finite. For infinite queues the backward series cannot be used.

It can be seen that the inverse of the backward matrix, $\hat{\underline{\mathbf{B}}}^{-1}$, shares exactly N eigenvalues and left eigenvectors with the solution set provided by generalized spectral expansion. Hence, taken together, the matrices $\hat{\underline{\mathbf{F}}}$ and $\hat{\underline{\mathbf{B}}}$ can offer a total of $2N$ distinct eigenvalues and eigenvectors (a total of N for an infinite queue).

3.3 Limitations of the MG method

Our experimentation with spectral expansion has revealed important characteristics of a class of queues which are missed when using matrix geometric methods in their current form. The problem lies in the spectrum of the solution. Matrices as used in MG have a fixed number of eigenvalues, equal to the number N of modulation states. Currently known algorithms provide either one or two of these matrices, depending on whether the queue is infinite of finite.

Infinite Queues

Infinite queues do not require the full set of eigenvalues, in particular only a number n^* of those which have magnitude strictly less than one. The value of n^* has been found to be directly related to the number of distinct arrival processes, n^{arr}, over all modulation states, N, and for an infinite queue that is not saturated, we have – in the absence of degenerate[2] arrival streams – $n^* = n^{arr}N$.

The MG method only provides one matrix, $\underline{\mathbf{F}}$, to represent the solution, and for $n^{arr} > 1$ it is therefore not able to accurately solve the problem.

Finite Queues

When queues have a finite buffer, contributions from eigenvalues outside the unit disc can also be normalized and need to be included in the solution process. As in the infinite case, the number of these depends on n^{arr}, the number of distinct arrival processes. In addition to these, n^{serv} and n^{kill}, the number of distinct service and killing processes need to be considered and we have $n^* = (n^{arr} + n^{serv} + n^{kill})N$ for the finite case. By way of forward and backward series the MG method can represent a total of $2N$ eigenvalues, so that a solution for a queue where $n^{arr} + n^{serv} + n^{kill} > 2$ is not possible in general.

In addition to this limitation, there is the practical obstacle that current matrix geometric methods are generally applied to third order matrix recurrence relations, leading to the solution of a quadratic matrix polynomial. Matrix analytic techniques seek the solution to a higher order problem, but still represent the solution using a single matrix, or pair of matrices (one forward and backward). It should be noted that the folding methods we describe in section 3.4 transform a system of higher order to a quadratic function, soluble using, for example, the Latouche-Ramaswami method [4]. The folding method is used to convert the problem into an equivalent one that has a complete matrix-geometric solution, but with some numerical stability issues. These numerical issues are not caused by the folding process: they are inherent in raising matrices to large powers.

[2] These are either unbatched arrival processes (*i.e.* batch parameter 0), or an arrival process whose batch parameter is equal to that of another arrival process

3.4 Folded Matrix geometric method

We will now show a transformation process that allows the MG solution process of a particular Markovian geometrically batched queue originally investigated in [7]. The queue features 3 distinct, geometrically batched streams for each of its modulation states; one for each of positive and negative customers and processing completions. In [7] the localized balance equations had been derived by hand. Using our automated approach [6], we have derived the following localized matrix recurrence of order 4 that holds within the repeating region

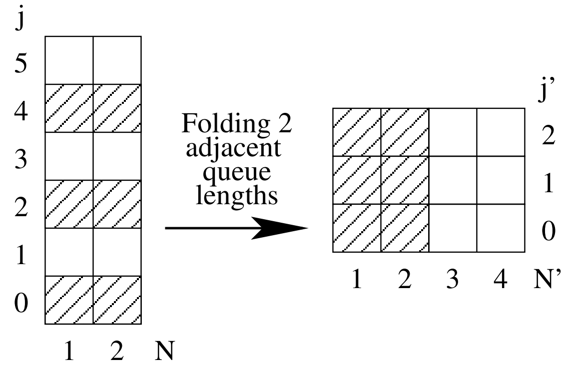

Figure 3: State correspondence for an example queue during the folding process.

Figure 3 shows how the states correspond between the original and the folded queue, for a queue with maximal queue length $L = 5$ and $N = 2$ modulation states. States at even queue lengths have been shaded to illustrate the folding effect: The folded level j is composed of the original levels $2j$ and $2j + 1$. As a result the queue length is halved, but the number of modulation states is doubled, with the folded modulation states 1 and 2 corresponding to even queue lengths and states 3 and 4 to odd queue lengths in the original queue.

When the maximum queue is not a multiple of 2, we have to add dummy states as pictured in Figure 4. These dummy states have no incoming or outgoing transitions, and will be given state occupation probability zero by the solution process.

4 Numerical Examples

We will first look at the solution of a finite queue of the basic form given in the previous section, and an infinite queue with an additional independent arrival process (corresponding to a queue in a network with arrivals from two other queues, for example). These particular examples are chosen to allow us to show concrete examples of circumstances in which it is not possible to solve using the original MG method, while at the same time demonstrating that eigenvalues and eigenvectors found are equal to those from the spectral expansion method. We then use

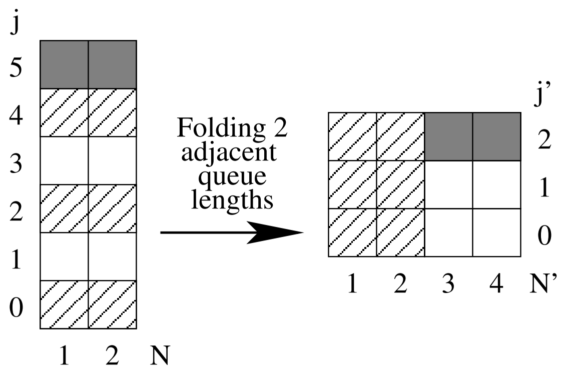

Figure 4: Folding using dummy states.

the folded queue expression derived in the previous section to provide MG with a spectrum of sufficient size to accommodate the solution.

For simplicity, we use queues with two modulation states. The total number of eigenvalues in the solution to such queues is found – in practice, and easily verified by examination of the determinant – to be equal to the number of modulation states, N, multiplied by the number of independent geometrically batched arrival and departure processes. The finite system we examine therefore has 6 eigenvalues, all of which are necessary to the solution. The infinite system has 8 eigenvalues, 4 of which lie within the unit disc.

4.1 Finite Queue Example

We consider a particular instance of a recurrence relation (2) where $M = 3$, that represents a two-state ($N = 2$) modulated queue with geometrically batched arrival, service and negative customer streams. The details on how to derive this (and other) recurrence relations is covered in [6]. Here, we are only considering the spectral properties of this recurrence relation.

$$\underline{\mathbf{Q}}_0 = \begin{pmatrix} \frac{96614646928067997729}{238418579101562500} & -\frac{2893401}{2000000000000} \\ -\frac{47515400128555803167}{59604644775390625} & \frac{124416243}{4000000000000} \end{pmatrix}$$

$$\underline{\mathbf{Q}}_1 = \begin{pmatrix} -\frac{258008622269807011119681}{238418579101562500} & \frac{20253807}{2500000000000} \\ \frac{19164544718518406106 9}{59604644775390625} & -\frac{743604057}{8000000000000} \end{pmatrix}$$

$$\underline{\mathbf{Q}}_2 = \begin{pmatrix} \frac{967888700618727105117 9}{1192092895507812500 0} & -\frac{176497461}{40000000000000} \\ -\frac{29142778745515592757 6}{14901161193847656 25} & \frac{2931015213}{40000000000000} \end{pmatrix}$$

$$\underline{\mathbf{Q}}_3 = \begin{pmatrix} -\frac{20748391389470340495 9}{1192092895507812500 0} & \frac{2893401}{80000000000000} \\ \frac{9503080025711606334}{298023223876953125} & -\frac{66548223}{40000000000000} \end{pmatrix}$$

4.2 Spectral Expansion approach

As $\underline{\mathbf{Q}}_3$ is non-singular, we can form the quantities $-\underline{\mathbf{Q}}_i\underline{\mathbf{Q}}_3^{-1}$ and linearlize the eigenvalue problem by populating the folowing matrix $\underline{\mathbf{A}}$, and finding *its* eigenvalues instead.

$$\underline{\mathbf{A}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{6967}{3001} & -\frac{219}{6002} \\ 0 & 0 & 0 & 0 & -\frac{348}{3001} & \frac{5603}{3001} \\ 1 & 0 & 0 & 0 & -\frac{184647}{30010} & \frac{21199}{60020} \\ 0 & 1 & 0 & 0 & \frac{2482}{3001} & -\frac{33425}{6002} \\ 0 & 0 & 1 & 0 & \frac{139089}{30010} & -\frac{9871}{60020} \\ 0 & 0 & 0 & 1 & \frac{4772}{15005} & \frac{131967}{30010} \end{pmatrix}$$

The 6 eigenvalues of this matrix are evaluated numerically and are given below to 8 digits precision.

$$\lambda^{SE} = \{0.49281008, 0.84141200, 1, 1.63072196, 2.40960738, 2.65763786\}$$

There are no repeated eigenvalues so that each eigenvalue corresponds to a distinct eigenvector, the set of which is linearly independent. Due to the particular structure of the linearization matrix $\underline{\mathbf{A}}$, its eigenvectors are of the form

$$\overrightarrow{\boldsymbol{\psi}}_i^{SE} = (\overrightarrow{\zeta}_i^{SE}, \lambda_i \overrightarrow{\zeta}_i^{SE}, \lambda_i^2 \overrightarrow{\zeta}_i^{SE})$$

where

$$\overrightarrow{\boldsymbol{\zeta}}^{SE} = \begin{pmatrix} -0.58750 & -0.90188 & -0.89442 & -0.81371 & -0.07298 & -0.92778 \\ 0.809222 & -0.43198 & -0.44721 & 0.58125 & 0.99733 & 0.37312 \end{pmatrix}$$

4.3 Matrix Geometric approach

Using the expressions gained for $\underline{\mathbf{Q}}_i$, we apply the folding methodology as introduced in section 3.4 and evaluate the numerical solutions $\hat{\underline{\mathbf{F}}}$ and $\hat{\underline{\mathbf{B}}}$ to the forward and backward series formed by the matrices $\underline{\mathbf{A}}_0$, $\underline{\mathbf{A}}_1$ and $\underline{\mathbf{A}}_2$. We find

$$\hat{\underline{\mathbf{F}}} = \begin{pmatrix} -0.88033194 & -0.01294500 & -1.75444663 & 0.05136837 \\ 0.02704049 & -0.76770941 & 0.25351728 & -1.51335404 \\ 1.99675926 & -0.08729268 & 3.12940253 & -0.35905559 \\ -0.25989511 & 1.96818477 & -1.00342909 & 3.12872885 \end{pmatrix}$$

$$\hat{\underline{\mathbf{B}}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.37509951 & 0.16049028 & 0.14065278 & 0.12699264 \\ -0.002920315 & 0.41617984 & -0.00231079 & 0.17315788 \end{pmatrix}$$

The eigenvalues of $\hat{\underline{\mathbf{F}}}$ are $\lambda^{\hat{F}} = \{0.24286178, 0.70797415, 1, 2.65925410\}$ and those of $\hat{\underline{\mathbf{B}}}$ are $\lambda^{\hat{B}} = \{0.17222946, 0.14158212, 0, 0\}$. To relate these to those found by the spectral expansion method, we note that our folding methodology grouped pairs of levels together. A multiplication by $\underline{\hat{\mathbf{F}}}$ now relates the probability vector for level j, $\overrightarrow{\mathbf{u}}_j = (\overrightarrow{\mathbf{v}}_{2j}, \overrightarrow{\mathbf{v}}_{2j+1})$, to the one at level $j+1$, $\overrightarrow{\mathbf{u}}_{j+1} = (\overrightarrow{\mathbf{v}}_{2j+2}, \overrightarrow{\mathbf{v}}_{2j+3})$, so that one multiplication within the folded representation gives a jump of 2 levels instead of just one. We therefore need to raise each eigenvalue to the power[3] $1/2$ to compare them with those found when using the system without folding. In addition, the matrix $\underline{\hat{\mathbf{B}}}$ relates the vector $\overrightarrow{\mathbf{u}}_j$ to $\overrightarrow{\mathbf{u}}_{j-1}$, a relation that runs in the opposite direction to that for $\underline{\hat{\mathbf{F}}}$ and the eigenmodes used in spectral expansion. If $\overrightarrow{\mathbf{u}}_j\underline{\hat{\mathbf{B}}} = \overrightarrow{\mathbf{u}}_{j-1}$, then – if $\underline{\hat{\mathbf{B}}}$ is non-singular – $\overrightarrow{\mathbf{u}}_{j-1}\underline{\hat{\mathbf{B}}}^{-1} = \overrightarrow{\mathbf{u}}_j$ is an equivalent relation that produces probabilities at *increasing* queue lengths. Thus, we need to calculate the eigenvalues and eigenvectors of $\underline{\hat{\mathbf{B}}}^{-1}$. We cannot form the inverse as two of $\underline{\hat{\mathbf{B}}}$'s eigenvalues are zero and it is therefore singular. We do, however know that the inverse of the linear transformation that the matrix $\underline{\hat{\mathbf{B}}}$ performs on the subspace that is spanned by its non-zero eigenmodes, is given by the reciprocal of the (non-zero) eigenvalues of $\underline{\hat{\mathbf{B}}}$ as well as the eigenvectors of $\underline{\hat{\mathbf{B}}}$:

$$\begin{aligned}
\sqrt{\lambda^{\hat{F}}} &= \{0.49281008, 0.84141200, 1, 1.63072196\} \\
\frac{1}{\sqrt{\lambda^{\hat{B}}}} &= \{2.40960738, 2.65763786\}
\end{aligned}$$

Note that the set $\lambda^{\hat{F}}$ contains eigenvalues of magnitude larger than one, so that the assertion in [5] that eigenvalues are within the unit disc does not hold in this case. The eigenvectors for the non-zero eigenvalues of both matrices are of the form $\overrightarrow{\psi}_i^{\hat{F}} = (\overrightarrow{\zeta}_i^{\hat{F}}, \sqrt{\lambda_i^{\hat{F}}}\overrightarrow{\zeta}_i^{\hat{F}})$ and $\overrightarrow{\psi}_i^{\hat{B}} = (\overrightarrow{\zeta}_i^{\hat{B}}, \sqrt{\lambda_i^{\hat{B}}}\overrightarrow{\zeta}_i^{\hat{B}})$. In addition, the matrix $\underline{\hat{\mathbf{B}}}$ has the two eigenvectors $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ associated with its two zero eigenvalues. Using a projection matrix on the $\overrightarrow{\psi}$'s, we find that

$$\overrightarrow{\zeta}^{\hat{F}} = \begin{pmatrix} 0.52698282 & -0.69009480 & -0.63245553 & 0.42538088 \\ -0.72586762 & -0.33054215 & -0.31622777 & -0.30385991 \end{pmatrix}$$

and

$$\overrightarrow{\zeta}^{\hat{B}} = \begin{pmatrix} 0.06741043 & -0.86834332 \\ -0.92115765 & 0.34922364 \end{pmatrix}$$

all of which are parallel to the eigenvectors corresponding to the same eigenvalue in $\overrightarrow{\zeta}^{SE}$. Comparison with the spectral expansion approach shows that $\underline{\hat{\mathbf{F}}}$ contains the spectral information for the smallest 4 eigenvalues, whereas $\underline{\hat{\mathbf{B}}}$ holds the information for the remaining 2 eigenmodes, as well as the two additional zero eigenvalues. A full solution process now needs to consider the boundary conditions of the queue, *i.e.* the balance equations that pertain to the processing and blocking regions. We do not provide these here, but instead directly give the the probability distributions found by the two methods when we set the queue length to $L = 15$. They can be seen in Figures 5 and 6.

[3] For k folded levels, use power $1/k$

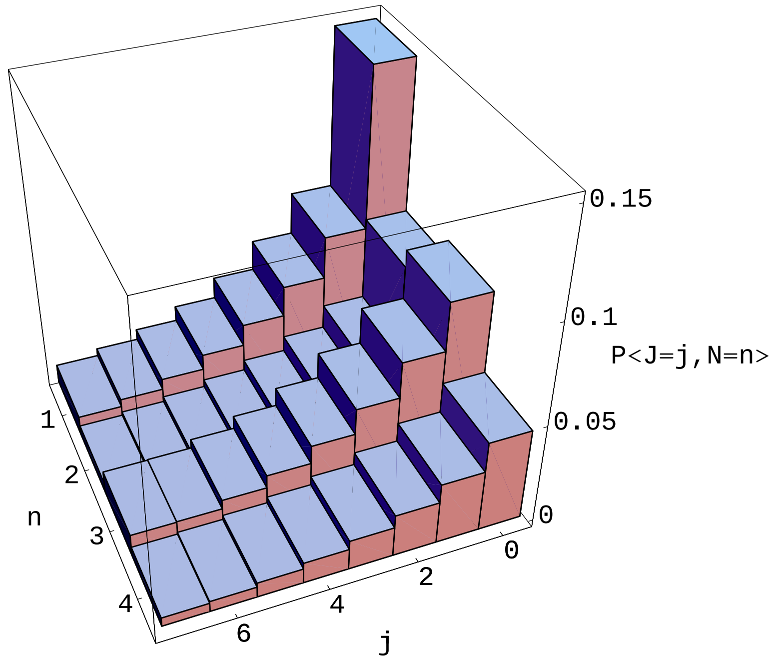

Figure 5: Finite queue example – steady state probability distribution of the *folded* queue. Modulation states 1 and 2 correspond to even levels in the original queue, states 3 and 4 to the odd levels.

References

[1] I. Mitrani and R. Chakka. Spectral expansion solution for a class of Markov models: Application and comparison with the matrix-geometric method, *Performance Evaluation* **23** pp. 241-260, 1995.

[2] Naoumov, Krieger and Wagner. *Analysis of a multi-server delay-loss system with a general Markovian arrival process.* Matrix-analytic methods in stochastic models, 1997.

[3] Marcel F. Neuts *Matrix-Geometric Solutions in Stochastic Models* John Hopkins University Press, Baltimore, 1981.

[4] G. Latouche and V. Ramaswami. *A logarithmic reduction algorithm for quasi-birth-death processes.* Journal of Applied Probability, 30: pp 650-674, 1993.

[5] D.A. Bini, G. Latouche, B. Meini, *Solving matrix polynomial equation arising in queueing problems*, Linear Algebra and its Applications 340 (2002) 225-244.

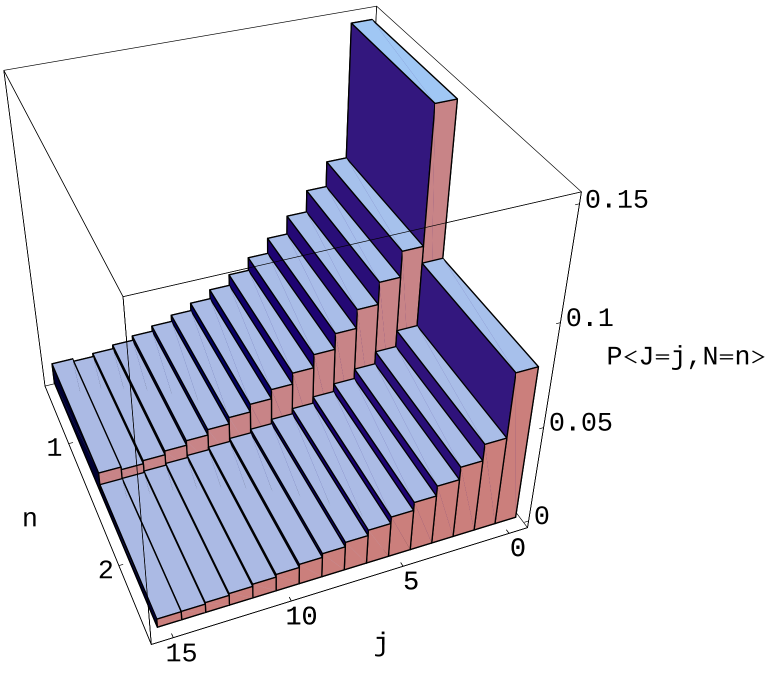

Figure 6: Finite queue example – steady state probability distribution found using spectral expansion or unfolding the matrix geometric solution.

[6] Thornley, D.J. and Zatschler, Harf and Harrison, P.G. *An automated formulation of queues with multiple geometric batch processes* , In Proceedings, FirstInternational Working Conference On Performance Modelling And Evaluation Of Heterogeneous Networks (HET-NETs 2003)

[7] Ram Chakka, Peter G. Harrison. *A Markov modulated multi-server queue with negative customers - The MM CPP/GE/c/L G-queue.* Acta Informatica 37(11-12): 881-919 (2001)