Modal Labelled Deductive Systems

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Abstract

We present a formalization of propositional modal logic in the framework of Labelled Deductive Systems (LDS) in which modal theory is presented as a "configuration" of several "local actual worlds". We define a natural deduction style proof system for a propositional modal labelled deductive system (MLDS). We describe a model–theoretical semantics (based on first–order logic) and we show that the natural deduction proof system is sound and complete with respect to this semantics. We also show that the semantics given here is equivalent to Kripke semantics for a normal modal logic whenever the initial configuration is a single point. Finally we discuss how this logic can be extended to the predicate case, we sketch some natural deduction rules for quantifiers and we discuss how such rules solve certain problems associated with the nesting of quantifiers within the scope of modal operators.

Contents

Introduction

Modal logic is considered to be a fundamental non–classical logic, from both a theoretical and an applicative point of view. Several extensions of modal logic have been developed as new logical systems, which have turned out to be knowledge representation formalisms suitable for a wide range of applications in Artificial Intelligence. As examples we can cite temporal logic, action logic [Ginsberg-88] and belief logic [Konolege-86].

However, most of the existing proof systems for predicate modal logics still have some limitations. This is true of both of the two traditional approaches of *implicit* and *explicit* formalizations. Proof systems based on a pure modal language (i.e. implicit formalizations) impose restrictions on the semantics in order to ensure soundness and completeness (see [Fitting-83]). This is because they are unable to keep track of the nesting of quantifiers in the scope of modal operators. On the other hand, the proof systems that belong to the second approach (explicit formalizations), even though they eliminate the above restrictions by translating modal logics into first order logic, require sofisticated unification algorithms with exponential complexity (see [Ohlbach-88], [Ohlbach-91]).

In this work, we define a new formalism for modal logic, following the idea presented in [Gabbay-92a], of compromising between the two implicit and explicit formalizations, with the hope that reasoning with a mixed representation will give us the benefits of both approaches without their restrictions. We will base this new formalization on the idea of a Labelled Deductive System (LDS) defined in [Gabbay-94]. The idea is to define a logical framework in which possible worlds and relationships bewteen possible worlds can be expressed declaratively, while retaining the conventional syntax of modal logic. This formalism eventually will allow us to develop a proof system in natural deduction style for predicate modal logic, which can be proved to be sound and complete with respect to a *varying domain* possible worlds semantics, and which will be able to keep track of the nesting of quantifiers in the scope of modal operators.

Therefore, it will be possible to distinguish proof theoretically between formulae of the form $\Diamond\exists x A(x)$ and $\exists x\Diamond A(x)$. This will also facilitate Skolemization procedures in the construction of automated theorem provers for modal logic, without requiring full translations into classical logic, thus eliminating the problem of complex unification algorithms.

Moreover the explicit declaration of relationships between possible worlds will allow us to generalize the notion of a modal theory from a single 'actual world' to a set of 'local' actual worlds with relations between them. In this way, we hope that this new formalism will be closer to the needs of many applications. The formalization of extensions of modal logic in this logical framework will be more flexible and simpler, in particular for applications based on notions of *status* and *status–transition*.

The overall research is in two main parts. The first part is concerned with formalization and development of a propositional modal labelled deductive system (MLDS, which specifies how to deal with modal operators. In the second part we intend to extend the propositional modal labelled deductive system to the predicate case. Rules for quantifiers will be introduced, together with particular rules called *Visa rules* that express the relationships between different possible worlds and elements of their associated domains. This report completely describes the first part of the work. Research is now focused on the second part. A work plan can be found in the last section.

In Section 1, we give a basic definition of a *Modal Propositional Labelled Deductive System* together with the definition of a MLDS theory as 'a structured set of local modal theories', called a *Configuration*. In Section 2, we define a natural deduction style proof system for MLDS, where inference rules are applied on configurations and the derivability relation is defined between theories (and not in the traditional way between theories and formulae). In Section 3, we first introduce some additional definitions and then we present a semantics, define the notion of semantic entailment and show that the proof system developed in Section 2 is sound with respect to this semantics. Note that this semantics is not the only way of defining a MLDS model theory. Nevertherless, it turns out to be quite straight-forward as illustrated in Section 3. In Section 4 we show that the proof system developed in Section 2 is also complete with respect to the semantics proposed in Section 3 and in Section 5 we prove that the model–theoretic semantics given in Section 3 is equivalent to a Kripke semantics for normal modal logic whenever the initial configuration is a single point.

We consider here a wide family of MLDS, corresponding to each of the modal logics K, T, $K4$, KB, $S4$, $S5$, D, $D4$ and DB.

Our approach is comparable with existing work using both implicit and explicit formalization. In Section 6 we compare a MLDS with four different existing proof systems for propositional modal logic.

Finally we also expect our work to have advantages in applications. We therefore, intend to undertake applicative case studies for MLDS which exploit its basic feature of dealing with "configurations".

Notation & Terminology

We introduce specific notation as and when necessary throughout the report. However, the reader might like to bear the following in mind. Constant and predicate symbols will often begin with an upper–case letter, whereas variables and function symbols will usually begin with a lower–case letter. We will sometimes use Greek–letter meta–variables to refer in general to terms and espressions in a logical language. Larger entities such as structures, sets, theories and languages will often be symbolised in caligraphic font, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$. The MLDS corresponding to the modal logic K will be referred to as the K–MLDS, the MLDS corresponding to the modal logic T will be referred to as T–MLDS, etc. We will use the symbol S to generally refer to an arbitrary MLDS within the family considered here.

1 Basic definitions concerning MLDS

In this section we present some basic definitions concerning MLDS to which we will refer in later sections.

Definition 1.1 (Labelling language $\mathcal{L}_L$)
A labelling language $\mathcal{L}_L$ is a first-order language composed of:

- $W_0, W_1, \ldots, W_n, \ldots$ a countable set of constant symbols
- $x, y, z, x_1, y_1, z_1, \ldots$ a countable set of variables
- $succ$ a unary function symbol
- R binary relation symbol
- $\neg, \wedge, \vee, \rightarrow, \equiv$ logical connectives
- $\forall$ universal quantifier

$\square$

Note that in the above definition the constants of a labelling language can be regarded as labels of possible worlds, and the relation symbol R as an accessibility relation between possible worlds. The unary function symbol $succ$ is used only in the MLDSs which require the seriality properties. Moreover we will see below that, given a particular modal language $\mathcal{L}_M$, the labelling language can be appropriately extended by including function symbols which will generate new accessible worlds from the initial set of constant symbols. The system presented here is most clearly analogous to traditional modal logics in the case when $\mathcal{L}_L$ has a single constant symbol, say W_0, which can then be regarded as representing the 'actual world'.

Definition 1.2 (Modal propositional language $\mathcal{L}_M$)
A modal language $\mathcal{L}_M$ is a modal propositional language composed of:

- $p, q, r, p_1, q_1, r_1, \ldots$ propositional letters
- $\neg, \wedge, \vee \rightarrow$ logical connectives[1]
- $\Box, \Diamond$ modal operators

$\Box$

Definition 1.3 (Semi–extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$)
Let $\mathcal{L}_L$ be a labelling language and $\mathcal{L}_M$ a modal propositional language. Let $\alpha_1,\ldots,\alpha_n,\ldots$ be the ordered set of all wffs of $\mathcal{L}_M$[2]. The first–order language $Func(\mathcal{L}_L, \mathcal{L}_M)$ is defined as the language $\mathcal{L}_L$ extended with:

- $f_{\alpha_1}, f_{\alpha_2}, f_{\alpha_3}, \ldots$ a unary function symbol corresponding to each wff α_i of $\mathcal{L}_M$
- $box_{\alpha_1}, box_{\alpha_2}, box_{\alpha_3}, \ldots$ a unary function symbol corresponding to each wff α_i of $\mathcal{L}_M$

$\Box$

The unary function symbols introduced in the above definition are used both in the semantics (Section 3) and to help define a natural deduction style proof system in Section 2. The function symbols f_{α_i} are used in a $\Diamond$–Elimination rule to generate a new accessible world uniquely associated with each $\Diamond$–formula and current world. In contrast, the function symbols box_{α_i} are used in a $\Box$–Introduction inference rule.

Definition 1.4 (Labelling algebra $\mathcal{A}$)
A labelling algebra $\mathcal{A}$ is a subset of the following axiom set, written in a semi–extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$:

$$\forall x(R(x,x)) \qquad \text{(T)}$$
$$\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z)) \qquad \text{(4)}$$
$$\forall x,y(R(x,y) \rightarrow R(y,x)) \qquad \text{(B)}$$
$$\forall x(R(x, succ(x))) \qquad \text{(D)}$$

$\Box$

The following table contains labelling algebrae corresponding to some of the normal modal logics:

[1] Given wffs α and β, we might sometimes, write $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$ as $\alpha \equiv \beta$

[2] We assume a canonical ordering. That such an ordering exists follows from the normal inductive definition of a wff in a modal language $\mathcal{L}_M$

T $\forall x(R(x,x))$

K4 $\forall x,y,z((R(x,y) \wedge R(y,z)) \to R(x,z))$

S4 $\forall x(R(x,x))$
$\forall x,y,z((R(x,y) \wedge R(y,z)) \to R(x,z))$

S5 $\forall x(R(x,x))$
$\forall x,y,z((R(x,y) \wedge R(y,z)) \to R(x,z))$
$\forall x,y(R(x,y) \to R(y,x))$

Definition 1.5 (The modal labelled deductive language (MLDL))
Given a labelling language $\mathcal{L}_L$ and a modal propositional language $\mathcal{L}_M$, a modal labelled deductive language (MLDL) is the ordered pair:

$$\langle \mathcal{L}_L, \mathcal{L}_M \rangle$$

□

An MLDL is composed of two languages in order to identify possible worlds with their inter–connecting accessibility relations and modal formulae syntactically. The basic unit of information in a MLDS is not a modal formula but a pair separated by colon — '*label* : *modal formula*'. The *label* component is a ground term of the semi–extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$ and the *modal formula* is a wff of the modal language $\mathcal{L}_M$. We call this basic unit of information a **declarative unit** and we define it as follows.

Definition 1.6 (Declarative unit)
Given the language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a declarative unit is a pair $\lambda : \alpha$ where λ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and α is a wff of $\mathcal{L}_M$.

□

Examples of declarative units are $W_0 : p$ and $f_{\Diamond q}(W_3) : p$, which are called *atomic declarative units* since the right side is an atomic modal formula (i.e. a single proposition), and others such as $W_1 : \Diamond p$ and $f_q(W_1) : p \to r$.

Definition 1.7 (R–literal)
Given the language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, an R–literal is a literal of the form $R(\lambda_1, \lambda_2)$ or $\neg R(\lambda_1, \lambda_2)$, where λ_1 and λ_2 are ground terms of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$

□

Definition 1.8 (Converse of an R-literal)
Let Δ be an R-literal. The converse of Δ, written $\overline{\Delta}$, is defined as

- $\neg R(\lambda_1, \lambda_2)$ if $\Delta = R(\lambda_1, \lambda_2)$
- $R(\lambda_1, \lambda_2)$ if $\Delta = \neg R(\lambda_1, \lambda_2)$

$\square$

Informally, a 'theory' written in an MLDL will consist of a set of declarative units together with a set of R-literals showing the accessibility relation between labels. We can sometimes represent this information graphically. For example, if we take the set of declarative units to be $\{W_0 : \Box(p \to q), W_0 : \Box r, W_1 : \Diamond p, f_p(W_1) : p, W_2 : q\}$ and the set of literals to be $\{R(W_0, W_1), R(W_0, W_2), R(W_1, f_p(W_1))\}$, this can be represented graphically as in Figure 1[3]. We call such a theory a *configuration*, and the associated structure of labels a *diagram*. In practise a configuration and its associated diagram will usually initially contain only constant symbols of $Func(\mathcal{L}_L, \mathcal{L}_M)$ as labels — configurations containing general ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ are generated by the application of inference rules. All of this is expressed formally in the definitions which follow.

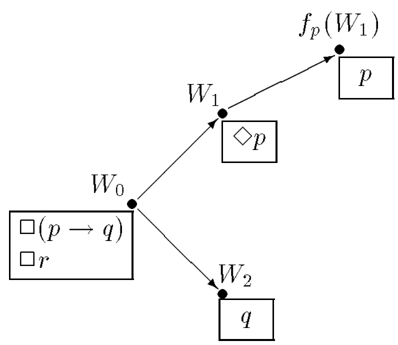

Figure 1 – A Configuration

Definition 1.9 (Diagram)
Given the language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a diagram $\mathcal{D}$ is a set of R-literals whose arguments, λ_i, λ_j, are ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$.

$\square$

An example of a diagram is the set

$$\{R(W_0, W_1), R(W_0, W_2), R(W_1, f_p(W_1))\}$$

which is represented graphically in Figure 2.

[3] It is less easy to represent R-literals of the form $\neg R(\lambda_i, \lambda_j)$ graphically

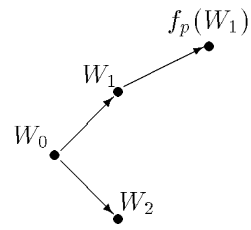

Figure 2 – A diagram

Definition 1.10 (Configuration)
Given a MLDL, a configuration is a tuple

$$\langle \mathcal{D}, \mathcal{F} \rangle$$

where:
- $\mathcal{D}$ is a diagram
- $\mathcal{F}$ is a function from the set of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ to the set $PW(wff(\mathcal{L}_M))$ of sets of wffs of $\mathcal{L}_M$

□

In the configuration represented in Figure 1, for instance, we have

$$\begin{array}{lcll}
\mathcal{D} & = & \{R(W_0, W_1), R(W_0, W_2), R(W_1, f_p(W_1))\} & \\
\mathcal{F}(\lambda) & = & \{\Box(p \to q), \Box r\} & \text{if } \lambda = W_0 \\
 & & \{\Diamond p\} & \lambda = W_1 \\
 & & \{p\} & \text{if } \lambda = f_p(W_1) \\
 & & \{q\} & \text{if } \lambda = W_2 \\
 & & \{\} & \text{otherwise}
\end{array}$$

Notation 1.1
Given a configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$, we say the R-literal Δ *is a member of* C, and write $\Delta \in \mathcal{C}$, if $\Delta \in \mathcal{D}$. We say that the declarative unit $\lambda : \alpha$ *is a member of* C, and write $\lambda : \alpha \in \mathcal{C}$, if $\alpha \in \mathcal{F}(\lambda)$.

◁

Finally, we are in a position to define a propositional modal labelled deductive system (MLDS).

Definition 1.11 (Modal labelled deductive system)
Given a MLDL=$\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a modal labelled deductive system (MLDS) S is a tuple:

$$S = \langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R} \rangle$$

where: • $\mathcal{A}$ is a labelling algebra written in $Func(\mathcal{L}_L, \mathcal{L}_M)$
• $\mathcal{R}$ is a set of inference rules which 'generate' one configuration from another

□

In the definition of a MLDS S above we have included the notion of a set of *inference rules*. Each inference rule is a relation defined over the set of configurations of S. The exact form which such a relation can take is the topic of the next section.

Discussion

This new formalism is clearly a compromise between implicit and explicit formalizations of modal logic. Like the implicit approach, it allows us to use modal formulae of the language $\mathcal{L}_M$. Moreover we can refer explicitly to possible worlds and relations between possible worlds, via the separate language $\mathcal{L}_L$. The idea of 'labelled modal formulae' (declarative units), is similar to the notion of *prefixed formulae* introduced in [Fitting-83]. However, as will be discussed in Section 5, our formalism allows us to deal with labels in a logical way. This will simplify the definition of the natural deduction rules for modal operators, as well as the proofs of soundness and completeness. Furthermore, the introduction of a separate binary first–order theory $\mathcal{A}$[4] makes the formalization modular. In fact, properties of the accessibility relation are all expressed in the theory $\mathcal{A}$ as first order axioms. Therefore, as we will see in Sections 3 and 4, the representation of different normal modal logics will only affect the labelling algebra $\mathcal{A}$[5], without causing any change in the proof system. Finally, the introduction of labels and R–literals has also allowed us to generalize the notion of a single current world theory to a 'structure of local actual world's theories', i.e. a configuration. In fact in a configuration, a set of declarative units that has the same label represents a local modal theory associated with that particular label (or possible world). The set of R–literals defines the relation between labels. In the next section we show how these local modal theories interact with each other.

[4]We use the word binary because the only predicate used in $\mathcal{A}$ is the binary predicate R.

[5]In the rest of the report we will refer to $\mathcal{A}$ as a labelling algebra or a labelling theory indistinctivly

2 A natural deduction system for MLDS

A difference between traditional modal systems and modal labelled deductive systems is that in the latter the inference rules are applied not to wffs but to *configurations*. In the inference system that we are going to define, all the inference rules 'generate' a new configuration from a given configuration. Thus an inference rule can be defined generally as follows.

Definition 2.1 (Inference Rule)
An inference rule $\mathcal{I}$ is a set of pairs of configurations, where each such pair is written as $\mathcal{C}/\mathcal{C}'$. If $\mathcal{C}/\mathcal{C}' \in \mathcal{I}$ then we say $\mathcal{C}$ is an *antecedent configuration* of $\mathcal{I}$, and $\mathcal{C}'$ is an *inferred (or consequence) configuration of* $\mathcal{I}$ *with respect to* $\mathcal{C}$. We also say $\mathcal{I}$ *generates* $\mathcal{C}'$ *from* $\mathcal{C}$, and $\mathcal{I}$ *infers* $\mathcal{C}'$ *from* $\mathcal{C}$.

□

In the rest of this section we assume a MLDL $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$. We will define the inference rules of an MLDS $S = \langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R} \rangle$ in a natural deduction style. *Introduction* and *Elimination* rules will be defined for each classical connective and modal operator of the language $\mathcal{L}_M$. For the remainder of this section, we assume that all terms referred to are ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and all wffs referred to are wffs of $\mathcal{L}_M$.

Definition 2.2 (Proof)
Let $S = \langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R} \rangle$ be a MLDS. A proof in S is a pair $\langle \mathcal{P}, m \rangle$, where $\mathcal{P}$ is a sequence of configurations $\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}$, with $n > 0$, and m is a mapping from the set $\{0, \ldots, n-1\}$ to $\mathcal{R}$ such that for each i, $0 \leq i < n$, $\mathcal{C}_i/\mathcal{C}_{i+1} \in m(i)$.

□

Definition 2.3 (Derivability)
Let $S = \langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R} \rangle$ be a MLDS and $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of S. $\mathcal{C}'$ is derivable from $\mathcal{C}$, written $\mathcal{C} \vdash_S \mathcal{C}'$, if there exists a proof $\langle \{\mathcal{C}, \ldots, \mathcal{C}'\}, m \rangle$.

□

Notation 2.1
Given a configuration $\mathcal{C}$, a declarative unit $\lambda : \alpha$ and an R-literal Δ we write

$$\mathcal{C} \vdash_S \lambda : \alpha$$

if there exists a configuration $\mathcal{C}'$ such that $\mathcal{C} \vdash_S \mathcal{C}'$ and $\lambda : \alpha \in \mathcal{C}'$. Similarly we write

$$\mathcal{C} \vdash_S \Delta$$

if there exists a configuration $\mathcal{C}'$ such that $\mathcal{C} \vdash_S \mathcal{C}'$ and $\Delta \in \mathcal{C}'$. We write

$$\mathcal{C} \vdash_S \lambda : \bot$$

if there exists a term λ and a wff α such that $\mathcal{C} \vdash_S \lambda : \alpha \wedge \neg\alpha$.

◁

Notation 2.2
Given the configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$, the declarative unit $\lambda : \alpha$ and the R-literal Δ, then

1. $\mathcal{C} + [\lambda : \alpha]$ is the configuration $\langle \mathcal{D}, \mathcal{F}' \rangle$, such that
   - $\mathcal{F}'(\lambda) = \mathcal{F}(\lambda) \cup \{\alpha\}$
   - $\mathcal{F}'(\lambda') = \mathcal{F}(\lambda')$ for each ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$
2. $\mathcal{C} + [\Delta]$ is the configuration $\langle \mathcal{D}', \mathcal{F} \rangle$, such that
   - $\mathcal{D}' = \mathcal{D} \cup \{\Delta\}$

$\triangleleft$

Definition 2.4 (Configuration Containment)
Given two configurations $\mathcal{C}_1 = \langle \mathcal{D}_1, \mathcal{F}_1 \rangle$ and $\mathcal{C}_2 = \langle \mathcal{D}_2, \mathcal{F}_2 \rangle$, we say $\mathcal{C}_2$ contains $\mathcal{C}_1$ and write $\mathcal{C}_1 \subseteq \mathcal{C}_2$ if

- $\mathcal{D}_1 \subseteq \mathcal{D}_2$
- $\mathcal{F}_1(\lambda) \subseteq \mathcal{F}_2(\lambda)$ for each ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$

$\square$

Definition 2.5 ($\wedge$–Elimination, $\mathcal{I}_{\wedge E}$)
For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha]$ and $\mathcal{C}/\mathcal{C} + [\lambda : \beta]$ are members of the inference rule $\wedge$–Elimination (sometimes written $\mathcal{I}_{\wedge E}$) if

- $\lambda : \alpha \wedge \beta \in \mathcal{C}$

We will sometimes write $\wedge$–Elimination as

$$\frac{\mathcal{C}\langle \lambda : \alpha \wedge \beta \rangle}{\mathcal{C}'\langle \lambda : \alpha \rangle} \qquad \frac{\mathcal{C}\langle \lambda : \alpha \wedge \beta \rangle}{\mathcal{C}'\langle \lambda : \beta \rangle}$$

$\square$

Definition 2.6 ($\wedge$–Introduction, $\mathcal{I}_{\wedge I}$)
For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha \wedge \beta]$ is a member of the inference rule $\wedge$-Introduction (sometimes written $\mathcal{I}_{\wedge I}$) if

- $\lambda : \alpha \in \mathcal{C}$
- $\lambda : \beta \in \mathcal{C}$

We will sometimes write $\wedge$–Introduction as

$$\frac{\mathcal{C}\langle \lambda : \alpha, \lambda : \beta \rangle}{\mathcal{C}'\langle \lambda : \alpha \wedge \beta \rangle}$$

$\square$

Definition 2.7 (**$\vee$–Introduction, $\mathcal{I}_{\vee I}$**)
For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha \vee \beta]$ and $\mathcal{C}/\mathcal{C} + [\lambda : \beta \vee \alpha]$ are members of the inference rule $\vee$–Introduction (sometimes written $\mathcal{I}_{\vee I}$) if

- $\lambda : \alpha \in \mathcal{C}$

We will sometimes write $\vee$–Introduction as

$$\frac{\mathcal{C}\langle \lambda : \alpha \rangle}{\mathcal{C}'\langle \lambda : \alpha \vee \beta \rangle} \qquad \frac{\mathcal{C}\langle \lambda : \alpha \rangle}{\mathcal{C}'\langle \lambda : \beta \vee \alpha \rangle}$$

□

Definition 2.8 (**$\vee$–Elimination, $\mathcal{I}_{\vee E}$**)
For all configurations $\mathcal{C}$, terms λ and wffs γ, $\mathcal{C}/\mathcal{C} + [\lambda : \gamma]$ is a member of the inference rule $\vee$–Elimination (sometimes written $\mathcal{I}_{\vee E}$) if there exist wffs α and β such that

- $\lambda : \alpha \vee \beta \in \mathcal{C}$
- $\mathcal{C} + [\lambda : \alpha] \vdash_S \lambda : \gamma$
- $\mathcal{C} + [\lambda : \beta] \vdash_S \lambda : \gamma$

We will sometimes write $\vee$–Elimination as

$$\frac{\mathcal{C}\langle \lambda : \alpha \vee \beta \rangle \quad \begin{array}{c} \mathcal{C}\langle [\lambda : \alpha] \rangle \\ \cdot \\ \cdot \\ \tilde{\mathcal{C}}\langle \lambda : \gamma \rangle \end{array} \quad \begin{array}{c} \mathcal{C}\langle [\lambda : \beta] \rangle \\ \cdot \\ \cdot \\ \tilde{\mathcal{C}}\langle \lambda : \gamma \rangle \end{array}}{\mathcal{C}'\langle \lambda : \gamma \rangle}$$

□

Definition 2.9 (**$\rightarrow$–Introduction, $\mathcal{I}_{\rightarrow I}$**)
For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\alpha \rightarrow \beta]$ is a member of the inference rule $\rightarrow$–Introduction (sometimes written $\mathcal{I}_{\rightarrow I}$) if

- $\mathcal{C} + [\lambda : \alpha] \vdash_S \lambda : \beta$

We will sometimes write $\rightarrow$–Introduction as

$$\frac{\begin{array}{c} \mathcal{C}\langle [\lambda : \alpha] \rangle \\ \cdot \\ \cdot \\ \tilde{\mathcal{C}}\langle \lambda : \beta \rangle \end{array}}{\mathcal{C}'\langle \lambda : \alpha \rightarrow \beta \rangle}$$

□

Definition 2.10 (**$\rightarrow$–Elimination, $\mathcal{I}_{\rightarrow E}$**)
For all configurations $\mathcal{C}$, terms λ and wffs β, $\mathcal{C}/\mathcal{C} + [\lambda : \beta]$ is a member of the inference rule $\rightarrow$–Elimination (sometimes written $\mathcal{I}_{\rightarrow E}$) if for some wff α,

- $\lambda : \alpha \rightarrow \beta \in \mathcal{C}$
- $\lambda : \alpha \in \mathcal{C}$

We will sometimes write $\rightarrow$–Elimination as

$$\frac{\mathcal{C}\langle\lambda:\alpha\rightarrow\beta,\lambda:\alpha\rangle}{\mathcal{C}'\langle\lambda:\beta\rangle}$$

□

Definition 2.11 (¬–Introduction, $\mathcal{I}_{\neg I}$)
For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C}+[\lambda:\neg\alpha]$ is a member of the inference rule ¬–Introduction (sometimes written $\mathcal{I}_{\neg I}$) if for some term λ'

- $\mathcal{C}+[\lambda:\alpha]\ \vdash_S\ \lambda':\perp$

We will sometimes write ¬–Introduction as

$$\begin{array}{c}\mathcal{C}\langle[\lambda:\alpha]\rangle\\ \vdots\\ \dfrac{\tilde{\mathcal{C}}\langle\lambda':\perp\rangle}{\mathcal{C}'\langle\lambda:\neg\alpha\rangle}\end{array}$$

□

Definition 2.12 (¬–Elimination, $\mathcal{I}_{\neg E}$)
For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C}+[\lambda:\alpha]$ is a member of the inference rule ¬–Elimination (sometimes written $\mathcal{I}_{\neg E}$) if

- $\lambda:\neg\neg\alpha\in\mathcal{C}$

We will sometimes write ¬–Elimination as

$$\frac{\mathcal{C}\langle\lambda:\neg\neg\alpha\rangle}{\mathcal{C}'\langle\lambda:\alpha\rangle}$$

□

Definition 2.13 (⊥–Introduction, $\mathcal{I}_{\perp I}$)
For all configurations $\mathcal{C}$, any R–literal Δ, and any declarative unit $\lambda:\alpha$, $\mathcal{C}/\mathcal{C}+[\lambda:\alpha]$ is a member of the inference rule ⊥–Introduction (sometimes written $\mathcal{I}_{\perp I}$) if

- $\Delta\in\mathcal{C}$
- $\overline{\Delta}\in\mathcal{C}$

We will sometimes write ⊥–Introduction as

$$\frac{\mathcal{C}\langle\Delta,\overline{\Delta}\rangle}{\mathcal{C}'\langle\lambda:\alpha\rangle}$$

□

The next group of inference rules concern the modal operators of the language $\mathcal{L}_M$ and are based on the structure of the diagrams within configurations. Indeed they are the only inference rules where R–literals of the form $R(\lambda_i,\lambda_j)$ are referred to explicitly in order to espress the inference of new formulae in an accessible world.

Definition 2.14 (◇–Elimination, $\mathcal{I}_{\diamond E}$)
For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C} + [f_\alpha(\lambda):\alpha] + [R(\lambda, f_\alpha(\lambda))]$ is a member of the inference rule ◇–Elimination (sometimes written $\mathcal{I}_{\diamond E}$) if

- $\lambda : \diamond\alpha \in \mathcal{C}$

We will sometimes write ◇–Elimination as

$$\frac{\mathcal{C}\langle \lambda : \diamond\alpha \rangle}{\mathcal{C}'\langle f_\alpha(\lambda):\alpha, R(\lambda, f_\alpha(\lambda))\rangle}$$

□

In Section 3 we will see that the inference of new labels and relations in the structure of a diagram within a configuration is consistent with the extended algebra $\mathcal{A}^+$ in which the process of defining new labels via function symbols is controlled by the axiom schema (Ax5). An example application of the ◇–Elimination rule is represented graphically in Figure 3.

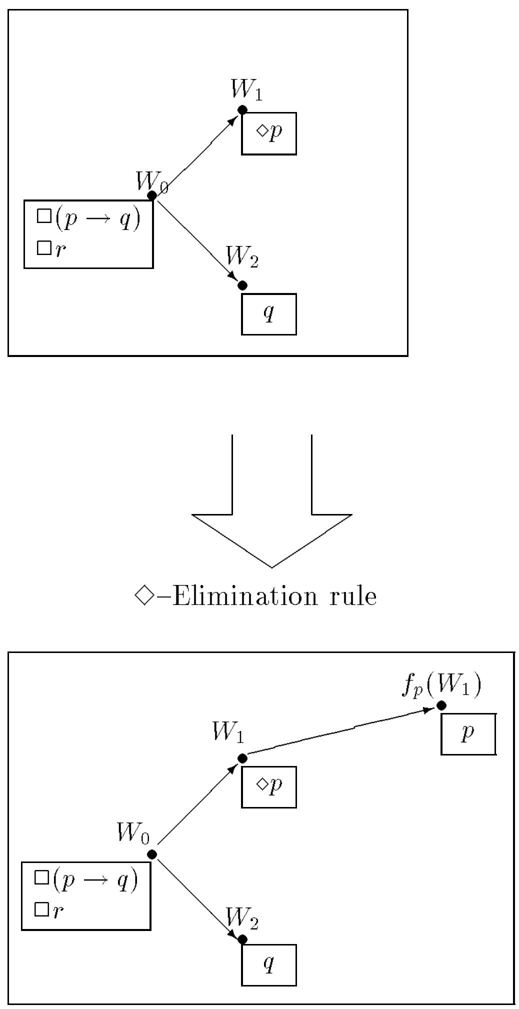

Figure 3 – ◇–Elimination rule

Definition 2.15 ($\Diamond$–Introduction, $\mathcal{I}_{\Diamond I}$)
For all configurations $\mathcal{C}$, terms λ_1 and λ_2 and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda_1 : \Diamond\alpha]$ is a member of the inference rule $\Diamond$–Introduction (sometimes written $\mathcal{I}_{\Diamond I}$) if

- $\lambda_2 : \alpha \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

We will sometimes write $\Diamond$–Introduction as

$$\frac{\mathcal{C}\langle \lambda_2 : \alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_1 : \Diamond\alpha\rangle}$$

$\Box$

Definition 2.16 ($\Box$–Introduction, $\mathcal{I}_{\Box I}$)
For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda : \Box\alpha]$ is a member of the inference rule $\Box$–Introduction (sometimes written $\mathcal{I}_{\Box I}$) if

- $\mathcal{C} + [R(\lambda, box_\alpha(\lambda))] \vdash_S box_\alpha(\lambda) : \alpha$

We will sometimes write $\Box$–Introduction as

$$\frac{\begin{array}{c}\mathcal{C} \cup \{[R(\lambda, box_\alpha(\lambda))]\} \\ \vdots \\ \tilde{\mathcal{C}}\langle box_\alpha(\lambda) : \alpha\rangle\end{array}}{\mathcal{C}'\langle \lambda : \Box\alpha\rangle}$$

$\Box$

Definition 2.17 ($\Box$–Elimination, $\mathcal{I}_{\Box E}$)
For all configurations $\mathcal{C}$, terms λ_1 and λ_2 and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda_2 : \alpha]$ is a member of the inference rule $\Box$–Elimination (sometimes written $\mathcal{I}_{\Box E}$) if

- $\lambda_1 : \Box\alpha \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

We will sometimes write $\Box$–Elimination as

$$\frac{\mathcal{C}\langle \lambda_1 : \Box\alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_2 : \alpha\rangle}$$

$\Box$

Definition 2.18 (R–Introduction, $\mathcal{I}_{R-I}$)
For all configurations $\mathcal{C}$, and R-literal Δ, $\mathcal{C}/\mathcal{C} + [\Delta]$ is a member of the inference rule R–Introduction (sometimes written $\mathcal{I}_{R-I}$) if for some term λ'

- $\mathcal{C} + [\overline{\Delta}] \vdash_S \lambda' : \bot$

We will sometimes write R–Introduction as

$$\frac{\begin{array}{c}\mathcal{C} + [\overline{\Delta}] \\ \cdot \\ \tilde{\mathcal{C}}\langle \lambda^{'} : \bot \rangle\end{array}}{\mathcal{C}^{'}\langle \Delta \rangle}$$

□

Definition 2.19 (R–**Assertion,** $\mathcal{I}_{R-A}$)
For all configurations $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$, and R-literal Δ, $\mathcal{C}/\mathcal{C} + [\Delta]$ is a member of the inference rule R–Assertion (sometimes written $\mathcal{I}_{R-A}$) if

- $\mathcal{D}, \mathcal{A} \vdash_{FOL} \Delta$

where $\mathcal{A}$ is the labelling algebra

□

Definition 2.20 ($\mathcal{C}$–**Reduction,** $\mathcal{I}_{C-R}$)
For all configurations $\mathcal{C}$ and $\mathcal{C}^{'}$, $\mathcal{C}/\mathcal{C}^{'}$ is a member of the inference rule $\mathcal{C}$–Reduction (sometimes written $\mathcal{I}_{C-R}$) if

- $\mathcal{C}^{'} \subseteq \mathcal{C}$

□

We conclude this section with some example proofs. It might be helpful to note that in the following examples the number of dashes ($'$) included as a superscript on each configuration symbol ($\mathcal{C}$) corresponds to the number of new assumptions introduced from the initial configuration $\mathcal{C}_0$. The subscript index (1, 2, etc.) corresponds to each step in each 'sub–proof' related to a new assumption.

Example 2.1 ($\mathcal{C}\langle W_0 : \Box p \rangle \vdash_S W_0 : \neg\Diamond\neg p$)
Let $\mathcal{C}_0 = \langle \mathcal{D}_0, \mathcal{F} \rangle$ where $\mathcal{D}_0 = \{\}$, $\mathcal{F}(W_0) = \{\Box p\}$ and $\mathcal{F}(\lambda) = \{\}$ for any $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $\lambda \neq W_0$. We will show that $\mathcal{C}_0 \vdash_S W_0 : \neg\Diamond\neg p$.

Let $\mathcal{C}_0^{'} = \mathcal{C}_0 + [W_0 : \Diamond\neg p]$ and let $\mathcal{C}_1^{'} = \mathcal{C}_0^{'} + [f_{\neg p}(W_0) : \neg p] + [R(W_0, f_{\neg p}(W_0))]$.

Then $\Diamond$–Elimination generates $\mathcal{C}_1^{'}$ from $\mathcal{C}_0^{'}$ (i)

Let $\mathcal{C}_2^{'} = \mathcal{C}_1^{'} + [f_{\neg p}(W_0) : p]$

Then $\Box$–Elimination generates $\mathcal{C}_2^{'}$ from $\mathcal{C}_1^{'}$ (ii)

Let $\mathcal{C}_3^{'} = \mathcal{C}_2^{'} + [f_{\neg p}(W_0) : p \wedge \neg p]$

Then $\wedge$–Introduction generates $\mathcal{C}_3^{'}$ from $\mathcal{C}_2^{'}$ (iii)

By (i), (ii) and (iii), $\mathcal{C}'_0, \mathcal{C}'_1, \mathcal{C}'_2, \mathcal{C}'_3$ is a proof, so that by (iii)

$$\mathcal{C}_0 + [W_0 : \Diamond\neg p] \vdash_S f_{\neg p}(W_0) : \bot \tag{iv}$$

Hence by (iv), $\neg$–Introduction generates $\mathcal{C}_0 + [W_0 : \neg\Diamond\neg p]$ from $\mathcal{C}_0$, so that $\mathcal{C}_0, \mathcal{C}_0 + [W_0 : \neg\Diamond\neg p]$ is a proof and therefore $\mathcal{C}_0 \vdash_S W_0 : \neg\Diamond\neg p$.

•

Example 2.2 ($\mathcal{C}\langle W_0 : \neg\Diamond\neg p\rangle \vdash_S W_0 : \Box p$)
Let $\mathcal{C}_0 = \langle \mathcal{D}_0, \mathcal{F}\rangle$, where $\mathcal{D}_0 = \{\}$, $\mathcal{F}(W_0) = \{\neg\Diamond\neg p\}$ and $\mathcal{F}(\lambda) = \{\}$ for any $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $\lambda \neq W_0$. We will show that $\mathcal{C}_0 \vdash_S W_0 : \Box p$.

Let $\mathcal{C}'_0 = \mathcal{C}_0 + [R(W_0, box_p(W_0))]$, let $\mathcal{C}''_0 = \mathcal{C}'_0 + [box_p(W_0) : \neg p]$[6], let $\mathcal{C}''_1 = \mathcal{C}''_0 + [W_0 : \Diamond\neg p]$ and let $\mathcal{C}''_2 = \mathcal{C}''_1 + [W_0 : (\Diamond\neg p) \wedge (\neg\Diamond\neg p)]$

Then $\wedge$–Introduction generates $\mathcal{C}''_2$ from $\mathcal{C}''_1$ and $\Diamond$–Introduction generates $\mathcal{C}''_1$ from $\mathcal{C}''_0$, so that $\mathcal{C}''_0, \mathcal{C}''_1, \mathcal{C}''_2$ is a proof. So

$$\mathcal{C}'_0 + [box_p(W_0) : \neg p] \vdash_S W_0 : \bot \tag{i}$$

Let $\mathcal{C}'_1 = \mathcal{C}'_0 + [box_p(W_0) : \neg\neg p]$

By (i), $\neg$–Introduction generates $\mathcal{C}'_1$ from $\mathcal{C}'_0$ (ii)

Let $\mathcal{C}'_2 = \mathcal{C}'_1 + [box_p(W_0) : p]$

By (ii), $\neg$–Elimination generates $\mathcal{C}'_2$ from $\mathcal{C}'_1$ (iii)

Hence $\mathcal{C}'_0, \mathcal{C}'_1, \mathcal{C}'_2$ is a proof and $\mathcal{C}_0 + [R(W_0, box_p(W_0))] \vdash_S box_p(W_0) : p$.

Let $\mathcal{C}_1 = \mathcal{C}_0 + [W_0 : \Box p]$

By (iii), $\Box$–Introduction generates $\mathcal{C}_1$ from $\mathcal{C}_0$

Hence $\mathcal{C}_0, \mathcal{C}_1$ is a proof and therefore $\mathcal{C}_0 \vdash_S W_0 : \Box p$.

•

[6] In the assumed world $box_p(W_0)$ we need to prove p in order to apply the $\Box$–Introduction rule. By the $\neg$–Elimination inference rule to prove $box_p(W_0) : p$ we need to prove $box_p(W_0) : \neg\neg p$, which requires a $\neg$–Introduction rule. Thus the second assumption $box_p(W_0) : \neg p$

We will sometimes represent proofs graphically. The two examples above can be represented as follows:

- **Example 2.1 :** $\mathcal{C}_0\langle W_0 : \Box p\rangle \vdash_S W_0 : \neg\Diamond\neg p$

$\underline{\mathcal{C}_0\langle W_0 : \Box p\rangle}$	
$\quad\underline{\mathcal{C}_0'\langle W_0 : \Box p, [W_0 : \Diamond\neg p]\rangle}$	(new assumption)
$\quad\underline{\mathcal{C}_1'\langle W_0 : \Box p, R(W_0, f_{\neg p}(W_0)), f_{\neg p}(W_0) : \neg p\rangle}$	($\Diamond$–E)
$\quad\underline{\mathcal{C}_2'\langle f_{\neg p}(W_0) : p, f_{\neg p}(W_0) : \neg p\rangle}$	($\Box$–E)
$\quad\underline{\mathcal{C}_3'\langle f_{\neg p}(W_0) : \bot\rangle}$	($\wedge$–I)
$\mathcal{C}_0\langle W_0 : \neg\Diamond\neg p\rangle$	($\neg$–I)

- **Example 2.2 :** $\mathcal{C}_0\langle W_0 : \neg\Diamond\neg p\rangle \vdash_S W_0 : \Box p$

$\underline{\mathcal{C}_0\langle W_0 : \neg\Diamond\neg p\rangle}$	
$\quad\underline{\mathcal{C}_0'\langle W_0 : \neg\Diamond\neg p, [R(W_0, box_p(W_0))]\rangle}$	(new assumption)
$\qquad\underline{\mathcal{C}_0''\langle W_0 : \neg\Diamond\neg p, [box_p(W_0) : \neg p]\rangle}$	(new assumption)
$\qquad\underline{\mathcal{C}_1''\langle W_0 : \neg\Diamond\neg p, W_0 : \Diamond\neg p\rangle}$	($\Diamond$–I)
$\qquad\underline{\mathcal{C}_2''\langle W_0 : \bot\rangle}$	($\wedge$–I)
$\quad\underline{\mathcal{C}_1'\langle box_p(W_0) : \neg\neg p\rangle}$	($\neg$–I)
$\quad\underline{\mathcal{C}_2'\langle box_p(W_0) : p\rangle}$	($\neg$–E)
$\mathcal{C}_0\langle W_0 : \Box p\rangle$	($\Box$–I)

Discussion

In this section, we have defined the set of introduction and elimination rules for classical connectives and modal operators. The rules for classical connectives faithfully reflect the natural deduction proof theory for propositional logic defined in [Prawitz-65]. The only difference is that in our system formulae are labelled. In fact, since these rules do not define new labels, they can be considered as 'local natural deduction rules' for propositional logic.

On the other hand, the inference rules for modal operators involve declarative units with different labels. They express the interaction between the local modal theories within a configuration. They allow us to infer specific labels for particular accessible worlds together with new relations (e.g. the $\Diamond$–Elimination rule), and to infer which formulae hold within these labels (e.g. the $\Diamond$–Introduction and $\Box$–Elimination rules). As an advantage, it is not necessary to introduce the extra notions of 'strict subordinate derivations' and 'strict iteration rules' defined in [Fitting-83]. Besides this, Fitting's distinction between 'I–style' and 'A-style' natural deduction proofs is unnecessary here. We will discuss the comparison between a MLDS and Fitting's modal natural deduction system in more detail in Section 5.

A MLDS also includes some inference rules which are related to the R–literals of a configuration. The R–Assertion rule allows us to infer new R–literals according to the labelling algebra $\mathcal{A}$. Because of this modularity, it is not necessary to differentiate modal rules according to the particular modal logic we want to represent, as is done in Fitting's modal natural deduction system. The R–Introduction rule is the equivalent of a $\neg$–Introduction rule for R-literals. The $\bot$–Introduction rule allows us to infer falsity (i.e. $\lambda : \bot$) whenever an R–literal and its negation are present in a configuration. This is necessary because no compound classical formulae with R–literals can be inferred in a configuration.

Finally, with the C–reduction rule it is possible to infer any configuration contained in an existing one. This rule must be imcluded because all the other rules have the effect of expanding their antecedent configurations.

3 A Semantics for MLDS and Soundness Theorem

In Section 1 we defined a semi–extended labelling language, $Func(\mathcal{L}_L, \mathcal{L}_M)$, and a set of axioms (a labelling algebra $\mathcal{A}$) written in this language. We also observed informally that the ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ can be regarded as labels of possible worlds and the relation symbol R as an accessibility relation between possible worlds. So a labelling algebra $\mathcal{A}$ can be considered as the first–order axiomatization that corresponds to a set of properties of the accessibility relation in a possible–world Kripke semantics. This is along the lines of a *correspondence theory* between modal logics and first–order logic, [van Benthem-84]. We propose a semantics of a MLDS S in terms of First–order semantics. In what follows Kripke semantic notions such as *a wff* α *is satisfied in a possible world* λ will be expressed in terms of first–order statements of the form $[\alpha]^*(\lambda)$, where $[\alpha]^*$ is a predicate symbol. In the same way, the Kripke semantic definition of satisfiability will be expressed as a set of first–order axiom schemas. We expand the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ with a monadic predicate $[\alpha]^*$ for each wff α of $\mathcal{L}_M$, and we expand the First–order theory $\mathcal{A}$ with an axiom schema for each type[7] of wff α of $\mathcal{L}_M$.

3.1 Semantics

Definition 3.1 (Extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$)
Let $Func(\mathcal{L}_L, \mathcal{L}_M)$ be a semi–extended labelling language. Let α_1,...,α_n,... be the ordered set of all wffs of $\mathcal{L}_M$. The first–order language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ is defined as the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ extended with:

- $[\alpha_1]^*, [\alpha_2]^*, [\alpha_3]^*, \ldots$ a unary predicate symbol corresponding to each wff α_i of $\mathcal{L}_M$

$\Box$

Definition 3.2 (Extended algebra $\mathcal{A}^+$)
Given an extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and a labelling algebra $\mathcal{A}$ written in $Func(\mathcal{L}_L, \mathcal{L}_M)$, the extended algebra $\mathcal{A}^+$ is the first–order theory in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ consisting of the following axiom schemas (Ax1)–(Ax9), together with the axioms of $\mathcal{A}$:

For any wffs α and β of $\mathcal{L}_M$:

[7] The type of a wff is given by the main connective of the wff itself, e.g. we say that the wff $\Diamond(p \rightarrow q)$ is a $\Diamond$–formula

$$\forall x([\alpha \wedge \beta]^*(x) \equiv ([\alpha]^*(x) \wedge [\beta]^*(x))) \tag{Ax1}$$

$$\forall x([\neg\alpha]^*(x) \equiv \neg[\alpha]^*(x)) \tag{Ax2}$$

$$\forall x([\alpha \vee \beta]^*(x) \equiv ([\alpha]^*(x) \vee [\beta]^*(x))) \tag{Ax3}$$

$$\forall x([\alpha \rightarrow \beta]^*(x) \equiv ([\alpha]^*(x) \rightarrow [\beta]^*(x))) \tag{Ax4}$$

$$\forall x([\Diamond\alpha]^*(x) \rightarrow (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x)))) \tag{Ax5}$$

$$\forall x(\exists y(R(x,y) \wedge [\alpha]^*(y)) \rightarrow [\Diamond\alpha]^*(x)) \tag{Ax6}$$

$$\forall x((R(x, box_\alpha(x)) \rightarrow [\alpha]^*(box_\alpha(x))) \rightarrow [\Box\alpha]^*(x)) \tag{Ax7}$$

$$\forall x([\Box\alpha]^*(x) \rightarrow (\forall y(R(x,y) \rightarrow [\alpha]^*(y)))) \tag{Ax8}$$

□

The first four axiom schemas express the distributive properties of the logical connectives among the monadic predicates of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. They cover the Kripke semantic definition of satisfiability of the logical connectives $\wedge$, $\neg$, $\vee$ and $\rightarrow$ respectively. (Ax5) forces the accessibility relation R on the labels generated by the application of function symbols f_{α_i} of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Axiom schemas (Ax5)–(Ax6) together cover the Kripke semantic definition of the modal operator $\Diamond$. (Note that from (Ax5)–(Ax6) we may derive $\forall x(\exists y(R(x,y) \wedge [\alpha]^*(y)) \equiv [\Diamond\alpha]^*(x))$ which reflects the traditional Kripke semantic meaning of $\Diamond$). Analogously axiom schemas (Ax7)–(Ax8) together cover the Kripke semantic definition of the modal operator $\Box$. (Note that from (Ax7)–(Ax8) we may derive $\forall x([\Box\alpha]^*(x) \equiv \forall y(R(x,y) \rightarrow [\alpha]^*(y)))$ which reflects the traditional Kripke semantic meaning of $\Box$). Then the axiom schemas (Ax1) – (Ax8) of $\mathcal{A}^+$ reflect the Kripke semantic definition of satisfiability of modal wffs. This is easy to see, by interpreting the truth of $[\alpha]^*(x)$ as the truth of the modal formula α in the possible world x.

In the next proposition we prove that traditional relation between $\Box$ and $\Diamond$ modal operators can be derived from the above set of axiom schemas.

Proposition 3.1
Given a modal language $\mathcal{L}_M$, a language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and an associated extended algebra $\mathcal{A}^+$, for any formula $\alpha \in \mathcal{L}_M$ the following equivalence is satisfied:

$$\mathcal{A}^+ \vdash_{FOL} \forall x([\Box\alpha]^*(x) \equiv [\neg\Diamond\neg\alpha]^*(x))$$

Proof*:*
It is sufficient to prove the following two results for an arbitrary ground term λ.

a) $\mathcal{A}^+ \vdash_{FOL} [\Box\alpha]^*(\lambda) \rightarrow [\neg\Diamond\neg\alpha]^*(\lambda)$
b) $\mathcal{A}^+ \vdash_{FOL} [\neg\Diamond\neg\alpha]^*(\lambda) \rightarrow [\Box\alpha]^*(\lambda)$

proof of a)

Assume:
$$[\Box\alpha]^*(\lambda) \qquad \text{(i)}$$
By (i) and (Ax8):
$$\forall y(R(\lambda, y) \rightarrow [\alpha]^*(y)) \qquad \text{(ii)}$$
By (ii) and (Ax2):
$$\forall y(R(\lambda, y) \rightarrow \neg[\neg\alpha]^*(y)) \qquad \text{(iii)}$$
By (iii):
$$\forall y\neg(R(\lambda, y) \wedge [\neg\alpha]^*(y)) \qquad \text{(iv)}$$
By (iv):
$$\neg(R(\lambda, f_{\neg\alpha}(x)) \wedge [\neg\alpha]^*(f_{\neg\alpha}(\lambda))) \qquad \text{(v)}$$
By (v) and (Ax5):
$$\neg[\Diamond\neg\alpha]^*(\lambda) \qquad \text{(vi)}$$
Finally by (vi) and (Ax2):
$$[\neg\Diamond\neg\alpha]^*(\lambda)$$

proof of b):

Assume:
$$[\neg\Diamond\neg\alpha]^*(\lambda) \qquad \text{(i)}$$
By (i), (Ax2) and (Ax6):
$$\neg\exists y(R(\lambda, y) \wedge [\neg\alpha]^*(y)) \qquad \text{(ii)}$$
By (ii):
$$\forall y(\neg R(\lambda, y) \vee \neg[\neg\alpha]^*(y)) \qquad \text{(iii)}$$
By (iii) and (Ax2):
$$\forall y(R(\lambda, y) \rightarrow [\alpha]^*(y)) \qquad \text{(iv)}$$
By (iv):
$$R(\lambda, box_\alpha(\lambda)) \rightarrow [\alpha]^*(box_\alpha(\lambda)) \qquad \text{(v)}$$
Finally by (v) and (Ax7):
$$[\Box\alpha]^*(\lambda)$$

∎

Definition 3.3 (Semantic Structure of a MLDS)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{A}^+$ be the associated extended algebra. $\mathcal{M}$ is a semantic structure of S if $\mathcal{M}$ is a model of $\mathcal{A}^+$.

□

Definition 3.4 (Satisfiability of a declarative unit)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{A}^+$ be the associated extended algebra and let $\lambda : \alpha$ be a declarative unit of S. $\lambda : \alpha$ is satisfiable if there exists a semantic structure $\mathcal{M}$ of S such that $\mathcal{M} \models_{FOL} [\alpha]^*(\lambda)$. In this case we say $\mathcal{M}$ satisfies $\lambda : \alpha$ and we write $\mathcal{M} \models_S \lambda : \alpha$.

□

Definition 3.5 (Satisfiability of an R–literal)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{A}^+$ be the associated extended algebra and let Δ be an R–literal of S. Δ is satisfiable if there exists a semantic structure $\mathcal{M}$ of S such that $\mathcal{M} \models_{FOL} \Delta$. In this case we say $\mathcal{M}$ satisfies Δ and we write $\mathcal{M} \models_S \Delta$.

□

Definition 3.6 (Satisfiability of a configuration)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{A}^+$ be the associated extended algebra and let $\mathcal{C}$ be a configuration of S. $\mathcal{C}$ is satisfiable if there exists a semantic structure $\mathcal{M}$ of S such that for each π, $\pi \in \mathcal{C}$, $\mathcal{M}$ satisfies π, (where π may be a declarative unit or an R–literal). In this case we say $\mathcal{M}$ satisfies $\mathcal{C}$ and we write $\mathcal{M} \models_S \mathcal{C}$.

□

On the basis of above definitions, it appears straightforward to translate configurations of a MLDS into first–order theories written in the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and to express semantic entailment between configurations in terms of classical semantic entailment between their respective first order translations. In this section, we present a formal definition of a first order translation of a configuration into the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, and we define the notion of logical entailment in a MLDS. We then prove the soundness property of the natural deduction system presented in Section 2 according to this semantics.

Definition 3.7 (First Order Translation of a Configuration)
Consider a MLDL $\langle\mathcal{L}_L, \mathcal{L}_M\rangle$ and the associated extended language $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Let $\mathcal{C} = \langle\mathcal{D}, \mathcal{F}\rangle$ be a configuration written in the language MLDL. The first order translation of $\mathcal{C}$, written $FOT(\mathcal{C})$, is the theory written in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and defined by the expression:

$$FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$$

where $[\alpha]^*(\lambda) \in \mathcal{DU}$ if and only if $\alpha \in \mathcal{F}(\lambda)$.

$\square$

A translation of a configuration is thus a first-order theory including the R–literals which are present in the diagram of the configuration, and the set of monadic formulae $[\alpha]^*(\lambda)$ that correspond to the declarative units present in the configuration. Therefore a first–order translation of a given configuration is a set of ground literals of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$.

Definition 3.8 (Semantic entailment in a MLDS)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{A}^+$ be the extended algebra of $\mathcal{A}$. Let $\mathcal{C} = \langle\mathcal{D}, \mathcal{F}\rangle$ and $\mathcal{C}' = \langle\mathcal{D}', \mathcal{F}'\rangle$ be two configurations of S and $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$ and $FOT(\mathcal{C}') = \mathcal{D}' \cup \mathcal{DU}'$ their respective first order translations. The configuration $\mathcal{C}$ semantically entails $\mathcal{C}'$, written $\mathcal{C} \models_S \mathcal{C}'$, if

- $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} \Delta$ for each $\Delta \in \mathcal{D}'$
- $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda)$ for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$

$\square$

Notation 3.1
Given two configurations $\mathcal{C}$ and $\mathcal{C}'$ and their first order translation $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$ and $FOT(\mathcal{C}') = \mathcal{D}' \cup \mathcal{DU}'$, we write $\mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$ if

$$\begin{array}{ll} \mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} \Delta & \text{for each } \Delta \in \mathcal{D}' \\ \mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} [\alpha]^*(\lambda) & \text{for each } [\alpha]^*(\lambda) \in \mathcal{DU}' \end{array}$$

and we write $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$ if

$$\begin{array}{ll} \mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} \Delta & \text{for each } \Delta \in \mathcal{D}' \\ \mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda) & \text{for each } [\alpha]^*(\lambda) \in \mathcal{DU}' \end{array}$$

$\triangleleft$

3.2 Soundness Theorem

We have now presented both the notions of proof and semantic entailment in a MLDS. Next, we will prove that if there exists a natural deduction proof of a configuration $\mathcal{C}'$ from a given configuration $\mathcal{C}$, then the configuration $\mathcal{C}$ semantically entails the configuration $\mathcal{C}'$. In other words, we will show that our natural deduction system is sound. We will take advantage of the soundness and completeness of first order classical logic, as shown informally in the diagram below:

$$\begin{tikzcd}
\mathcal{C} \vdash_S \mathcal{C}' \arrow[r, "(1)"] \arrow[d, "(2)"'] & \mathcal{C} \models_S \mathcal{C}' \\
\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}') \arrow[r, "(3)"'] & \mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}') \arrow[u, "(4)"']
\end{tikzcd}$$

Figure 1 – Proof of the Soundness Theorem

The arrow labelled with (1) corresponds to the soundness statement. It is also given by the composition of the arrows (2), (3), (4). In this composition, arrow (4) is given by Definition 3.8 and arrow (3) is given by Proposition 3.2 (below) based on soundness of first order logic. Arrow (2) will be proved in Lemma 3.1 (below).

Proposition 3.2
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C} = \langle\mathcal{D}, \mathcal{F}\rangle$ and $\mathcal{C}' = \langle\mathcal{D}', \mathcal{F}'\rangle$ be two configurations of S. If $\mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$ then $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$.

> **Proof**:
> The hypothesis $\mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$ means that $\mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} \Delta$ for each $\Delta \in \mathcal{D}'$ and $\mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} [\alpha]^*(\lambda)$ for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$. Then by soundness of first order logic, $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} \Delta$ for each $\Delta \in \mathcal{D}'$ and $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda)$ for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$. Hence $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$.

■

By Proposition 3.2 and Definition 3.8, it is sufficient to prove (Lemma 3.1 below) that if a configuration $\mathcal{C}'$ is derivable from a configuration $\mathcal{C}$, then all the formulae of its first order translation are derivable from the first order translation of $\mathcal{C}$, together with the extended algebra $\mathcal{A}^+$ (Arrow (2) in the previous diagram).

We first introduce the notions of *size of a member of an inference rule* and *size of a derivation* in a MLDS, which will be used in Lemma 3.1. The definition of size of a member of an inference rule is based on the following notation.

Notation 3.2
In the MLDS's defined in Section 2, four sets of inference rules can be identified. The first one is the singleton set $\{\mathcal{I}_{C-R}\}$, which we shall write as $\mathcal{I}^{00}$. $\mathcal{I}_{C-R}$ is the only inference that does not infer new declarative units or new R–literals and does not use any subderivation in MLDS as condition.

The second set consists of inference rules that infer new declarative units and/or new R–literals without using any subderivation as conditions. We denote it as $\mathcal{I}^0$, so that

$\mathcal{I}^0 = \{\mathcal{I}_{\wedge E}, \mathcal{I}_{\wedge I}, \mathcal{I}_{\vee I}, \mathcal{I}_{\rightarrow E}, \mathcal{I}_{\neg E}, \mathcal{I}_{\diamond E}, \mathcal{I}_{\diamond I}, \mathcal{I}_{\Box E}, \mathcal{I}_{R-A}\}$.

The third set consists of those inference rules that require one subderivation as a condition. We denote it as $\mathcal{I}^+$, so that

$\mathcal{I}^+ = \{\mathcal{I}_{\rightarrow I}, \mathcal{I}_{\neg I}, \mathcal{I}_{\Box I}, \mathcal{I}_{R-I}\}$.

The fourth set refers to inference rules that use two subderivations as conditions. We denote it as $\mathcal{I}^{++}$, so that

$\mathcal{I}^{++} = \{\mathcal{I}_{\vee E}\}$.

We also observe that $\mathcal{I}^{00} \cup \mathcal{I}^0 \cup \mathcal{I}^+ \cup \mathcal{I}^{++} = \mathcal{R}$.

$\triangleleft$

Definition 3.9 (Size of a member of an inference rule in MLDS)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{I}_i \in \mathcal{R}$ and let $\mathcal{C}/\mathcal{C}' \in \mathcal{I}_i$. The size of $\mathcal{C}/\mathcal{C}'$ with respect to $\mathcal{I}_i$, written $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i)$, is defined as follows:

- If $\mathcal{I}_i \in \mathcal{I}^{00}$ then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_{C-R}) = 0$
- If $\mathcal{I}_i \in \mathcal{I}^0$, then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1$.
- If $\mathcal{I}_i \in \mathcal{I}^+$, then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1 + l_1$ where l_1 is the smallest of the sizes of all subderivations (definded below) that can be used as condition of the rule.
- If $\mathcal{I}_i \in \mathcal{I}^{++}$, then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_{\vee E}) = 1 + l_1 + l_2$ where l_1 is the smallest of the sizes of all subderivations that can be used as its first condition and l_2 is the smallest of the sizes of all subderivations that can be used as its second condition.

$\Box$

Definition 3.10 (Size of a proof)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}\mathcal{R}\rangle$ be a MLDS. The size of a proof $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$, written $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle)$, is defined as follows:

$$l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = \sum_{k=0}^{n-1} l(\mathcal{C}_k/\mathcal{C}_{k+1}, m(k))$$

$\Box$

Proposition 3.3
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{A}^+$ be the associated extended algebra and let $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_k, \ldots, \mathcal{C}_n\}, m\rangle$ be a proof where $k \geq 0$ and $n > k$. Let $m(j) = \mathcal{I}_{C-R}$ for all $k \leq j < n$ and let $\mathcal{A}^+, FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_k)$. Then $\mathcal{A}^+, FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$.

Proof:
For all j, $k \leq j < n$, $\mathcal{C}_{j+1} \subseteq \mathcal{C}_j$, since $m(j) = \mathcal{I}_{C-R}$. Then $\mathcal{C}_n \subseteq \mathcal{C}_k$ and, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_k) \vdash_{FOL} FOT(\mathcal{C}_n)$. Moreover by hypothesis and reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_0) \vdash_{FOL} \mathcal{A}^+, FOT(\mathcal{C}_k)$. Hence, by transitivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$.

■

Lemma 3.1
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{A}^+$ be the extended algebra of $\mathcal{A}$. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of S and let $FOT(\mathcal{C})$ and $FOT(\mathcal{C}')$ be their respective first order translations. If $\mathcal{C} \vdash_S \mathcal{C}'$ then $\mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$.

Proof:
Proof is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$.
In what follows, $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ is a proof of this smallest size.

Base Case
The base case is when $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = 0$. Then $\mathcal{C}_n \subseteq \mathcal{C}_0$, by Definition 3.10, and $FOT(\mathcal{C}_n) \subseteq FOT(\mathcal{C}_0)$. Hence by reflexivity of $\vdash_{FOL}$ $\mathcal{A}^+, FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$.

Inductive Step
Suppose that $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = L$, $L > 0$, and the lemma is true whenever there is a smallest derivation of size less than L.

Without loss of generality, we can assume $m(n-1) \neq \mathcal{I}_{C-R}$. (This is because Proposition 3.3 allows us to extend the end of any proof with any finite number of applications of $\mathcal{I}_{C-R}$ without affecting the statement of the lemma, or the size of the proof). Then there are two cases to consider:

(i) $n = 1$. In this case $n - 1 = 0$ and $l(\mathcal{C}_{n-1}/\mathcal{C}_n, m(n-1)) = L$. It remains to show by cases that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_n)$ for any rule $m(n-1) \in \mathcal{R} - \{\mathcal{I}_{C-R}\}$.

(ii) $n > 1$. In this case $0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, m(n-1)) \leq L$ and $0 \leq l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_{n-1}\}, m'\rangle) < L$ (where $m'(i) = m(i)$ for all $0 \leq i \leq n-2$). Hence by the inductive hypothesis and by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_0) \vdash_{FOL} \mathcal{A}^+, FOT(\mathcal{C}_{n-1})$. So as in case ($i$), it remains to show by cases that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_n)$ for any rule $m(n-1) \in \mathcal{R} - \{\mathcal{I}_{C-R}\}$, since then, by transitivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$.

$\wedge$–Elimination
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\wedge E}$. Then there exists a declarative unit of the form $\lambda : \alpha \wedge \beta \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is either equal to $\mathcal{C}_{n-1} + [\lambda : \alpha]$ or to $\mathcal{C}_{n-1} + [\lambda : \beta]$. We consider only the first case since the argument for the second case is analogous. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $(\alpha \wedge \beta) \in \mathcal{F}_{n-1}(\lambda)$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $[\alpha \wedge \beta]^*(\lambda) \in FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$. This is proved applying axiom schema $(Ax1)$, as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots \\ \forall x([\alpha \wedge \beta]^*(x) \rightarrow ([\alpha]^*(x) \wedge [\beta]^*(x)))\end{array}}{[\alpha \wedge \beta]^*(\lambda) \rightarrow ([\alpha]^*(\lambda) \wedge [\beta]^*(\lambda))} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\alpha \wedge \beta]^*(\lambda)\end{array}}{[\alpha]^*(\lambda) \wedge [\beta]^*(\lambda)}}{[\alpha]^*(\lambda)}$$

$\wedge$–Introduction
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\wedge I}$. Then there exist declarative units of the form $\lambda : \alpha \in \mathcal{C}_{n-1}$, and $\lambda : \beta \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha \wedge \beta]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\{\alpha, \beta\} \subseteq \mathcal{F}_{n-1}(\lambda)$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\alpha \wedge \beta\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $\{[\alpha]^*(\lambda), [\beta]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha \wedge \beta]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \wedge \beta]^*(\lambda)$. This is proved applying axiom schema $(Ax1)$, as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots \\ \forall x([\alpha \wedge \beta]^*(x) \equiv ([\alpha]^*(x) \wedge [\beta]^*(x)))\end{array}}{\forall x(([\alpha]^*(x) \wedge [\beta]^*(x)) \rightarrow [\alpha \wedge \beta]^*(x))}}{([\alpha]^*(\lambda) \wedge [\beta]^*(\lambda)) \rightarrow [\alpha \wedge \beta]^*(\lambda)} \qquad \dfrac{\begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\alpha]^*(\lambda)\end{array} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\beta]^*(\lambda)\end{array}}{[\alpha]^*(\lambda) \wedge [\beta]^*(\lambda)}}{[\alpha \wedge \beta]^*(\lambda)}$$

$\vee$–Introduction
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\vee I}$. Then there exists a declarative unit of the form $\lambda : \alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is either equal to $\mathcal{C}_{n-1} + [\lambda : \alpha \vee \beta]$ or to $\mathcal{C}_{n-1} + [\lambda : \beta \vee \alpha]$. We consider only the first case since the argument for the second case is analogous. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\{\alpha\} \subseteq \mathcal{F}_{n-1}(\lambda)$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\alpha \vee \beta\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $\{[\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha \vee \beta]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that
$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \vee \beta]^*(\lambda)$.
This is proved applying axiom schema $(Ax3)$, as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{A}^+\\ \vdots\\ \forall x([\alpha \vee \beta]^*(x) \equiv ([\alpha]^*(x) \vee [\beta]^*(x)))\end{matrix}}{\forall x([\alpha]^*(x) \vee [\beta]^*(x) \rightarrow [\alpha \vee \beta]^*(x))}}{([\alpha]^*(\lambda) \vee [\beta]^*(\lambda)) \rightarrow [\alpha \vee \beta]^*(\lambda)} \qquad \dfrac{\begin{matrix}FOT(\mathcal{C}_{n-1})\\ \vdots\\ [\alpha]^*(\lambda)\end{matrix}}{[\alpha]^*(\lambda) \vee [\beta]^*(\lambda)}}{[\alpha \vee \beta]^*(\lambda)}$$

$\rightarrow$–Elimination
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\rightarrow E}$. Then there exist declarative units of the form $\lambda : \alpha \rightarrow \beta \in \mathcal{C}_{n-1}$, and $\lambda : \alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \beta]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\{\alpha \rightarrow \beta, \alpha\} \subseteq \mathcal{F}_{n-1}(\lambda)$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\beta\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $\{[\alpha \rightarrow \beta]^*(\lambda), [\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\beta]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\beta]^*(\lambda)$. This is proved applying axiom schema $(Ax4)$, as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{A}^+\\ \vdots\\ \forall x([\alpha \rightarrow \beta]^*(x) \equiv ([\alpha]^*(x) \rightarrow [\beta]^*(x)))\end{matrix}}{[\alpha \rightarrow \beta]^*(\lambda) \rightarrow ([\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda))} \qquad \begin{matrix}FOT(\mathcal{C}_{n-1})\\ \vdots\\ [\alpha \rightarrow \beta]^*(\lambda)\end{matrix}}{[\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda)} \qquad \begin{matrix}FOT(\mathcal{C}_{n-1})\\ \vdots\\ [\alpha]^*(\lambda)\end{matrix}}{[\beta]^*(\lambda)}$$

$\neg$–Elimination

Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\neg E}$. Then there exists a declarative unit of the form $\lambda : \neg\neg\alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1}\rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n\rangle$. Then $\{\neg\neg\alpha\} \subseteq \mathcal{F}_{n-1}(\lambda)$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $\{[\neg\neg\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$. This is proved applying axiom schema $(Ax2)$ as shown in the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots\\ [\neg\neg\alpha]^*(\lambda)\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots\\ [\neg\neg\alpha]^*(\lambda) \equiv \neg[\neg\alpha]^*(\lambda)\end{array}}{[\neg\neg\alpha]^*(\lambda) \rightarrow \neg[\neg\alpha]^*(\lambda)}}{\neg[\neg\alpha]^*(\lambda)} \qquad \dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots\\ \neg[\neg\alpha]^*(\lambda) \equiv \neg\neg[\alpha]^*(\lambda)\end{array}}{\neg[\neg\alpha]^*(\lambda) \rightarrow \neg\neg[\alpha]^*(\lambda)}}{\dfrac{\neg\neg[\alpha]^*(\lambda)}{[\alpha]^*(\lambda)}}$$

$\perp$–Introduction

Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\perp I}$. Then there exists an R–literal Δ such that $\Delta \in \mathcal{C}_{n-1}$ and $\neg\Delta \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha]$ where $\lambda : \alpha$ is any declarative unit. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1}\rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n\rangle$. Then $\{\Delta, \neg\Delta\} \subseteq \mathcal{D}_{n-1}$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $\{\Delta, \neg\Delta\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to prove that

$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$.

Since $\{\Delta, \neg\Delta\} \subseteq FOT(\mathcal{C}_{n-1})$, $FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \perp$. Then $FOT(\mathcal{C}_{n-1})$ is inconsistent and so $FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$. Hence, by monotonicity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$.

$\Diamond$–Elimination

Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\Diamond E}$. Then there exists a declarative unit of the form $\lambda : \Diamond\alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [f_\alpha(\lambda) : \alpha] +$

$[R(\lambda, f_\alpha(\lambda)]$ (where $f_\alpha(\lambda)$ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$). Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\{\Diamond\alpha\} \subseteq \mathcal{F}_{n-1}(\lambda)$, $\mathcal{D}_n = \mathcal{D}_{n-1} \cup \{R(\lambda, f_\alpha(\lambda))\}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(f_\alpha(\lambda)) = \mathcal{F}_{n-1}(f_\alpha(\lambda)) \cup \{\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq f_\alpha(\lambda)$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $\{[\Diamond\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{R(\lambda, f_\alpha(\lambda)), [\alpha]^*(f_\alpha(\lambda))\}$.
Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} R(\lambda, f_\alpha(\lambda))$ and $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(f_\alpha(\lambda))$. These are proved applying axiom schema $(Ax6)$, as shown in the following derivations:

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots \\ \forall x([\Diamond\alpha]^*(x) \to (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x))))\end{array}}{([\Diamond]^*(\lambda) \to (R(\lambda, f_\alpha(\lambda)) \wedge [\alpha]^*(f_\alpha(\lambda))))} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\Diamond\alpha]^*(\lambda)\end{array}}{R(\lambda, f_\alpha(\lambda)) \wedge [\alpha]^*(f_\alpha(\lambda))}
$$

$\Diamond$–Introduction
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\Diamond I}$. Then there exists a declarative unit of the form $\lambda_2 : \alpha \in \mathcal{C}_{n-1}$, an R-literal of the form $R(\lambda_1, \lambda_2) \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda_1 : \Diamond\alpha]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\alpha \in \mathcal{F}_{n-1}(\lambda_2)$, and $R(\lambda_1, \lambda_2) \in \mathcal{D}_{n-1}$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda_1) = \mathcal{F}_{n-1}(\lambda_1) \cup \{\Diamond\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda_1$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore
$\{[\alpha]^*(\lambda_2), R(\lambda_1, \lambda_2)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\Diamond\alpha]^*(\lambda_1)\}$. Since, by reflexivity of $\vdash_{FOL}$,
$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$,
it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\Diamond\alpha]^*(\lambda_1)$. This is proved applying axiom schema $(Ax7)$, as shown in the following derivations:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ R(\lambda_1, \lambda_2)\end{array} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\alpha]^*(\lambda_2)\end{array}}{R(\lambda_1, \lambda_2) \wedge [\alpha]^*(\lambda_2)}}{\exists y(R(\lambda_1, y) \wedge [\alpha]^*(y))} \qquad \begin{array}{c}\mathcal{A}^+\\ \vdots \\ \exists y(R(\lambda_1, y) \wedge [\alpha]^*(y)) \to [\Diamond\alpha]^*(\lambda_1)\end{array}}{[\Diamond\alpha]^*(\lambda_1)}
$$

$\Box$–Elimination
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\Box E}$. Then there exists a declarative unit of the form $\lambda_1 : \Box\alpha \in \mathcal{C}_{n-1}$, an R-literal of the form $R(\lambda_1, \lambda_2) \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda_2 : \alpha]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1}\rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n\rangle$. Then $\Box\alpha \in \mathcal{F}_{n-1}(\lambda_1)$, and $R(\lambda_1, \lambda_2) \in \mathcal{D}_{n-1}$, $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda_2) = \mathcal{F}_{n-1}(\lambda_2) \cup \{\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda_2$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $\{[\Box\alpha]^*(\lambda_1), R(\lambda_1, \lambda_2)\} \subseteq FOT(\mathcal{C}_{n-1})$, and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda_2)\}$.
Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda_2)$. This is proved applying axiom schema (Ax8), as shown in the following derivation:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots \\ [\Box\alpha]^*(\lambda_1) \to \forall y(R(\lambda_1, y) \to [\alpha]^*(y))\end{array} \quad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\Box\alpha]^*(\lambda_1)\end{array}}{\forall y(R(\lambda_1, y) \to [\alpha]^*(y))}}{R(\lambda_1, \lambda_2) \to [\alpha]^*(\lambda_2)} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ R(\lambda_1, \lambda_2)\end{array}}{[\alpha]^*(\lambda_2)}
$$

***R*-Assertion**
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{R-A}$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1}\rangle$. Then there exists an R–literal Δ such that there exists a first–order derivation $\mathcal{D}_{n-1}, \mathcal{A} \vdash_{FOL} \Delta$ and such that $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\Delta]$. Let $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n\rangle$. Then for each $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda)$, and $\mathcal{D}_n = \mathcal{D}_{n-1} \cup \{\Delta\}$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{\Delta\}$.
Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \Delta$. Since $\mathcal{D}_{n-1} \subseteq FOT(\mathcal{C}_{n-1})$, $\mathcal{A} \subseteq \mathcal{A}^+$ and $\mathcal{D}_{n-1}, \mathcal{A} \vdash_{FOL} \Delta$, by monotonicity of $\vdash_{FOL}$,
$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \Delta$.

$\to$–Introduction
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\to I}$. Then there exist declarative units of the form $\lambda : \alpha$, $\lambda : \beta$, $\lambda : \alpha \to \beta$ such that $\mathcal{C}_{n-1} + [\lambda : \alpha] \vdash_S \lambda : \beta$ and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha \to \beta]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1}\rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n\rangle$. Then $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\alpha \to \beta\}$,

and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha \rightarrow \beta]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \rightarrow \beta]^*(\lambda)$.
Let $\langle\{\mathcal{C}_{n-1} + [\lambda:\alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle$, with $\lambda:\beta \in \tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda:\alpha] \vdash_S \lambda:\beta$, condition of $\mathcal{I}_{\rightarrow I}$. By Definition 3.9 and by hypothesis of the inductive step, $0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{\rightarrow I}) = 1 + l_1 \leq L$. Then $0 \leq l_1 = l(\langle\{\mathcal{C}_{n-1} + [\lambda:\alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$. By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} [\beta]^*(\lambda)$. By the Deduction Theorem of first order logic, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda)$. So by axiom schema $(Ax4)$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \rightarrow \beta]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \cdot\\ \cdot\\ \cdot\\ \forall x(([\alpha]^*(x) \rightarrow [\beta]^*(x)) \rightarrow [\alpha \rightarrow \beta]^*(x))\end{array}}{([\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda)) \rightarrow [\alpha \rightarrow \beta]^*(\lambda)} \qquad \dfrac{\begin{array}{c}\mathcal{A}^+\ ,\ \ FOT(\mathcal{C}_{n-1})\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{array}}{[\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda)}}{[\alpha \rightarrow \beta]^*(\lambda)}$$

$\neg$–Introduction
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\neg I}$. Then there exists a declarative unit of the form $\lambda:\alpha$ and a term $\overline{\lambda}$ of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $\mathcal{C}_{n-1} + [\lambda:\alpha] \vdash_S \overline{\lambda}:\perp$ and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda:\neg\alpha]$. Let $\mathcal{C}_{n-1} = \langle\mathcal{D}_{n-1}, \mathcal{F}_{n-1}\rangle$, and $\mathcal{C}_n = \langle\mathcal{D}_n, \mathcal{F}_n\rangle$. Then $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\neg\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $f'(\lambda') = \overline{f}(\lambda')$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\neg\alpha]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that

$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\neg\alpha]^*(\lambda)$.

Let $\langle\{\mathcal{C}_{n-1} + [\lambda:\alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle$, with $\overline{\lambda}:\perp \in \tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda:\alpha] \vdash_S \overline{\lambda}:\perp$, condition of $\mathcal{I}_{\neg I}$. By Definition 3.9 and by hypothesis of the inductive step, $0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{\neg I}) = 1 + l_1 \leq L$. Then $0 \leq l_1 = l(\langle\{\mathcal{C}_{n-1} + [\lambda:\alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$. By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} \perp$. By the Deduction Theorem of first order logic, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda) \rightarrow \perp$. So, since $[\alpha]^*(\lambda) \rightarrow \bot$ is equivalent to $\neg[\alpha]^*(\lambda)$, by axiom schema (Ax2), $\mathcal{A}^+, FOT(\mathcal{C}_n) \vdash_{FOL} [\neg\alpha]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\\ \vdots \\ \neg[\alpha]^*(\lambda)\end{array} \qquad \begin{array}{c}\mathcal{A}^+\\ \vdots \\ \neg[\alpha]^*(\lambda) \rightarrow [\neg\alpha]^*(\lambda)\end{array}}{[\neg\alpha]^*(\lambda)}$$

$\Box$–Introduction

Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\Box I}$. Then there exists an R–literal of the form $R(\lambda, box_\alpha(\lambda))$ and a wff α such that $\mathcal{C}_{n-1} + [R(\lambda, box_\alpha(\lambda))] \vdash_S box_\alpha(\lambda) : \alpha$ and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \Box\alpha]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\Box\alpha\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\Box\alpha]^*(\lambda)\}$.
Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\Box\alpha]^*(\lambda)$. Let $\langle \{\mathcal{C}_{n-1} + [R(\lambda, box_\alpha(\lambda))], \ldots, \tilde{\mathcal{C}}\}, \tilde{m} \rangle$, with $box_\alpha(\lambda) : \alpha \in \tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [R(\lambda, box_\alpha(\lambda))] \vdash_S box_\alpha(\lambda) : \alpha$. By Definition 3.9 and by hypothesis of the inductive step,

$0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{\Box I}) = 1 + l_1 \leq L$. Then

$0 \leq l_1 = l(\langle \{\mathcal{C}_{n-1} + [R(\lambda, box_\alpha(\lambda))], \ldots, \tilde{\mathcal{C}}\}, \tilde{m} \rangle) < L$.

By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{R(\lambda, box_\alpha(\lambda))\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{R(\lambda, box_\alpha(\lambda))\} \vdash_{FOL} [\alpha]^*(box_\alpha(\lambda))$.
By the Deduction Theorem of first order logic,

$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} R(\lambda, box_\alpha(\lambda)) \rightarrow [\alpha]^*(box_\alpha(\lambda))$.

So by axiom schema (Ax8), $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\Box\alpha]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots\end{array}}{(R(\lambda, box_\alpha(\lambda)) \rightarrow [\alpha]^*(box_\alpha(\lambda))) \rightarrow [\Box\alpha]^*(\lambda)} \qquad \dfrac{\begin{array}{c}\mathcal{A}^+,\ FOT(\mathcal{C}_{n-1})\\ \vdots\end{array}}{R(\lambda, box_\alpha(\lambda)) \rightarrow [\alpha]^*(box_\alpha(\lambda))}}{[\Box\alpha]^*(\lambda)}$$

R–**Introduction**
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{R-I}$. Then there exists an R–literal Δ such that $\mathcal{C}_{n-1} + [\overline{\Delta}] \vdash_S \lambda' : \perp$ for some term λ', and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\Delta]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\mathcal{D}_n = \mathcal{D}_{n-1} \cup \{\Delta\}$ and $\mathcal{F}_n$ is such that for each $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda)$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{\Delta\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \Delta$. Let $\langle \{\mathcal{C}_{n-1} + [\overline{\Delta}], \ldots, \tilde{\mathcal{C}}\}, \tilde{m} \rangle$, with $\lambda' : \perp \in \tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\overline{\Delta}] \vdash_S \lambda' : \perp$. By Definition 3.9 and by hypothesis of the inductive step,

$0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{R-I}) = 1 + l_1 \leq L$. Then

$0 \leq l_1 = l(\langle \{\mathcal{C}_{n-1} + [\overline{\Delta}], \ldots, \tilde{\mathcal{C}}\}, \tilde{m} \rangle) < L$.

By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{\overline{\Delta}\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{\overline{\Delta}\} \vdash_{FOL} \perp$. By the Deduction Theorem of first order logic, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \overline{\Delta} \rightarrow \perp$.
Since $\overline{\Delta} \rightarrow \perp$ is equivalent to $\neg\overline{\Delta}$, which is also equivalent to Δ, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \Delta$.

$\vee$–**Elimination**
Suppose that $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\vee E}$. Then there exist declarative units of the form $\lambda : \alpha \vee \beta$, $\lambda : \alpha$, $\lambda : \beta$, and $\lambda : \gamma$, such that $\lambda : \alpha \vee \beta \in \mathcal{C}_{n-1}$, $\mathcal{C}_{n-1} + [\lambda : \alpha] \vdash_S \lambda : \gamma$, $\mathcal{C}_{n-1} + [\lambda : \beta] \vdash_S \lambda : \gamma$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \gamma]$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$, and $\mathcal{C}_n = \langle \mathcal{D}_n, \mathcal{F}_n \rangle$. Then $\mathcal{D}_n = \mathcal{D}_{n-1}$ and $\mathcal{F}_n$ is such that $\mathcal{F}_n(\lambda) = \mathcal{F}_{n-1}(\lambda) \cup \{\gamma\}$, and for each $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}_n(\lambda') = \mathcal{F}_{n-1}(\lambda')$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\gamma]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\gamma]^*(\lambda)$. Let $\langle \{\mathcal{C}_{n-1} + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m} \rangle$, with $\lambda : \gamma \in \tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda : \alpha] \vdash_S \lambda : \gamma$ (the first condition of $\mathcal{I}_{\Box I}$). Analogously, let $\langle \{\mathcal{C}_{n-1} + [\lambda : \beta], \ldots, \tilde{\mathcal{C}}'\}, \tilde{m}' \rangle$, with $\lambda : \gamma \in \tilde{\mathcal{C}}'$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda : \beta] \vdash_S \lambda : \gamma$ (the second condition of $\mathcal{I}_{\Box I}$). By Definition 3.9 and by hypothesis of the inductive step,

$0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{\vee I}) = 1 + l_1 + l_2 \leq L$. Then

$0 \leq l_1 = l(\langle \{\mathcal{C}_{n-1} + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m} \rangle) < L$, and

$0 \leq l_2 = l(\langle \{\mathcal{C}_{n-1} + [\lambda : \beta], \ldots, \tilde{\mathcal{C}}'\}, \tilde{m}' \rangle) < L$.

By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\beta]^*(\lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$. In particular

$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} [\gamma]^*(\lambda)$ and

$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\beta]^*(\lambda)\} \vdash_{FOL} [\gamma]^*(\lambda)$.

By the Deduction Theorem of first order logic,

$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda) \rightarrow [\gamma]^*(\lambda)$ and
$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\beta]^*(\lambda) \rightarrow [\gamma]^*(\lambda)$.

So by axiom schema $(Ax3)$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\gamma]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\\ \vdots\end{array}}{[\alpha]^*(\lambda) \rightarrow [\gamma]^*(\lambda)} \qquad \dfrac{\begin{array}{c}\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\\ \vdots\end{array}}{[\beta]^*(\lambda) \rightarrow [\gamma]^*(\lambda)}}{(([\alpha]^*(\lambda) \rightarrow [\gamma]^*(\lambda)) \wedge ([\beta]^*(\lambda) \rightarrow [\gamma]^*(\lambda)))}}{([\alpha]^*(\lambda) \vee [\beta]^*(\lambda)) \rightarrow [\gamma]^*(\lambda)} \qquad \dfrac{\begin{array}{c}\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\\ \vdots\end{array}}{[\alpha]^*(\lambda) \vee [\beta]^*(\lambda)}}{[\gamma]^*(\lambda)}$$

■

Theorem 3.1 (Soundness)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}$, $\mathcal{C}'$ be two configurations of S. If $\mathcal{C} \vdash_S \mathcal{C}'$ then $\mathcal{C} \models_S \mathcal{C}'$.

Proof:
If $\mathcal{C} \vdash_S \mathcal{C}'$, by Lemma 3.1 $\mathcal{A}^+, FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$, where $\mathcal{A}^+$ is the extended labelling algebra of $\mathcal{A}$, and $FOT(\mathcal{C})$, $FOT(\mathcal{C}')$ are the respective first order translations of $\mathcal{C}$ and $\mathcal{C}'$. By Proposition 3.2 $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$. Rewriting $FOT(\mathcal{C}')$ as $\mathcal{D}' \cup \mathcal{DU}'$, $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$ means that $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} \Delta$ for each $\Delta \in \mathcal{D}'$ and $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda)$ for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$. Hence by Definition 3.8, $\mathcal{C} \models_S \mathcal{C}'$.

■

Discussion

In this section, we have defined a semantics for a MLDS in terms of first–order semantics. The traditional notions of a Kripke model together with the associated satisfiability conditions are embedded in the axiomatization of the extended algebra $\mathcal{A}^+$. In this way, a semantic structure of a MLDS is given by a classical model of $\mathcal{A}^+$. The axiom schemas (Ax7) and (Ax8) semantically characterize the labels generated by the function symbol box_α as 'arbitrary accessible worlds', and the axiom schemas (Ax5) and (Ax6) semantically characterize the labels generated by the function symbol f_α as 'particular accessible worlds'. Corresponding proof theoretical characterizations will be provided in the next section.

The notion of the size of an inference rule has been introduced to simplify the proof of the soundness theorem. The typical features of making and discharging assumptions in a natural deduction proof (see [Fitting-83]) is realized in a MLDS by defining conditions on inference rules (in the two classes of inference rules $\mathcal{I}^{+}$ and $\mathcal{I}^{++}$). Since these conditions are subderivations[8], the definition of a proof, as sequence of rules, is recursive. Nevertheless, a MLDS also includes rules (all members of the class $\mathcal{I}^{0}$) that do not rely on subderivations and so are 'single step' rules. Using the definition of size of inference rules, the former type of rules can be expressed as sequences of single step rules. In the same way, using the definition of size of proof, we can consider a derivation as a sequence of single step rules. As a consequence, the recursive feature of a derivation is eliminated and induction on the size of a proof can be used to prove properties of derivations, as we have shown in the proof of the soundness theorem. In this type of induction, we often use the configuration reduction rule as the base case. In fact, since it infers information already contained in the initial configuration, it is considered as an 'empty'[9] application of single step rules.

[8] They correspond indeed to the notion of 'boxes' defined in [Prawitz-65]

[9] Hence its size has been defined as zero

4 Completeness of a MLDS

In this section we show that the proof system developed in Section 2 is complete with respect to the semantics described in Section 3. In the proof of completeness, we will only consider entailments $\mathcal{C} \models_S \mathcal{C}'$ in which the *configuration difference* (written $\mathcal{C}' - \mathcal{C}$ and defined formally below) is finite [10].

The completeness proof which follows is based on a Henkin style methodology (see for example [Hughes-68]).

We give now a proof theoretical definition of *Consistency*. This definition holds for any MLDS S.

Definition 4.1 (Inconsistent configuration)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}$ be a configuration of S. $\mathcal{C}$ is inconsistent if $\mathcal{C} \vdash \lambda{:}\perp$ for some ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$.

$\square$

Definition 4.2 (Consistent configuration)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}$ be a configuration of S. $\mathcal{C}$ is consistent if it is not inconsistent.

$\square$

In the definition of a configuration, labels are terms of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$. Terms generated by function symbols, f_α and box_α, can be consistentely used in an initial configuration only in certain cases. The following proposition provides syntactic 'constraints' on consistent configurations with respect to the use of such function symbols.

Proposition 4.1 ('Constraints' on consistent configuration)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}\rangle$ be a consistent configuration of S. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. The following two statements hold.

1. If $\Diamond\alpha \in \mathcal{F}(\lambda)$ then $\neg R(\lambda, f_\alpha(\lambda)) \notin \mathcal{D}$ and $\neg\alpha \notin \mathcal{F}(f_\alpha(\lambda))$
2. If $\neg\Box\alpha \in \mathcal{F}(\lambda)$ then $\alpha \notin \mathcal{F}(box_\alpha(\lambda))$ and $\neg R(\lambda, box_\alpha(\lambda)) \notin \mathcal{D}$

   Proof:
   The proof is by contradiction.

[10] We could prove completeness in a more general case, by adjusting the definition of the derivability relation $\vdash_S$

1. Suppose that $\Diamond\alpha \in \mathcal{F}(\lambda)$ and that $\neg R(\lambda, f_\alpha(\lambda)) \in \mathcal{D}$. Then $\mathcal{C} \vdash_S \lambda : \bot$, as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda : \Diamond\alpha, \neg R(\lambda, f_\alpha(\lambda))\rangle & \Diamond\text{-E} \\ \hline \mathcal{C}'\langle R(\lambda, f_\alpha(\lambda)), \neg R(\lambda, f_\alpha(\lambda)), f_\alpha(\lambda) : \alpha\rangle & \bot\text{-I} \\ \hline \mathcal{C}''\langle \lambda : \bot\rangle & \end{array}$$

Now suppose that $\Diamond\alpha \in \mathcal{F}(\lambda)$ and that $\neg\alpha \in \mathcal{F}(f_\alpha(\lambda))$. Then $\mathcal{C} \vdash_S f_\alpha(\lambda) : \bot$ as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda : \Diamond\alpha, f_\alpha(\lambda) : \neg\alpha\rangle & \Diamond\text{-E} \\ \hline \mathcal{C}'\langle R(\lambda, f_\alpha(\lambda)), f_\alpha(\lambda) : \alpha, f_\alpha(\lambda) : \neg\alpha\rangle & \wedge\text{-I} \\ \hline \mathcal{C}''\langle f_\alpha(\lambda) : \bot\rangle & \end{array}$$

2. Suppose that $\neg\Box\alpha \in \mathcal{F}(\lambda)$ and that $\alpha \in \mathcal{F}(box_\alpha(\lambda))$. Then $\mathcal{C} \vdash_S \lambda : \bot$ as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda : \neg\Box\alpha, box_\alpha(\lambda) : \alpha\rangle & \Box\text{-I} \\ \hline \mathcal{C}'\langle \lambda : \Box\alpha, \lambda : \neg\Box\alpha\rangle & \wedge\text{-I} \\ \hline \mathcal{C}''\langle \lambda : \bot\rangle & \end{array}$$

Now suppose that $\neg\Box\alpha \in \mathcal{F}(\lambda)$ and that $\neg R(\lambda, box_\alpha(\lambda)) \in \mathcal{D}$. Then $\mathcal{C} \vdash_S \lambda : \bot$ as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda : \neg\Box\alpha, \neg R(\lambda, box_\alpha(\lambda))\rangle & \\ \hline \mathcal{C}'\langle \neg R(\lambda, box_\alpha(\lambda))\rangle + [R(\lambda, box_\alpha(\lambda))] & \text{(new assumption)} \\ \hline \mathcal{C}'_1\langle box_\alpha(\lambda) : \alpha\rangle & (\bot\text{–I}) \\ \hline \mathcal{C}_1\langle \lambda : \neg\Box\alpha, \lambda : \Box\alpha\rangle & (\Box\text{–I}) \\ \hline \mathcal{C}_2\langle \lambda : \bot\rangle & (\wedge\text{–I}) \end{array}$$

■

We prove now that, given a MLDS S, the derivability relation $\vdash_S$ is monotonic, reflexive and transitive.

Definition 4.3 (Finite configuration)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}\rangle$ be a configuration of S. $\mathcal{C}$ is finite if

- $\mathcal{D}$ is finite
- there exists a finite set $\mathcal{T}$ of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that, for each ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, if $\lambda \in \mathcal{T}$ then $\mathcal{F}(\lambda)$ is finite, and if $\lambda \notin \mathcal{T}$ then $\mathcal{F}(\lambda) = \{\}$

$\square$

Notation 4.1
We write $\mathcal{C} + [\pi_1, \pi_2, \pi_3, \ldots, \pi_n]$ for the configuration $\mathcal{C} + [\pi_1] + [\pi_2] + \ldots + [\pi_n]$, where each π_i, $1 \leq i \leq n$, is a declarative unit or an R–literal.

$\triangleleft$

Definition 4.4 (Configuration difference)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}\rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}'\rangle$ be two configurations of S. The configuration difference $\mathcal{C}' - \mathcal{C}$ is defined as the configuration $\langle \mathcal{D}_{\mathit{diff}}, \mathcal{F}_{\mathit{diff}}\rangle$ where

- $\mathcal{D}_{\mathit{diff}} = \mathcal{D}' - \mathcal{D}$[11]
- $\mathcal{F}_{\mathit{diff}}(\lambda) = \mathcal{F}'(\lambda) - \mathcal{F}(\lambda)$ for each ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$

$\square$

The difference between two configurations, in which one is finite, can be espressed via the following notation.

Notation 4.2
Given a configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}\rangle$, the declarative unit $\lambda : \alpha$ and the R–literal Δ, then

1. $\mathcal{C} - [\lambda : \alpha]$ is the configuration $\langle \mathcal{D}, \mathcal{F}'\rangle$, such that
   - $\mathcal{F}'(\lambda) = \mathcal{F}(\lambda) - \{\alpha\}$
   - $\mathcal{F}'(\lambda') = \mathcal{F}(\lambda')$ for each ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$
2. $\mathcal{C} - [\Delta]$ is the configuration $\langle \mathcal{D}', \mathcal{F}\rangle$, such that
   - $\mathcal{D}' = \mathcal{D} - \{\Delta\}$

$\triangleleft$

[11] We assume the following set theoretic definition of $A - B$ for sets A and B.
$e \in A - B \;\; iff_{def} \;\; e \in A \; and \; e \notin B$.

Definition 4.5 (Configuration union)[12]
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}\rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}'\rangle$, be two configurations of S. The configuration union $\mathcal{C} \cup \mathcal{C}'$ is defined as the configuration $\langle \mathcal{D}_u, \mathcal{F}_u\rangle$ where

- $\mathcal{D}_u = \mathcal{D} \cup \mathcal{D}'$
- $\mathcal{F}_u(\lambda) = \mathcal{F}(\lambda) \cup \mathcal{F}'(\lambda)$ for each ground term λ in $Func(\mathcal{L}_L, \mathcal{L}_M)$

□

Proposition 4.2 (Monotonicity)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be three configurations of S such that $\mathcal{C} \vdash_S \mathcal{C}'$ and $\mathcal{C} \subseteq \mathcal{C}''$. Then $\mathcal{C}'' \vdash_S \mathcal{C}'$.

> **Proof**:
> Let $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ (where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$) be a proof of $\mathcal{C} \vdash_S \mathcal{C}'$. By hypothesis $\mathcal{C} \subseteq \mathcal{C}''$, so $\mathcal{C}''/\mathcal{C} \in \mathcal{I}_{C-R}$. Let m' be a mapping from the set $\{0, \ldots, n\}$ to the set $\mathcal{R}$ such that $m'(0) = \mathcal{I}_{C-R}$ and for each i, $1 \leq i \leq n$, $m'(i) = m(i-1)$. Then the pair $\langle\{\mathcal{C}'', \mathcal{C} \ldots, \mathcal{C}'\}, m'\rangle$ is a proof in S. Hence by Definition 2.3, $\mathcal{C}'' \vdash_S \mathcal{C}'$.

■

Proposition 4.3 (Reflexivity of $\vdash_S$)
Let S be a MLDS, and let $\mathcal{C}$ be a configuration of S, then $\mathcal{C} \vdash_S \mathcal{C}$.

> **Proof**:
> By Definition 2.20, $\mathcal{C}/\mathcal{C} \in \mathcal{I}_{C-R}$. Then let m be a mapping from the set $\{0\}$ to the set $\mathcal{R}$ such that $m(0) = \mathcal{I}_{C-R}$. The pair $\langle\{\mathcal{C}, \mathcal{C}\}, m\rangle$ is a proof in S. Hence by Definition 2.3 $\mathcal{C} \vdash_S \mathcal{C}$.

■

Proposition 4.4 (Transitivity of $\vdash_S$)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}$,$\mathcal{C}'$,$\mathcal{C}''$ be three configurations of S such that $\mathcal{C} \vdash_S \mathcal{C}'$ and $\mathcal{C}' \vdash_S \mathcal{C}''$. Then $\mathcal{C} \vdash_S \mathcal{C}''$.

> **Proof**:
> Let the pair $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_h\}, m\rangle$ (where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_h = \mathcal{C}'$) be a proof of $\mathcal{C} \vdash_S \mathcal{C}'$ and the pair $\langle\{\mathcal{C}'_0, \ldots, \mathcal{C}'_k\}, m'\rangle$ (where $\mathcal{C}'_0 = \mathcal{C}'$ and $\mathcal{C}'_k = \mathcal{C}''$) be a proof of $\mathcal{C}' \vdash_S \mathcal{C}''$. Let $\overline{m}$ be a mapping from the set $\{0, \ldots, h+k-1\}$ to the set $\mathcal{R}$ such that for each i, $0 \leq i \leq h-1$, $\overline{m}(i) = m(i)$, and for each i, $h \leq i \leq h+k-1$, $\overline{m}(i) = m'(i-h)$. Then the pair $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_h, \ldots, \mathcal{C}'_k\}, \overline{m}\rangle$ is a proof of S. Hence by Definition 2.3 $\mathcal{C} \vdash_S \mathcal{C}''$.

■

[12] A union of configurations, in which one is finite, can be expressed via the notation '+' already defined in Section 2

We observe that monotonicity, reflexivity and transitivity properties of the derivability relation $\vdash_S$ hold for both finite and infinite configurations.

Lemma 4.1
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{I}$ be an inference rule and let $\mathcal{C}$, $\mathcal{C}'$ and $\overline{\mathcal{C}}$ be three configurations of S. Let $\mathcal{C}/\mathcal{C}' \in \mathcal{I}$. Then $(\mathcal{C} \cup \overline{\mathcal{C}})/(\mathcal{C}' \cup \overline{\mathcal{C}}) \in \mathcal{I}$.

> **Proof**:
> This follows directly from the definition of $\mathcal{I}$ and from Proposition 4.2 (monotonicity).
>
> ■

Theorem 4.1 (Characterization of derivability)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, and let $\mathcal{C}$, $\mathcal{C}'$ be two configurations of S such that the configuration $\mathcal{C}' - \mathcal{C}$ is finite. $\mathcal{C} \vdash_S \mathcal{C}'$ if and only if for each $\pi \in \mathcal{C}' - \mathcal{C}$, $\mathcal{C} \vdash_S \pi$, where π is either a declarative unit or an R–literal.

> **Proof**:
> 'Only if' half:
> By hypothesis $\mathcal{C} \vdash_S \mathcal{C}'$. Notice that if $\pi \in \mathcal{C}' - \mathcal{C}$ then $\pi \in \mathcal{C}'$. Hence by Notation 2.1 for each $\pi \in \mathcal{C}' - \mathcal{C}$, $\mathcal{C} \vdash_S \pi$.
> 'If' half:
> In order to prove that $\mathcal{C} \vdash_S \mathcal{C}'$, we need to show that there exists a proof $\langle\{\mathcal{C}, \ldots, \mathcal{C}'\}, m\rangle$. Let $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$ be a (possibly empty) enumeration of all the elements (declarative units and R–literals) of the configuration $\mathcal{C}' - \mathcal{C}$. The proof is by induction on n.
>
> **Base Case**
> The base case is when $n = 0$ (empty enumeration). Then $\mathcal{C}' \subseteq \mathcal{C}$. By Definition 2.20, $\mathcal{C}/\mathcal{C}' \in \mathcal{I}_{C-R}$. Then $\langle\{\mathcal{C}_0, \mathcal{C}_1\}, m\rangle$ is a proof, where $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_1 = \mathcal{C}'$ and $m(0) = \mathcal{I}_{C-R}$. So by Definition 2.3 $\mathcal{C} \vdash_S \mathcal{C}'$.
>
> **Inductive Step**
> Assume, by the inductive hypothesis, that for any pair of configurations $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$, such that $\tilde{\mathcal{C}}' - \tilde{\mathcal{C}} = [\tilde{\pi}_1, \ldots, \tilde{\pi}_{n-1}]$, and such that for each $\tilde{\pi} \in \tilde{\mathcal{C}}' - \tilde{\mathcal{C}}$, $\tilde{\mathcal{C}} \vdash_S \tilde{\pi}$, then $\tilde{\mathcal{C}} \vdash_S \tilde{\mathcal{C}}'$.
>
> Let $\overline{\mathcal{C}}$ be the configuration $\mathcal{C} + [\pi_1, \ldots, \pi_{n-1}]$ and let $\mathcal{C}'' = \overline{\mathcal{C}} + [\pi_n]$, so that for each $\pi \in \mathcal{C}'' - \mathcal{C}$, $\mathcal{C} \vdash_S \pi$. Notice that by the inductive hypothesis $\mathcal{C} \vdash_S \overline{\mathcal{C}}$, and, since $\mathcal{C}' \subseteq \mathcal{C}''$, $\mathcal{C}'' \vdash_S \mathcal{C}'$. Therefore, by the transitivity property of $\vdash_S$, in order to prove that $\mathcal{C} \vdash_S \mathcal{C}'$ it is sufficient to prove that $\overline{\mathcal{C}} \vdash_S \mathcal{C}''$. By the original hypothesis, $\mathcal{C} \vdash_S \pi_n$.

Since $\mathcal{C} \subseteq \overline{\mathcal{C}}$, by the monotonicity property of $\vdash_S$, $\overline{\mathcal{C}} \vdash_S \pi_n$. Then by Notation 2.1, there exists a configuration $\mathcal{C}_{\pi_n}$ such that $\overline{\mathcal{C}} \vdash_S \mathcal{C}_{\pi_n}$ and $\pi_n \in \mathcal{C}_{\pi_n}$. Let

$$\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_h\}, m\rangle \tag{i}$$

be a proof of $\overline{\mathcal{C}} \vdash_S \mathcal{C}_{\pi_n}$, where $\mathcal{C}_0 = \overline{\mathcal{C}}$ and $\mathcal{C}_h = \mathcal{C}_{\pi_n}$ and m is a mapping from the set $\{0, \ldots, (h-1)\}$ to the set $\mathcal{R}$. We construct now a corresponding proof

$$\langle\{\tilde{\mathcal{C}_0}, \ldots, \tilde{\mathcal{C}_h}\}, m\rangle \tag{ii}$$

in the following way. $\tilde{\mathcal{C}_0} = \mathcal{C}_0$ and for each $0 \leq i \leq (h-1)$, if $m(i) = \mathcal{I}_{C-R}$ then $\tilde{\mathcal{C}}_{i+1} = \tilde{\mathcal{C}_i}$, otherwise $\tilde{\mathcal{C}}_{i+1} = \tilde{\mathcal{C}_i} \cup \mathcal{C}_{i+1}$. By Lemma 4.1, (ii) is a proof. Moreover, since in any inference rule different from $\mathcal{I}_{C-R}$ the infered configuration contains the antecedent configuration, it follows that $\mathcal{C}_0 \subseteq \tilde{\mathcal{C}_h}$. Since $\mathcal{C}_0 = \overline{\mathcal{C}}$ and $\pi_n \in \tilde{\mathcal{C}_h}$, then $\mathcal{C}'' \subseteq \tilde{\mathcal{C}_h}$. So by Definition 2.20 $\tilde{\mathcal{C}_h}/\mathcal{C}' \in \mathcal{I}_{C-R}$. Then from the proof (ii) we construct a proof

$$\langle\{\tilde{\mathcal{C}_0}, \ldots, \tilde{\mathcal{C}_h}, \tilde{\mathcal{C}}_{h+1}\}, \overline{m}\rangle \tag{iii}$$

where $\tilde{\mathcal{C}}_{h+1} = \mathcal{C}''$ and $\overline{m}$ is a mapping from the set $\{0, \ldots, h\}$ to the set $\mathcal{R}$ such that for each i, $0 \leq i \leq (h-1)$, $\overline{m}(i) = m(i)$ and $\overline{m}(h) = \mathcal{I}_{C-R}$. Hence by Definition 2.3, $\overline{\mathcal{C}} \vdash_S \mathcal{C}''$.

■

We show now two important properties of a consistent configuration.

Proposition 4.5
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. Let $\mathcal{C}$ be a consistent configuration of S and let π be a declarative unit or an R–literal. If $\mathcal{C} + [\pi]$ is a consistent configuration then for any configuration $\mathcal{C}'$, $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' + [\pi]$ is consistent too.

Proof:
The proof is by contradiction. Suppose $\mathcal{C}' + [\pi]$ is not consistent. Then by Definition 4.1 $\mathcal{C}' + [\pi] \vdash_S \lambda : \perp$, for some ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Since $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' + [\pi] \subseteq \mathcal{C} + [\pi]$. Then, by the monotonicity property of $\vdash_S$, Proposition 4.2, $\mathcal{C} + [\pi] \vdash_S \lambda : \perp$. Hence $\mathcal{C} + [\pi]$ is inconsistent which is in contradiction with the original hypothesis.

■

Proposition 4.5, shows a property of the consistency of a configuration $\mathcal{C}$ with respect to the set of its 'sub–configurations' $\mathcal{C}'$ ($\mathcal{C}' \subseteq \mathcal{C}$). No assumption is made about the finiteness of $\mathcal{C}'$. A second property of a consistent configuration $\mathcal{C}$ is proved in Proposition 4.6 (see below) which shows that a configuration is consistent if every finite sub–configuration is consistent. Before proving Proposition 4.6, it is important to prove the following theorem.

Theorem 4.2
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}$ be a configuration of S. Let π be a declarative unit or an R–literal. If $\mathcal{C} \vdash_S \pi$ then there exists a configuration $\mathcal{C}'$ such that $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' \vdash_S \pi$ and $\mathcal{C}'$ is finite.

Proof:
It is only necessary to prove the case when $\mathcal{C}$ is not finite. By hypothesis $\mathcal{C} \vdash_S \pi$. Then by Notation 2.1, there exists a configuration $\overline{\mathcal{C}}$ such that $\mathcal{C} \vdash_S \overline{\mathcal{C}}$ and $\pi \in \overline{\mathcal{C}}$. The proof is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \overline{\mathcal{C}}$. In what follows, $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ is a proof of the smallest size.

Base Case
The base case is when $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = 0$. Then $\mathcal{C}_n \subseteq \mathcal{C}_0$, by Definition 3.10. Since $\pi \in \mathcal{C}_n$, then $\pi \in \mathcal{C}_0$. If π is an R–literal then let $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}'\rangle$, where $\mathcal{D}' = \{\pi\}$ and for each ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}'(\lambda) = \{\}$. If π is a declarative unit, $\lambda : \alpha$, then let $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}'\rangle$, where $\mathcal{D}' = \{\}$, $\mathcal{F}'(\lambda) = \{\alpha\}$ and for each ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}'(\lambda') = \{\}$. In both cases (declarative unit or R–literal), $\mathcal{C}'$ is finite and $\mathcal{C}' \subseteq \mathcal{C}_0$. Notice that $\pi \in \mathcal{C}'$ and $\mathcal{C}'/\mathcal{C}' \in \mathcal{I}_{C-R}$. Then $\langle\{\mathcal{C}'_0, \mathcal{C}'_1\}, m'\rangle$ is a proof, where $\mathcal{C}'_0 = \mathcal{C}'_1 = \mathcal{C}'$ and $m'(0) = \mathcal{I}_{C-R}$. Then by Definition 2.3, $\mathcal{C}' \vdash_S \mathcal{C}'$. Hence by Notation 2.1, $\mathcal{C}' \vdash_S \pi$.

Inductive Step
Assume, by the inductive hypothesis, that for any configuration $\mathcal{C}^*$ and π^* (declarative unit or R–literal) such that there exists a smallest derivation $\langle\{\mathcal{C}^*_0, \ldots, \mathcal{C}^*_n\}, m^*\rangle$ (where $\mathcal{C}^*_0 = \mathcal{C}^*$ and $\pi^* \in \mathcal{C}^*_n$) of size $l(\langle\{\mathcal{C}^*_0, \ldots, \mathcal{C}^*_n\}, m^*\rangle) < L$, then there exists a finite configuration $\mathcal{C}^*_f$ such that $\mathcal{C}^*_f \subseteq \mathcal{C}^*_0$ and $\mathcal{C}^*_f \vdash_S \pi^*$.

Suppose that $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = L$, $L > 0$. We assume without loss of generality that $m(0) \neq \mathcal{I}_{C-R}$ and that $\pi \notin \mathcal{C}_0$. Then $\pi \in \mathcal{C}_n - \mathcal{C}_0$. For all j, $0 \leq j \leq n-1$, and for all $m(j) \in \mathcal{R} - \mathcal{I}_{C-R}$, we write $[\,]_j = \mathcal{C}_{j+1} - \mathcal{C}_j$, where $[\,]_j$ represents the new declarative unit(s) and/or R–literal(s) infered at the step j of the proof. We also assume, without loss of generality, that $n > 1$. Then $0 < l(\mathcal{C}_0/\mathcal{C}_1, m(0)) \leq L$, and $0 \leq l(\langle\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}, m'\rangle) < L$ (where $m'(i) = m(i)$ for all $1 \leq i \leq n-1$)[13]. By the inductive hypothesis there exists a finite configuration $\mathcal{C}_1' \subseteq \mathcal{C}_1$, such that $\mathcal{C}_1' \vdash_S \pi$. Notice that $\mathcal{C}_1 = \mathcal{C}_0 + [\ ]_0$. So if $\mathcal{C}_1' \subseteq \mathcal{C}_0$ then the theorem is proved. Suppose now that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$. Then, by the transitivity property of $\vdash_S$, it remains to show that there exists a finite configuration $\mathcal{C}' \subseteq \mathcal{C}_0$ such that $\mathcal{C}' \vdash_S \mathcal{C}_1'$. We prove it by cases on $m(0)$.

$m(0) = \wedge$–**Elimination**
Suppose that $m(0) = \mathcal{I}_{\wedge-E}$. $\mathcal{C}_0/\mathcal{C}_1 \in \mathcal{I}_{\wedge-E}$. Then there exists a declarative unit of the form $\lambda : \alpha \wedge \beta \in \mathcal{C}_0$ and $\mathcal{C}_1$ is either equal to $\mathcal{C}_0 + [\lambda : \alpha]$ or equal to $\mathcal{C}_0 + [\lambda : \beta]$. We consider the first case since the argument for the second case is analogous. $[\ ]_0 = [\lambda : \alpha]$. By the inductive hypothesis, $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. By assumption $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$. So $[\lambda : \alpha] \in \mathcal{C}_1'$, but it is not necessary the case that $\lambda : \alpha \wedge \beta \in \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\lambda : \alpha]) + [\lambda : \alpha \wedge \beta]$. Then $\mathcal{C}'$ is a finite configuration and $\langle\{\mathcal{C}', \mathcal{C}' + [\lambda : \alpha], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\wedge E}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_S \mathcal{C}_1'$.

R–**Assertion**
Suppose that $m(0) = \mathcal{I}_{R-A}$ so that $\mathcal{C}_0/\mathcal{C}_1 \in \mathcal{I}_{R-A}$. Let $\mathcal{C}_0 = \langle \mathcal{D}_0, \mathcal{F}_0 \rangle$. Then there exists an R–literal Δ such that $\mathcal{D}_0, \mathcal{A} \vdash_{FOL} \Delta$. $\mathcal{C}_1 = \mathcal{C}_0 + [\Delta]$. Let $\Gamma \subseteq \mathcal{D}_0$ be the set of R–literals assumptions used in the first–order derivation $\mathcal{D}_0, \mathcal{A} \vdash_{FOL} \Delta$. Since a proof in classical logic is a finite sequence of inference rules and each inference rule uses a finite set of assumptions, then Γ is finite. Notice now that $[\ ]_0 = [\Delta]$. By the inductive hypothesis, $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. By assumption $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$. So $[\Delta] \in \mathcal{C}_1'$, but it is not necessary the case that $\Gamma \subseteq \mathcal{C}_1'$. Let $\mathcal{C}' = (\mathcal{C}_1' - [\Delta]) + \Gamma$. Then $\mathcal{C}'$ is a finite configuration and $\langle\{\mathcal{C}', \mathcal{C}' + [\Delta], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{R-A}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_S \mathcal{C}_1'$.

The argument for any other inference rule $\mathcal{I}_i \in \mathcal{I}^0$, such that $m(0) = \mathcal{I}_i$ and $\mathcal{C}_0/\mathcal{C}_1 \in m(0)$, is analougous to $\wedge$–Elimination.

$\rightarrow$–**Introduction**
Suppose that $m(0) = \mathcal{I}_{\rightarrow I}$. $\mathcal{C}_0/\mathcal{C}_1 \in \mathcal{I}_{\rightarrow I}$. Then there exist declarative

[13]For $n = 1$ it is always possible to extend a smallest size proof $\langle\{\mathcal{C}_0, \mathcal{C}_1\}, m(0)\rangle$, where $m(0) \in \mathcal{R}$, to a proof of the same size of the form $\langle\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_1\}, m'\rangle$, where $m'(0) = m(0)$ and $m'(1) = \mathcal{I}_{C-R}$

units of the form $\lambda:\alpha$, $\lambda:\beta$, $\lambda:\alpha\rightarrow\beta$ such that $\mathcal{C}_0+[\lambda:\alpha]\vdash_S\lambda:\beta$ and $\mathcal{C}_1=\mathcal{C}_0+[\lambda:\alpha\rightarrow\beta]$. Then $[\;]_0=[\lambda:\alpha\rightarrow\beta]$. By the inductive hypothesis $\mathcal{C}_1^{'}\subseteq\mathcal{C}_1$ and $\mathcal{C}_1^{'}$ is finite. By assumption $\mathcal{C}_1^{'}\not\subseteq\mathcal{C}_0$, so that $[\lambda:\alpha\rightarrow\beta]\in\mathcal{C}_1^{'}$. Let $\langle\{\mathcal{C}_0+[\lambda:\alpha],\ldots,\tilde{\mathcal{C}}_n\},\tilde{m}\rangle$ (with $\lambda:\beta\in\tilde{\mathcal{C}}_n$) be a proof of the smallest size of $\mathcal{C}_0+[\lambda:\alpha]\vdash_S\lambda:\beta$. Since by hypothesis $0<l(\mathcal{C}_0/\mathcal{C}_1,\mathcal{I}_{\rightarrow I})\leq L$, then by Definition 3.9, $0\leq l(\langle\{\mathcal{C}_0+[\lambda:\alpha],\ldots,\tilde{\mathcal{C}}_n\},\tilde{m}\rangle)<L$. Then by the inductive hypothesis, there exists a finite configuration $\mathcal{C}_0^{*}\subseteq\mathcal{C}_0+[\lambda:\alpha]$ such that $\mathcal{C}_0^{*}\vdash_S\lambda:\beta$. Let $\mathcal{C}^{'}$ be the configuration $(\mathcal{C}_1^{'}-[\lambda:\alpha\rightarrow\beta])\cup(\mathcal{C}_0^{*}-[\lambda:\alpha])$. Then $\mathcal{C}^{'}$ is a finite configuration and $\mathcal{C}^{'}\subseteq\mathcal{C}_0$. Since $\mathcal{C}_0^{*}\subseteq\mathcal{C}^{'}+[\lambda:\alpha]$, by the monotonicity property of $\vdash_S$, $\mathcal{C}^{'}+[\lambda:\alpha]\vdash_S[\lambda:\beta]$. So by Definition 2.9, $\mathcal{C}^{'}/\mathcal{C}^{'}+[\lambda:\alpha\rightarrow\beta]\in\mathcal{I}_{\rightarrow I}$. Therefore $\langle\{\mathcal{C}^{'},\mathcal{C}^{'}+[\lambda:\alpha\rightarrow\beta],\mathcal{C}_1^{'}\},\overline{m}\rangle$ is a proof, where $\overline{m}(0)=\mathcal{I}_{\rightarrow I}$ and $\overline{m}(1)=\mathcal{I}_{C-R}$. Hence $\mathcal{C}^{'}\vdash_S\mathcal{C}_1^{'}$.

The argument for any other inference rule $\mathcal{I}_i\in\mathcal{I}^{+}$, such that $m(0)=\mathcal{I}_i$ and $\mathcal{C}_0/\mathcal{C}_1\in m(0)$, is analogous to $\rightarrow$–Introduction.

$\vee$–**Elimination**

Suppose that $m(0)=\mathcal{I}_{\vee E}$. $\mathcal{C}_0/\mathcal{C}_1\in\mathcal{I}_{\vee E}$. Then there exist declarative units of the form $\lambda:\alpha$, $\lambda:\beta$, $\lambda:\alpha\vee\beta$ and $\lambda:\gamma$ such that $\lambda:\alpha\vee\beta\in\mathcal{C}_0$, $\mathcal{C}_0+[\lambda:\alpha]\vdash_S\lambda:\gamma$, $\mathcal{C}_0+[\lambda:\beta]\vdash_S\lambda:\gamma$ and $\mathcal{C}_1$ is equal to $\mathcal{C}_0+[\lambda:\gamma]$. Then $[\;]_0=[\lambda:\gamma]$. By the inductive hypothesis $\mathcal{C}_1^{'}\subseteq\mathcal{C}_1$ and $\mathcal{C}_1$ is finite. By assumption $\mathcal{C}_1^{'}\not\subseteq\mathcal{C}_0$, so that $[\lambda:\gamma]\in\mathcal{C}_1^{'}$. Let $\langle\{\mathcal{C}_0+[\lambda:\alpha],\ldots,\tilde{\mathcal{C}}\},\tilde{m}\rangle$ (with $\lambda:\gamma\in\tilde{\mathcal{C}}$) be a proof of the smallest size of $\mathcal{C}_0+[\lambda:\alpha]\vdash_S\lambda:\gamma$. Analogously let $\langle\{\mathcal{C}_0+[\lambda:\beta],\ldots,\tilde{\mathcal{C}}^{'}\},\tilde{m}^{'}\rangle$ (with $\lambda:\gamma\in\tilde{\mathcal{C}}^{'}$) be a proof of the smallest size of $\mathcal{C}_0+[\lambda:\beta]\vdash_S\lambda:\gamma$. Since by hypothesis $0<l(\mathcal{C}_0/\mathcal{C}_1,\mathcal{I}_{\vee I})\leq L$, by Definition 3.9

$$0\leq l(\langle\{\mathcal{C}_0+[\lambda:\alpha],\ldots,\tilde{\mathcal{C}}\},\tilde{m}\rangle)<L \text{ and}$$
$$0\leq l(\langle\{\mathcal{C}_0+[\lambda:\beta],\ldots,\tilde{\mathcal{C}}^{'}\},\tilde{m}^{'}\rangle)<L.$$

By the inductive hypothesis, there exist finite configurations $\mathcal{C}^{*}\subseteq\mathcal{C}_0+[\lambda:\alpha]$ and $\tilde{\mathcal{C}}^{*}\subseteq\mathcal{C}_0+[\lambda:\beta]$ such that $\mathcal{C}^{*}\vdash_S\lambda:\gamma$ and $\tilde{\mathcal{C}}^{*}\vdash_S\lambda:\gamma$. Let $\mathcal{C}^{'}$ be the configuration

$$((\mathcal{C}_1^{'}-[\lambda:\gamma])+[\lambda:\alpha\vee\beta])\cup(\mathcal{C}^{*}-[\lambda:\alpha])\cup(\tilde{\mathcal{C}}^{*}-[\lambda:\beta]).$$

Then $\mathcal{C}^{'}$ is a finite configuration, $\mathcal{C}^{'}\subseteq\mathcal{C}_0$ and $\lambda:\alpha\vee\beta\in\mathcal{C}^{'}$. Moreover since $\mathcal{C}^{*}\subseteq\mathcal{C}^{'}+[\lambda:\alpha]$, by the monotonicity property of $\vdash_S$, $\mathcal{C}^{'}+[\lambda:\alpha]\vdash_S\lambda:\gamma$. Analogously, since $\tilde{\mathcal{C}}^{*}\subseteq\mathcal{C}^{'}+[\lambda:\beta]$, by the monotonicity property of $\vdash_S$, $\mathcal{C}^{'}+[\lambda:\beta]\vdash_S\lambda:\gamma$. By Definition 2.8, $\mathcal{C}^{'}/\mathcal{C}^{'}+[\lambda:\gamma]\in\mathcal{I}_{\vee E}$. Therefore $\langle\{\mathcal{C}^{'},\mathcal{C}^{'}+[\lambda:\gamma],\mathcal{C}_1^{'}\},\overline{m}\rangle$ is a proof, where $\overline{m}(0)=\mathcal{I}_{\vee E}$ and $\overline{m}(1)=\mathcal{I}_{C-R}$.

∎

Proposition 4.6
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. Let $\mathcal{C}$ be a configuration of S. If for any finite configuration $\mathcal{C}'$, $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}'$ is consistent then $\mathcal{C}$ is consistent.

> **Proof**:
> We prove the contrapositive statement. Suppose that $\mathcal{C}$ is inconsistent. By definition of inconsistency, $\mathcal{C} \vdash_S \lambda : \bot$, for some ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Then, by Theorem 4.2, there exists a finite configuration $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathcal{C}' \vdash_S \lambda : \bot$, so that $\mathcal{C}'$ is inconsistent.

■

So far we have defined the notion of a consistent configuration of a MLDS S and we have proved some consistency properties. We are now interested in showing that the MLDS S proposed in Section 2 is complete with respect to the semantics defined in Section 3. Informally the completeness theorem, whose proof is given below, shows that given a MLDS S and two configurations $\mathcal{C}$, $\mathcal{C}'$ of S, such that their configuration difference $\mathcal{C}' - \mathcal{C}$ is finite, if $\mathcal{C}'$ is semantically entailed from $\mathcal{C}$ then $\mathcal{C}'$ is also derived from $\mathcal{C}$. We prove this following the Henkin–style methodology [Hughes-68]. We give here an informal description of the proof, underlining the main steps and theorems that will be used. The overall structure of the proof is illustrated in Figure 4.

The proof will be done by contrapositive. The contrapositive equivalent statement of the completeness theorem says that given two configurations $\mathcal{C}$ and $\mathcal{C}'$, such that their configuration difference $\mathcal{C}' - \mathcal{C}$ is finite, if $\mathcal{C}'$ is not derivable from $\mathcal{C}$ then $\mathcal{C}'$ is not semantically entailed from $\mathcal{C}$. This statement corresponds to the arrow labelled with (1) in the diagram below. Arrow (1) is also given by the composition of the arrows (2) and (3). In this composition, arrow (2) is already given by Definition 3.8, while arrow (3) represents the main part of the completeness theorem. Its proof is based on the statement *if $\mathcal{C}$ is a consistent configuration then $\mathcal{C}$ is satisfiable*, known as 'Model Existence Lemma', and informally it will be proved in the following way.

The hypothesis that $\mathcal{C}'$ is not derivable from $\mathcal{C}$, $\mathcal{C} \not\vdash_S \mathcal{C}'$, implies, by Theorem 4.1, that there exists a $\pi \in \mathcal{C}' - \mathcal{C}$ (where π is a declarative unit or an R–literal) such that $\mathcal{C} \not\vdash_S \pi$. So $\mathcal{C} + [\neg\pi]$ is a consistent configuration (see Proposition 4.7 below). So, by the *Model Existence Lemma* (see Lemma 4.2 below) and by Corollary 4.1, the configuration $\mathcal{C} + [\neg\pi]$ is satisfiable. This means, by the definition of satisfiability of a configuration, that there exists a semantic structure $\mathcal{M}$ of S (i.e. a model of the associated extended algebra $\mathcal{A}^+$) which satisfies $\mathcal{C}$ and that also satisfies $\neg\pi$. Then, as will be shown below, the semantic structure

$\mathcal{M}$ does not satisfy π. So, since $\pi \in \mathcal{C}'$, by the definition of satisfiability of a configuration, $\mathcal{M}$ does not satisfy $\mathcal{C}'$. This proves arrow (3) in the diagram below.

Arrow (3) and Definition 3.8 (arrow(2) in the diagram below), allow us to conclude that the configuration $\mathcal{C}'$ is not semantically entailed from $\mathcal{C}$.

$$\begin{tikzcd}
\mathcal{C} \nvdash_S \mathcal{C}' \arrow[r, "(1)"] \arrow[dr, "(2)"] & \mathcal{C} \not\models_S \mathcal{C}' \\
 & \mathcal{A}^+ \cup FOT(\mathcal{C}) \not\models_{FOL} FOT(\mathcal{C}') \arrow[u, "(3)"']
\end{tikzcd}$$

Figure 2 - Proof of the Completeness Theorem

As pointed out above, the main step of the completeness proof is then arrow (3) (Figure 4) whose proof is based on the *Model Existence Lemma* 4.2. The proof of this lemma is based on the construction of a *maximal consistent configuration*. By the term 'maximal consistent configuration', we mean a consistent configuration of a MLDS S, such that any declarative unit or R–literal not already part of it, if added, would make it inconsistent. Then 'model existence lemma' will be proved by showing that for any consistent configuration it is possible to construct a maximal consistent configuration (Proposition 4.8 below) which includes it and for which there exists a semantic structure of S that satisfies it. Then such semantic structure will also satisfy the initial consistent configuration (Remark 4.1 below).

Proposition 4.7
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}$ be a configuration of S. Let π be a declarative unit or an R–literal such that $\pi \notin \mathcal{C}$. If $\mathcal{C} \nvdash_S \pi$ then $\mathcal{C} + [\neg\pi]$ is a consistent configuration.

> **Proof**:
> There are two cases to consider: (i) π is a declarative unit and (ii) π is an R–literal.
>
> (i) Suppose that π is a declarative unit of the form $\lambda : \alpha$. We prove the contrapositive of the proposition statement. Assume that $\mathcal{C} + [\lambda : \neg\alpha]$ is not consistent. By Definition 4.1, $\mathcal{C} + [\lambda : \neg\alpha] \vdash_S \lambda' : \perp$, for some ground term λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Let $\mathcal{C}' = \mathcal{C} + [\lambda : \neg\neg\alpha]$ and $\mathcal{C}^* = \mathcal{C}' + [\lambda : \alpha]$. By Definition 2.11, the pair of configurations

$\mathcal{C}/\mathcal{C}' \in \mathcal{I}_{\neg I}$ and by Definition 2.12, the pair of configurations $\mathcal{C}'/\mathcal{C}^* \in \mathcal{I}_{\neg E}$. So $\langle\{\mathcal{C},\mathcal{C}',\mathcal{C}^*\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{\neg I}$ and $m(1) = \mathcal{I}_{\neg E}$. Since $\lambda:\alpha \in \mathcal{C}^*$, by Notation 2.1, $\mathcal{C} \vdash_S \lambda:\alpha$.

(ii) Suppose that π is an R–literal Δ. We prove the contrapositive of the proposition statement. Assume that $\mathcal{C} + [\neg\Delta]$ is not consistent. By Definition 4.1, $\mathcal{C} + [\neg\Delta] \vdash_S \lambda' : \bot$, for some ground term λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Then by Definition 2.18, the pair of configurations $\mathcal{C}/\mathcal{C} + [\Delta] \in \mathcal{I}_{R-I}$. So $\langle\{\mathcal{C}, \mathcal{C} + [\Delta]\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{R-I}$. Hence by Notation 2.1, $\mathcal{C} \vdash_S \Delta$.

■

Definition 4.6 (Maximal Consistent Configuration, $\mathcal{C}_{mcc}$)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. $\mathcal{C}_{mcc}$ is a maximal consistent configuration relative to S, if it is a consistent configuration and if for any $\pi \notin \mathcal{C}_{mcc}$ (where π is a declarative unit or an R–literal), the configuration $\mathcal{C}_{mcc} + [\pi]$ is not consistent.
□

We are going now to show that given a consistent configuration $\mathcal{C}$ of S, we can always construct a maximal consistent configuration $\mathcal{C}_{mcc}$ which contains it. To do this we assume that the set of all declarative units and R–literals of S is ordered so that we can speak of the 1st, 2nd, 3rd, ..., nth, ..., etc. element π of S (where π is a declarative unit or an R–literal). Using this assumption, we show in Definition 4.7 and Proposition 4.8, how to expand an initial consistent configuration $\mathcal{C}$ into a maximal consistent configuration $\mathcal{C}_{mcc}$. Informally, the construction is based on the following procedure. We start from the configuration $\mathcal{C}$, and we go through all the elements π_i of S (where π_i is a declarative unit or an R–literal) in the ordering we have chosen, adding each in turn if and only if it can consistently be done. This is formally defined as follows.

Definition 4.7 (Contruction of $MCC(\mathcal{C})$)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. Let $\pi_1, \pi_2, \pi_3, \ldots, \pi_n, \ldots$ be an ordering on the set of all declarative units and R–literals of S[14]. Let $\mathcal{C}$ be a consistent configuration of S. Let $\mathcal{C}_0 = \mathcal{C}$.
Consider the first element π_1 in the chosen ordering. If $\mathcal{C}_0 + [\pi_1]$ is consistent then let $\mathcal{C}_1 = \mathcal{C}_0 + [\pi_1]$, otherwise let $\mathcal{C}_1 = \mathcal{C}_0$. Then take the second element π_2 of the chosen ordering. If $\mathcal{C}_1 + [\pi_2]$ is consistent then let $\mathcal{C}_2 = \mathcal{C}_1 + [\pi_2]$, otherwise let $\mathcal{C}_2 = \mathcal{C}_1$. Then apply the same process on each element π_i of S in turn according to the chosen ordering.

[14]For each $i \geq 0$, π_i is a declarative unit or an R–literal

$$
\begin{array}{lll}
\mathcal{C}_0 = & \mathcal{C} & \\
\mathcal{C}_1 = & \mathcal{C}_0 + [\pi_1] & \text{if } \mathcal{C}_0 + [\pi_1] \text{ is consistent} \\
\mathcal{C}_1 = & \mathcal{C}_0 & \text{otherwise} \\
\ldots & \ldots & \\
\mathcal{C}_n = & \mathcal{C}_{n-1} + [\pi_n] & \text{if } \mathcal{C}_{n-1} + [\pi_n] \text{ is consistent} \\
\mathcal{C}_n = & \mathcal{C}_n & \text{otherwise} \\
\ldots & \ldots &
\end{array}
$$

Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n, \ldots$ be the sequence of configurations constructed above. $MCC(\mathcal{C})$ is the configuration which contains all the elements π_i (declarative units and R–literals) which are in any configuration $\mathcal{C}_i$.

$$MCC(\mathcal{C}) \;=\; \bigcup_{i \geq 0} \mathcal{C}_i$$

□

We will usually write $MCC(\mathcal{C})$ as $\mathcal{C}_{mcc}$.

Remark 4.1
The sequence of configurations $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n, \ldots$, described in Definition 4.7, is such that for each $i \geq 0$, $\mathcal{C}_i \subseteq \mathcal{C}_{mcc}$. Moreover, the configuration $\mathcal{C}_0$ is consistent by assumption ($\mathcal{C}_0 = \mathcal{C}$), and for each $i \geq 0$, if $\mathcal{C}_i$ is consistent then by 'construction' also $\mathcal{C}_{i+1}$ is consistent. Hence for each $i \geq 0$, $\mathcal{C}_i$ is a consistent configuration.

∘

We prove now that the configuration $\mathcal{C}_{mcc}$, described in Definition 4.7, is a maximal consistent configuration.

Proposition 4.8
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{C}$ be a consistent configuration and let $\mathcal{C}_{mcc}$ be the configuration specified in Definition 4.7. The two following statements hold

1. $\mathcal{C}_{mcc}$ is consistent
2. $\mathcal{C}_{mcc}$ is maximal

Proof:

1. To prove that $\mathcal{C}_{mcc}$ is a consistent configuration we use Proposition 4.6. Let $\mathcal{C}'$ be a finite configuration such that $\mathcal{C}' \subseteq \mathcal{C}_{mcc}$. Enumerate all the elements (declarative units and R–literals) of $\mathcal{C}'$ according to the ordering chosen in Definition 4.7. Let π_n be the last element of $\mathcal{C}'$. Then $\mathcal{C}' \subseteq \mathcal{C}_n$. By Remark 4.1, $\mathcal{C}_n$ is a consistent configuration. Then, by Proposition 4.5, $\mathcal{C}'$ is also consistent. Hence, by Proposition 4.6, $\mathcal{C}_{mcc}$ is a consistent configuration.

2. To prove that $\mathcal{C}_{mcc}$ is a maximal configuration, it is sufficient to prove, by Definition 4.6, that for any π of S (where π is a declarative unit or an R–literal), which is consistent with $\mathcal{C}_{mcc}$, $\pi \in \mathcal{C}_{mcc}$. Let π_n be a declarative unit or an R–literal of S, in the chosen ordering specified in Definition 4.7, such that π_n is consistent with $\mathcal{C}_{mcc}$. Since $\mathcal{C}_{mcc}$ is a consistent configuration, by Proposition 4.5, for any configuration $\mathcal{C}' \subseteq \mathcal{C}_{mcc}$, $\mathcal{C}' + [\pi_n]$ is also a consistent configuration. Let $\mathcal{C}_{n-1}$ be the configuration specified in Definition 4.7. $\mathcal{C}_{n-1} \subseteq \mathcal{C}_{mcc}$, by Remark 4.1. Then $\mathcal{C}_{n-1} + [\pi_n]$ is a consistent configuration. Therefore by Definition 4.7, $\mathcal{C}_{n-1} + [\pi_n] = \mathcal{C}_n$, where by Remark 4.1 $\mathcal{C}_n \subseteq \mathcal{C}_{mcc}$. Hence $\pi_n \in \mathcal{C}_{mcc}$.

■

So far we have shown that given a consistent configuration $\mathcal{C}$, it is always possible to construct a maximal consistent configuration $\mathcal{C}_{mcc}$ relative to S, such that $\mathcal{C} \subseteq \mathcal{C}_{mcc}$. We are now going to prove some properties of a maximal consistent configuration.

Proposition 4.9
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. Then for any declarative unit $\lambda : \alpha$, $\lambda : \alpha$ and $\lambda : \neg\alpha$ are not both in $\mathcal{C}_{mcc}$.

Proof:
The proof follows directly from the definition of $\wedge$–Introduction rule.

■

Proposition 4.10
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. Then for any R–literal Δ, Δ and $\neg\Delta$ are not both in $\mathcal{C}_{mcc}$.

Proof:
The proof follows directly from the definition of $\perp$–Introduction rule.

■

Proposition 4.11
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. For any declarative unit $\lambda : \alpha$, either $\lambda : \alpha \in \mathcal{C}_{mcc}$ or $\lambda : \neg\alpha \in \mathcal{C}_{mcc}$.

Proof:

The proof is by contradiction. Suppose that $\lambda:\alpha \notin \mathcal{C}_{mcc}$ and that $\lambda:\neg\alpha \notin \mathcal{C}_{mcc}$. Then, since $\mathcal{C}_{mcc}$ is a maximal consistent configuration, $\mathcal{C}_{mcc} + [\lambda:\alpha]$ and $\mathcal{C}_{mcc} + [\lambda:\neg\alpha]$ are inconsistent configurations. By Definition 4.1 $\mathcal{C}_{mcc} + [\lambda:\alpha] \vdash_S \lambda^{'}:\bot$, and $\mathcal{C}_{mcc} + [\lambda:\neg\alpha] \vdash_S \lambda^{''}:\bot$, for some ground terms $\lambda^{'}$, $\lambda^{''}$ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. By Theorem 4.2, there exist two finite configurations $\mathcal{C}_1 \subseteq \mathcal{C}_{mcc}$ and $\mathcal{C}_2 \subseteq \mathcal{C}_{mcc}$, such that $\mathcal{C}_1 + [\lambda:\alpha] \vdash_S \lambda^{'}:\bot$, and $\mathcal{C}_2 + [\lambda:\neg\alpha] \vdash_S \lambda^{''}:\bot$. Let $\mathcal{C}^{'}$ be the configuration $\mathcal{C}^{'} = \mathcal{C}_1 \cup \mathcal{C}_2$. By monotonicity, $\mathcal{C}^{'} + [\lambda:\alpha] \vdash_S \lambda^{'}:\bot$. So $\mathcal{C}^{'}/\mathcal{C}^{'} + [\lambda:\neg\alpha] \in \mathcal{I}_{\neg I}$ and $\langle\{\mathcal{C}^{'}, \mathcal{C}^{'} + [\lambda:\neg\alpha]\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{\neg I}$. So $\mathcal{C}^{'} \vdash_S \mathcal{C}^{'} + [\lambda:\neg\alpha]$. By the monotonicity property of $\vdash_S$, $\mathcal{C}^{'} + [\lambda:\neg\alpha] \vdash_S \lambda^{''}:\bot$. Then by the transitivity property of $\vdash_S$, $\mathcal{C}^{'} \vdash_S \lambda^{''}:\bot$. Since $\mathcal{C}^{'} \subseteq \mathcal{C}_{mcc}$, by the monotonicity property of $\vdash_S$, $\mathcal{C}_{mcc} \vdash_S \lambda^{''}:\bot$. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

■

Proposition 4.12
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. For any R–literal Δ either $\Delta \in \mathcal{C}_{mcc}$ or $\neg\Delta \in \mathcal{C}_{mcc}$.

Proof:

The proof is by contradiction. Suppose that $\Delta \notin \mathcal{C}_{mcc}$ and that $\neg\Delta \notin \mathcal{C}_{mcc}$. Since $\mathcal{C}_{mcc}$ is a maximal consistent configuration, $\mathcal{C}_{mcc} + [\Delta]$ and $\mathcal{C}_{mcc} + [\neg\Delta]$ are inconsistent configurations. Then by Definition 4.1, $\mathcal{C}_{mcc} + [\Delta] \vdash_S \lambda^{'}:\bot$ and $\mathcal{C}_{mcc} + [\neg\Delta] \vdash_S \lambda^{''}:\bot$, for some ground terms $\lambda^{'}$, $\lambda^{''}$ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. By Theorem 4.2, there exist two finite configurations $\mathcal{C}_1 \subseteq \mathcal{C}_{mcc}$ and $\mathcal{C}_2 \subseteq \mathcal{C}_{mcc}$, such that $\mathcal{C}_1 + [\Delta] \vdash_S \lambda^{'}:\bot$, and $\mathcal{C}_2 + [\neg\Delta] \vdash_S \lambda^{''}:\bot$. Let $\mathcal{C}^{'}$ be the configuration $\mathcal{C}^{'} = \mathcal{C}_1 \cup \mathcal{C}_2$. By the monotonicity property of $\vdash_S$, $\mathcal{C}^{'} + [\Delta] \vdash_S \lambda^{'}:\bot$. Then $\mathcal{C}^{'}/\mathcal{C}^{'} + [\neg\Delta] \in \mathcal{I}_{R-I}$ and $\langle\{\mathcal{C}^{'}, \mathcal{C}^{'} + [\neg\Delta]\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{R-I}$. Then $\mathcal{C}^{'} \vdash_S \mathcal{C}^{'} + [\neg\Delta]$. Moreover by the monotonicity property of $\vdash_S$, $\mathcal{C}^{'} + [\neg\Delta] \vdash_S \lambda^{''}:\bot$. Then by the transitivity property of $\vdash_S$, $\mathcal{C}^{'} \vdash_S \lambda^{''}:\bot$. Since $\mathcal{C}^{'} \subseteq \mathcal{C}_{mcc}$, by the monotonicity property of $\vdash_S$, $\mathcal{C}_{mcc} \vdash_S \lambda^{''}:\bot$. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

■

Proposition 4.13
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α and β be two wffs of $\mathcal{L}_M$. Then $\lambda{:}\,\alpha \wedge \beta \in \mathcal{C}_{mcc}$, if and only if $\lambda{:}\,\alpha \in \mathcal{C}_{mcc}$ and $\lambda{:}\,\beta \in \mathcal{C}_{mcc}$.

Proof:
<u>'Only if' half:</u>
The proof is by contradiction. By hypothesis $\lambda{:}\,\alpha \wedge \beta \in \mathcal{C}_{mcc}$, and we assume that either $\lambda{:}\,\alpha \notin \mathcal{C}_{mcc}$ or $\lambda{:}\,\beta \notin \mathcal{C}_{mcc}$. We consider only the first case since the argument for the second case is analogous. If $\lambda{:}\,\alpha \notin \mathcal{C}_{mcc}$, then by Proposition 4.11 $\lambda{:}\,\neg\alpha \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc}$ is inconsistent as shown in the following derivation. This is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle\lambda{:}\,\alpha\wedge\beta, \lambda{:}\,\neg\alpha\rangle}{\mathcal{C}'\langle\lambda{:}\,\alpha, \lambda{:}\,\neg\alpha\rangle}\ \wedge\text{–E}}{\mathcal{C}''\langle\lambda{:}\,\bot\rangle}\ \wedge\text{–I}$$

<u>'If' half:</u>
By hypothesis $\lambda : \alpha \in \mathcal{C}_{mcc}$ and $\lambda : \beta \in \mathcal{C}_{mcc}$. We assume that $\lambda : \alpha \wedge \beta \notin \mathcal{C}_{mcc}$. Then, by Proposition 4.11 $\lambda{:}\,\neg(\alpha \wedge \beta) \in \mathcal{C}_{mcc}$. Hence $\mathcal{C}_{mcc}$ is inconsistent, as shown in the following derivation, which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle\lambda{:}\,\neg(\alpha\wedge\beta), \lambda{:}\,\alpha, \lambda{:}\,\beta\rangle}{\mathcal{C}'\langle\lambda{:}\,\alpha\wedge\beta, \lambda{:}\,\neg(\alpha\wedge\beta)\rangle}\ \wedge\text{–I}}{\mathcal{C}''\langle\lambda{:}\,\bot\rangle}\ \wedge\text{–I}$$

■

Proposition 4.14
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. Let λ be any ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α, β be two wffs of $\mathcal{L}_M$. Then $\lambda : \alpha \vee \beta \in \mathcal{C}_{mcc}$, if and only if $\lambda : \alpha \in \mathcal{C}_{mcc}$ or $\lambda{:}\,\beta \in \mathcal{C}_{mcc}$.

Proof:
<u>'Only if' half</u>
The proof is by contradiction. By hypothesis $\lambda : \alpha \vee \beta \in \mathcal{C}_{mcc}$ and we assume that $\lambda : \alpha \notin \mathcal{C}_{mcc}$ and that $\lambda : \beta \notin \mathcal{C}_{mcc}$. Then by Proposition 4.11, $\lambda : \neg\alpha \in \mathcal{C}_{mcc}$ and $\lambda : \neg\beta \in \mathcal{C}_{mcc}$. Therefore $\mathcal{C}_{mcc} + [\lambda : \alpha] \vdash_S \lambda : \bot$ and $\mathcal{C}_{mcc} + [\lambda : \beta] \vdash_S \lambda : \bot$. So, by the $\vee$–Elimination rule, $\mathcal{C}_{mcc} \vdash_S \lambda{:}\,\bot$. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in

contradiction with the original hypothesis.

<u>'If' half</u>
The proof is by contradiction. By hypothesis either $\lambda : \alpha \in \mathcal{C}_{mcc}$ or $\lambda : \beta \in \mathcal{C}_{mcc}$. We consider the first case since the argument for the second case is analogous. Assume, by contradiction, that $\lambda : \alpha \vee \beta \notin \mathcal{C}_{mcc}$. Then by Proposition 4.11 $\lambda : \neg(\alpha \vee \beta) \in \mathcal{C}_{mcc}$. So $\mathcal{C}_{mcc}$ is inconsistent as shown in the following derivation. This is in contradiction with the original hypothesis.

$$
\begin{array}{ll}
\mathcal{C}_{mcc}\langle \lambda : \neg(\alpha \vee \beta), \lambda : \alpha \rangle & \\
\hline
\mathcal{C}'\langle \lambda : \alpha \vee \beta, \lambda : \neg(\alpha \vee \beta) \rangle & \vee\text{–I} \\
\hline
\mathcal{C}''\langle \lambda : \bot \rangle & \wedge\text{–I}
\end{array}
$$

■

Proposition 4.15
Let $S = \langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R} \rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and α, β be two wffs of $\mathcal{L}_M$. If $\lambda : \neg\alpha \in \mathcal{C}_{mcc}$ or $\lambda : \beta \in \mathcal{C}_{mcc}$ then $\lambda : \alpha \rightarrow \beta \in \mathcal{C}_{mcc}$.

Proof:
Suppose that $\lambda : \neg\alpha \in \mathcal{C}_{mcc}$. Assume by contradiction that $\lambda : \alpha \rightarrow \beta \notin \mathcal{C}_{mcc}$. Then by Proposition 4.11, $\lambda : \neg(\alpha \rightarrow \beta) \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc}$ is inconsistent as shown in the following derivation. This is in contradiction with the original assumption.

$$
\begin{array}{ll}
\mathcal{C}_{mcc}\langle \lambda : \neg(\alpha \rightarrow \beta), \lambda : \neg\alpha \rangle & \\
\hline
\tilde{\mathcal{C}}\langle \lambda : \neg(\alpha \rightarrow \beta), \lambda : \neg\alpha \vee \beta \rangle & (\vee\text{–I}) \\
\hline
\quad \mathcal{C}'_0\langle \lambda : \neg\alpha \vee \beta, \lambda : \alpha \rangle & (\text{new assumption}) \\
\hline
\qquad \mathcal{C}''_0\langle \lambda : \neg\alpha \vee \beta, \lambda : \alpha, \lambda : \neg\beta \rangle & (\text{new assumption}) \\
\hline
\qquad \mathcal{C}''_1\langle \lambda : \neg\alpha \vee \beta, \lambda : \alpha, \lambda : \neg\beta, \lambda : \bot \rangle & (\vee\text{–E}) \\
\hline
\quad \mathcal{C}'_1\langle \lambda : \neg\alpha \vee \beta, \lambda : \alpha, \lambda : \neg\neg\beta \rangle & (\neg\text{–I}) \\
\hline
\quad \mathcal{C}'_2\langle \lambda : \neg\alpha \vee \beta, \lambda : \alpha, \lambda : \beta \rangle & (\neg\text{–E}) \\
\hline
\tilde{\mathcal{C}}_1\langle \lambda : \neg(\alpha \rightarrow \beta), \lambda : \alpha \rightarrow \beta \rangle & (\rightarrow\text{–I}) \\
\hline
\tilde{\mathcal{C}}_2\langle \lambda : \bot \rangle & (\wedge\text{–I})
\end{array}
$$

Now suppose that $\lambda : \beta \in \mathcal{C}_{mcc}$. Assume by contradiction that $\lambda : \alpha \rightarrow \beta \notin \mathcal{C}_{mcc}$. Then by Proposition 4.11, $\lambda : \neg(\alpha \rightarrow \beta) \in \mathcal{C}_{mcc}$. Hence $\mathcal{C}_{mcc}$ is inconsistent as shown in the following derivation. This is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\underline{\mathcal{C}_{mcc}\langle\lambda:\neg(\alpha\rightarrow\beta),\lambda:\beta\rangle} & \\
\quad \underline{\mathcal{C}'\langle\lambda:\neg(\alpha\rightarrow\beta),\lambda:\beta,\lambda:\alpha\rangle} & \text{(new assumption)}\\
\quad \underline{\mathcal{C}'_1\langle\lambda:\beta\rangle} & \text{(C-R)}\\
\underline{\mathcal{C}_1\langle\lambda:\neg(\alpha\rightarrow\beta),\lambda:\alpha\rightarrow\beta\rangle} & (\rightarrow\text{–I})\\
\mathcal{C}_2\langle\lambda:\bot\rangle & (\wedge\text{–I})
\end{array}$$

■

Proposition 4.16
Let $S = \langle\langle\mathcal{L}_L,\mathcal{L}_M\rangle,\mathcal{A},\mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. Let λ be a ground term of $Func(\mathcal{L}_L,\mathcal{L}_M)$ and let α, β, be two wffs of $\mathcal{L}_M$. If $\lambda:\alpha\in\mathcal{C}_{mcc}$ and $\lambda:\alpha\rightarrow\beta\in\mathcal{C}_{mcc}$ then $\lambda:\beta\in\mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $\lambda:\beta\notin\mathcal{C}_{mcc}$. Then by Proposition 4.11 $\lambda:\neg\beta\in\mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc}\vdash_S\lambda:\bot$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent and this is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\underline{\mathcal{C}_{mcc}\langle\lambda:\alpha,\lambda:\alpha\rightarrow\beta,\lambda:\neg\beta\rangle} & \rightarrow\text{–E}\\
\underline{\mathcal{C}'\langle\lambda:\beta,\lambda:\neg\beta\rangle} & \wedge\text{–I}\\
\mathcal{C}''\langle\lambda:\bot\rangle &
\end{array}$$

■

Proposition 4.17
Let $S = \langle\langle\mathcal{L}_L,\mathcal{L}_M\rangle,\mathcal{A},\mathcal{R}\rangle$ be a MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S and let λ be a ground term of $Func(\mathcal{L}_L,\mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. If $\lambda:\Diamond\alpha\in\mathcal{C}_{mcc}$ then $f_\alpha(\lambda):\alpha\in\mathcal{C}_{mcc}$ and $R(\lambda,f_\alpha(\lambda))\in\mathcal{C}_{mcc}$.

Proof:
Let $\mathcal{C}_{mcc}$ be the configuration $\langle\mathcal{D}_{mcc},\mathcal{F}_{mcc}\rangle$. By hypothesis $\lambda:\Diamond\alpha\in\mathcal{C}_{mcc}$, or equivalently, $\Diamond\alpha\in\mathcal{F}_{mcc}(\lambda)$. Then by Proposition 4.1, $\neg R(\lambda,f_\alpha(\lambda))\notin\mathcal{D}_{mcc}$ and $\neg\alpha\notin\mathcal{F}_{mcc}(f_\alpha(\lambda))$[15]. Hence by Proposition 4.12, $R(\lambda,f_\alpha(\lambda))\in\mathcal{C}_{mcc}$ and by Proposition 4.11, $f_\alpha(\lambda):\alpha\in\mathcal{C}_{mcc}$.

■

[15]Equivalently $\neg R(\lambda,f_\alpha(\lambda))\notin\mathcal{C}_{mcc}$ and $f_\alpha(\lambda):\neg\alpha\notin\mathcal{C}_{mcc}$

Proposition 4.18
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S, let λ, λ' be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. If $\lambda : \Box\alpha \in \mathcal{C}_{mcc}$ and $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$, then $\lambda' : \alpha \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $\lambda' : \alpha \notin \mathcal{C}_{mcc}$. Then by Proposition 4.11, $\lambda' : \neg\alpha \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc} \vdash_S \lambda' : \bot$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent and this is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}_{mcc}\langle \lambda : \Box\alpha, R(\lambda, \lambda'), \lambda' : \neg\alpha\rangle & \Box\text{–E} \\ \hline \mathcal{C}'\langle \lambda' : \alpha, \lambda' : \neg\alpha\rangle & \wedge\text{–I} \\ \hline \mathcal{C}''\langle \lambda' : \bot\rangle & \end{array}$$

■

Proposition 4.19
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S, let λ_1 and λ_2 be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. If $R(\lambda_1, \lambda_2) \in \mathcal{C}_{mcc}$ and $\lambda_2 : \alpha \in \mathcal{C}_{mcc}$ then $\lambda_1 : \Diamond\alpha \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $\lambda_1 : \Diamond\alpha \notin \mathcal{C}_{mcc}$. Then by Proposition 4.11, $\lambda_1 : \neg\Diamond\alpha \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc} \vdash_S \lambda_1 : \bot$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent and this is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}_{mcc}\langle R(\lambda_1, \lambda_2), \lambda_2 : \alpha, \lambda_1 : \neg\Diamond\alpha\rangle & \Diamond\text{–I} \\ \hline \mathcal{C}'\langle \lambda_1 : \Diamond\alpha, \lambda_1 : \neg\Diamond\alpha\rangle & \wedge\text{–I} \\ \hline \mathcal{C}''\langle \lambda_1 : \bot\rangle & \end{array}$$

■

Proposition 4.20
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S, let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. If $\neg R(\lambda, box_\alpha(\lambda)) \in \mathcal{C}_{mcc}$ or $box_\alpha(\lambda) : \alpha \in \mathcal{C}_{mcc}$ then $\lambda : \Box\alpha \in \mathcal{C}_{mcc}$.

Proof:
Let $\mathcal{C}_{mcc}$ be the configuration $\langle \mathcal{D}_{mcc}, \mathcal{F}_{mcc}\rangle$. Suppose by contradiction that $\lambda : \Box\alpha \notin \mathcal{C}_{mcc}$. Then by Proposition 4.11, $\lambda : \neg\Box\alpha \in \mathcal{C}_{mcc}$, or equivalently $\neg\Box\alpha \in \mathcal{F}_{mcc}(\lambda)$. Then by Proposition 4.1, $\alpha \notin \mathcal{F}_{mcc}(box_\alpha(\lambda))$ and $\neg R(\lambda, Box_\alpha(\lambda)) \notin \mathcal{D}_{mcc}$. Hence $box_\alpha(\lambda) : \alpha \notin \mathcal{C}_{mcc}$, and $\neg R(\lambda, Box_\alpha(\lambda)) \notin \mathcal{C}_{mcc}$, which are together in contradiction with the original hypothesis. ■

So far we have proved properties of a maximal consistent configuration that hold in any MLDS S. However there exist some extra properties that depend on the particular labelling algebra $\mathcal{A}$ of a MLDS S. These properties refer only to the R–literals of a maximal consistent configuration. As shown in Definition 1.4, a labelling algebra $\mathcal{A}$ is a finite set of axioms on the predicate R of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$. Regarding the predicate R as the Kripke accessibility relation between possible worlds, each labelling algebra $\mathcal{A}$ defines a particular MLDS S in the class of normal modal logics (K, T, $K4$, KB, $S4$, $S5$, D, $D4$, DB). For example, a MLDS K is defined by an empty-set labelling algebra, $\mathcal{A} = \{\}$[16]. In a MLDS $K4$ the labelling algebra $\mathcal{A}$ is the singleton set given by axiom (4) of Definition 1.4 [17]. Analogously, in a MLDS T the labelling algebra $\mathcal{A}$ is the singleton set given by axiom (T) of Definition 1.4 [18]. So, for each MLDS S different from K, there are further properties of a maximal consistent configuration to be proved.

Proposition 4.21
Let $T = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x, x)\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to T. Then for each ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, the R–literal $R(\lambda, \lambda) \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Let λ' be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda', \lambda') \notin \mathcal{C}_{mcc}$. Then by Proposition 4.12, $\neg R(\lambda', \lambda') \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc} \vdash_T \lambda'' : \perp$, for some ground term $\lambda'' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle\neg R(\lambda', \lambda')\rangle}{\mathcal{C}'\langle\neg R(\lambda', \lambda'), R(\lambda', \lambda')\rangle}\ R\text{–Assertion}}{\mathcal{C}''\langle\lambda'' : \perp\rangle}\ \perp\text{–I}$$

■

Proposition 4.22
Let $D = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x, succ(x))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to D. Then for each ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, the R–literal $R(\lambda, succ(\lambda)) \in \mathcal{C}_{mcc}$.

[16] In a modal logic K the accessibility relation does not have any property
[17] In a modal logic $K4$ the accessibility relation is transitive
[18] In a modal logic T the accessibility relation is reflexive

Proof:
The proof is by contradiction. Let λ' be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda', succ(\lambda')) \notin \mathcal{C}_{mcc}$. Then by Proposition 4.12, $\neg R(\lambda', succ(\lambda')) \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc} \vdash_D \lambda'' : \perp$, for some ground term $\lambda'' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle \neg R(\lambda', succ(\lambda'))\rangle}{\mathcal{C}'\langle \neg R(\lambda', succ(\lambda')), R(\lambda', succ(\lambda'))\rangle}\ R\text{–Assertion}}{\mathcal{C}''\langle \lambda'' : \perp\rangle}\ \perp\text{–I}$$

■

Proposition 4.23
Let $K4 = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x, y, z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $K4$. Let λ_1,λ_2,λ_3 be three ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1, \lambda_2) \in \mathcal{C}_{mcc}$ and $R(\lambda_2, \lambda_3) \in \mathcal{C}_{mcc}$. Then $R(\lambda_1, \lambda_3) \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $R(\lambda_1, \lambda_3) \notin \mathcal{C}_{mcc}$. Then by Proposition 4.12, $\neg R(\lambda_1, \lambda_3) \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc} \vdash_{K4} \lambda' : \perp$, for some ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle \neg R(\lambda_1, \lambda_3)\rangle}{\mathcal{C}'\langle \neg R(\lambda_1, \lambda_3), R(\lambda_1, \lambda_3)\rangle}\ R\text{–Assertion}}{\mathcal{C}''\langle \lambda' : \perp\rangle}\ \perp\text{–I}$$

■

Proposition 4.24
Let $KB = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x, y(R(x,y) \rightarrow R(y,x))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to KB. Let λ_1,λ_2 be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1, \lambda_2) \in \mathcal{C}_{mcc}$. Then $R(\lambda_2, \lambda_1) \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $R(\lambda_2, \lambda_1) \notin \mathcal{C}_{mcc}$. Then by Proposition 4.12, $\neg R(\lambda_2, \lambda_1) \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc} \vdash_{KB} \lambda' : \perp$, for some ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle\neg R(\lambda_2,\lambda_1)\rangle}{\mathcal{C}'\langle\neg R(\lambda_2,\lambda_1), R(\lambda_2,\lambda_1)\rangle}\quad R\text{–Assertion}}{\mathcal{C}''\langle\lambda':\perp\rangle}\quad \perp\text{–I}$$

■

Proposition 4.25
Let $S4 = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x,x),\ \forall x,y,z((R(x,y)\wedge R(y,z)) \rightarrow R(x,z))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $S4$. Then the two following statements hold

1. For any ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $R(\lambda,\lambda) \in \mathcal{C}_{mcc}$
2. Given three ground terms λ_1,λ_2,λ_3 of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1,\lambda_2) \in \mathcal{C}_{mcc}$ and $R(\lambda_2,\lambda_3) \in \mathcal{C}_{mcc}$, then $R(\lambda_1,\lambda_3) \in \mathcal{C}_{mcc}$.

Proof:
1. This follows from Proposition 4.21
2. This follows from Proposition 4.23

■

Proposition 4.26
Let $S5 = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x,x),\ \forall x,y,z((R(x,y)\wedge R(y,z)) \rightarrow R(x,z)),\ \forall x,y(R(x,y) \rightarrow R(y,x))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $S5$. Then the following statements hold

1. For any ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $R(\lambda,\lambda) \in \mathcal{C}_{mcc}$
2. Given three ground terms λ_1,λ_2,λ_3 of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1,\lambda_2) \in \mathcal{C}_{mcc}$ and $R(\lambda_2,\lambda_3) \in \mathcal{C}_{mcc}$, then $R(\lambda_1,\lambda_3) \in \mathcal{C}_{mcc}$.
3. Given two ground terms λ_1,λ_2 of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1,\lambda_2) \in \mathcal{C}_{mcc}$, then $R(\lambda_2,\lambda_1) \in \mathcal{C}_{mcc}$.

Proof:

1. This follows from Proposition 4.21
2. This follows from Proposition 4.23
3. This follows from Proposition 4.24

■

We have already pointed out earlier in this section, that, given a MLDS S, the proof of the completeness theorem is based on the Model Existence Lemma 4.2. We have also said that the proof of this lemma requires two main steps. The first step consists of proving that it is always possible to expand a given consistent configuration into a maximal consistent configuration. The second step consists of constructing a semantic structure of S which satisfies the maximal consistent configuration. So far we have proved the first step in Proposition 4.8. We are now going to show how to construct a semantic structure which satisfies a maximal consistent configuration $\mathcal{C}_{mcc}$.

Definition 4.8 (Maximal Interpretation, $\mathcal{M}(\mathcal{C}_{mcc})$)
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. Let $\mathcal{C}_{mcc} = \langle\mathcal{D}_{mcc}, \mathcal{F}_{mcc}\rangle$ be a maximal consistent configuration relative to S. Let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. $FOT(\mathcal{C}_{mcc}) = \mathcal{D}_{mcc} \cup \mathcal{DU}_{mcc}$, where $\mathcal{DU}_{mcc} = \{[\alpha]^*(\lambda) \mid \lambda : \alpha \in \mathcal{C}_{mcc}\}$. Let $\mathcal{U}$ be the Herbrand Universe of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$. $I(\mathcal{C}_{mcc})$ is the interpretation function on the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, defined as follows.

- For each ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$
  $\parallel \lambda \parallel_{I(\mathcal{C}_{mcc})} = \ \lambda \in \mathcal{U}$.
- For the binary predicate $R \in Mon(\mathcal{L}_L, \mathcal{L}_M)$
  $\parallel R \parallel_{I(\mathcal{C}_{mcc})} = \ \{\langle\lambda_i, \lambda_j\rangle \mid R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{mcc})\}$[19].
- For each monadic predicate $[\alpha]^* \in Mon(\mathcal{L}_L, \mathcal{L}_M)$
  $\parallel [\alpha]^* \parallel_{I(\mathcal{C}_{mcc})} = \{\lambda_i \mid [\alpha]^*(\lambda_i) \in FOT(\mathcal{C}_{mcc})\}$

$\mathcal{M}(\mathcal{C}_{mcc}) = \langle\mathcal{U}, I(\mathcal{C}_{mcc})\rangle$ is a maximal interpretation.

□

Remark 4.2
Let S be a MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S, let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. Then we observe that for any ground atomic formula of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ of the form $R(\lambda_i, \lambda_j)$, $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} R(\lambda_i, \lambda_j)$ if and only if $R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{mcc})$; analogously for any ground atomic formula of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ of the form $[\alpha]^*(\lambda)$, $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} [\alpha]^*(\lambda)$ if and only if $[\alpha]^*(\lambda) \in FOT(\mathcal{C}_{mcc})$.
Now let V be a variable assignment from the set of variables of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ to the Herbrand Universe $\mathcal{U}$. The truth value of any wff of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ is defined as follows.

- $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(x, y)$ if and only if $\langle V(x), V(y)\rangle \in\parallel R \parallel_{I(\mathcal{C}_{mcc})}$

[19] Notice that $FOT(\mathcal{C}_{mcc})$ contains only ground literals

- $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(x)$ if and only if $V(x) \in \| [\alpha]^* \|_{I(\mathcal{C}_{mcc})}$
- for any wff ϕ of $Mon(\mathcal{L}_L, \mathcal{L}_M)$, the truth value of ϕ with respect to $\mathcal{M}(\mathcal{C}_{mcc})$ and V is defined in the usuall way (see for example [Genesereth-88]).

$\circ$

We are now going to prove that, given a MLDS $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$, the maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc}) = \langle \mathcal{U}, I(\mathcal{C}_{mcc})\rangle$ of Definition 4.8 is a model of the associated extended algebra $\mathcal{A}^+$. Then by Definition 3.3, $\mathcal{M}(\mathcal{C}_{mcc})$ is a semantic structure of S. The proof is based on two steps. In the first step, we prove that $\mathcal{M}(\mathcal{C}_{mcc})$ satisfies all the axioms of the labelling algebra $\mathcal{A}$. Then we prove that $\mathcal{M}(\mathcal{C}_{mcc})$ satisfies all the axiom schemas $(Ax1)$–$(Ax8)$ of the associated extended algebra $\mathcal{A}^+$. The first step depends on the particular MLDS S; we prove it for S equal to T, D, $K4$, $S4$, $S5$[20]. The second step, instead, is common to any MLDS S.

For any variable assignment V and any variable x, of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, $V(x)$ refers to some ground term in the Herbrand Universe $\mathcal{U}$. Hence, for simplicity, we will use $V(x)$ to stand for an arbitrary ground term in the arguments that follow.

Proposition 4.27
Let $T = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x,x)\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to T and let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x R(x,x)$.

> **Proof**:
> Let V be an arbitrary variable assignment. It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), V(x))$. By Proposition 4.21, $R(V(x), V(x)) \in \mathcal{C}_{mcc}$ and so $R(V(x), V(x)) \in FOT(\mathcal{C}_{mcc})$. So, by Definition 4.8, $\langle V(x), V(x)\rangle \in \| R \|_{I(\mathcal{C}_{mcc})}$. Hence, by Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), V(x))$.

$\blacksquare$

Proposition 4.27 shows that given a MLDS $T = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$, a maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc})$ is a model of the labelling algebra $\mathcal{A} = \{\forall x R(x,x)\}$. We prove similar results for the MLDS D, $K4$, $S4$ and $S5$. Analogous results can be easly shown for the $D4$–MLDS, the DB–MLDS and the KB–MLDS.

[20]For S is equal K, we don't need to prove the first step, since the associated labelling algebra $\mathcal{A}$ is the empty set

Proposition 4.28
Let $D = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x, succ(x))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to D and let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x R(x, succ(x))$.

> **Proof**:
> Let V be an arbitrary variable assignment. It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), succ(V(x)))$. By Proposition 4.22, $R(V(x), succ(V(x))) \in \mathcal{C}_{mcc}$ and so $R(V(x), succ(V(x))) \in FOT(\mathcal{C}_{mcc})$. So, by Definition 4.8, $\langle V(x), succ(V(x))\rangle \in\| R \|_{I(\mathcal{C}_{mcc})}$. Hence, by Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), succ(V(x)))$.

$\blacksquare$

Proposition 4.29
Let $K4 = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x, y, z((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $K4$ and let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x, y, z((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$.

> **Proof**:
> Let V be an arbitrary variable assignment. It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} (R(V(x), V(y)) \wedge R(V(y), V(z))) \rightarrow R(V(x), V(z))$.
> By the truth table of $\rightarrow$, we assume that
> $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} (R(V(x), V(y)) \wedge R(V(y), V(z)))$
> and it remains to show that
> $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), V(z))$.
> By assumption, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), V(y))$ and $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(y), V(z))$. Then by Remark 4.2, $\langle V(x), V(y)\rangle \in\| R \|_{I(\mathcal{C}_{mcc})}$ and $\langle V(y), V(z)\rangle \in\| R \|_{I(\mathcal{C}_{mcc})}$. By Definition 4.8, $R(V(x), V(y)) \in FOT(\mathcal{C}_{mcc})$ and $R(V(y), V(z)) \in FOT(\mathcal{C}_{mcc})$. Then $R(V(x), V(y)) \in \mathcal{C}_{mcc}$ and $R(V(y), V(z)) \in \mathcal{C}_{mcc}$. By Proposition 4.23, $R(V(x), V(z)) \in \mathcal{C}_{mcc}$. Then $R(V(x), V(z)) \in FOT(\mathcal{C}_{mcc})$. So, by Definition 4.8, $\langle V(x), V(z)\rangle \in\| R \|_{I(\mathcal{C}_{mcc})}$. Hence, by Remark 4.2 $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), V(z))$.

$\blacksquare$

Proposition 4.30
Let $S4 = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x, x),\ \forall x, y, z((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $S4$ and let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. Then

1. $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x R(x,x)$

2. $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$

Proof:

1. This follows from Proposition 4.27
2. This follows from Proposition 4.29

■

Proposition 4.31
Let $S5 = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS where $\mathcal{A} = \{\forall x R(x,x),\ \forall x,y,z((R(x,y)\wedge R(y,z)) \rightarrow R(x,z)),\ \forall x,y(R(x,y) \rightarrow R(y,x))\}$. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $S4$ and let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. Then

1. $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x R(x,x)$

2. $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$

3. $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \forall x,y(R(x,y) \rightarrow R(y,x))$

Proof:

1. This follows from Proposition 4.27

2. This follows from Proposition 4.29

3. Let V be an arbitrary variable assignment. It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} (R(V(x),V(y)) \rightarrow R(V(y),V(x)))$. By the truth table of $\rightarrow$, we assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} (R(V(x),V(y))$ and we show that
$\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(y),V(x)))$.
By assumption and Remark 4.2, $\langle V(x), V(y)\rangle \in \parallel R \parallel_{I(\mathcal{C}_{mcc})}$. Then by Definition 4.8, $R(V(x),V(y)) \in FOT(\mathcal{C}_{mcc})$. Therefore $R(V(x),V(y)) \in \mathcal{C}_{mcc}$. Then by Proposition 4.25,
$R(V(y),V(x)) \in \mathcal{C}_{mcc}$. So $R(V(y),V(x)) \in FOT(\mathcal{C}_{mcc})$ and by Definition 4.8, $\langle V(y), V(x)\rangle \in \parallel R \parallel_{I(\mathcal{C}_{mcc})}$. Hence, by Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(y),V(x)))$.

■

We now show that, given a MLDS S, any maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc})$ of Definition 4.8 is a model of the axiom schemas $(Ax1)$–$(Ax8)$ of the associated extended algebra $\mathcal{A}^+$.

Theorem 4.3 (Model existence for $\mathcal{A}^+ - \mathcal{A}$)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. Let $\mathcal{A}^+$ be the extended algebra associated with S and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to S. and let $FOT(\mathcal{C}_{mcc})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{mcc})$ is a model of $\mathcal{A}^+ - \mathcal{A}$.

Proof:
Let U, V be arbitrary variable assignments, and let α, β be two wffs of $\mathcal{L}_M$. The proof is by cases on each of the axiom schemas $(Ax1)$–$(Ax8)$.

- **(Ax1):** $\quad \forall x([\alpha \wedge \beta]^*(x) \equiv ([\alpha]^*(x) \wedge [\beta]^*(x))$
  By the truth table of the logical connectives $\equiv$, $\wedge$, it is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha \wedge \beta]^*(V(x))$ if and only if $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$ and $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$.

  <u>'Only if' half</u>
  We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha \wedge \beta]^*(V(x))$. Then, by Definition 4.8, $[\alpha \wedge \beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Then $V(x) : \alpha \wedge \beta \in \mathcal{C}_{mcc}$. By Proposition 4.13, $V(x) : \alpha \in \mathcal{C}_{mcc}$ and $V(x) : \beta \in \mathcal{C}_{mcc}$. Then $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ and $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Therefore $V(x) \in \| [\alpha]^* \|_{I(\mathcal{C}_{mcc})}$ and $V(x) \in \| [\beta]^* \|_{I(\mathcal{C}_{mcc})}$. Hence by Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$ and $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$.

  <u>'If' half</u>
  We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$ and that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$. By Definition 4.8, $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ and $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Then $V(x) : \alpha \in \mathcal{C}_{mcc}$ and $V(x) : \beta \in \mathcal{C}_{mcc}$. Then, by Proposition 4.13, $V(x) : \alpha \wedge \beta \in \mathcal{C}_{mcc}$. Then $[\alpha \wedge \beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and by Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha \wedge \beta]^*(V(x))$.

- **(Ax2):** $\quad \forall x([\neg\alpha]^*(x) \equiv \neg[\alpha]^*(x))$
  By the truth table of $\equiv$, it is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\neg\alpha]^*(V(x))$ if and only if $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} \neg[\alpha]^*(V(x))$.

  <u>'Only if' half</u>

We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\neg\alpha]^*(V(x))$. By Definition 4.8 $[\neg\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ and then $V(x): \neg\alpha \in \mathcal{C}_{mcc}$. Then, by Proposition 4.11, $V(x): \alpha \notin \mathcal{C}_{mcc}$. Then $[\alpha]^*(V(x)) \notin FOT(\mathcal{C}_{mcc})$. So $V(x) \notin \| [\alpha]^* \|_{I(\mathcal{C}_{mcc})}$. So $\mathcal{M}(\mathcal{C}_{mcc}), V \not\models_{FOL} [\alpha]^*(V(x))$. Hence $\mathcal{M}_m, V \models_{FOL} \neg[\alpha]^*(V(x))$.

<u>'If' half</u>
We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} \neg[\alpha]^*(V(x))$. Then by the truth table of $\neg$, $\mathcal{M}(\mathcal{C}_{mcc}), V \not\models_{FOL} [\alpha]^*(V(x))$, which implies that $V(x) \notin \| [\alpha]^* \|_{I(\mathcal{C}_{mcc})}$. Then $[\alpha]^*(V(x)) \notin FOT(\mathcal{C}_{mcc})$ and then $V(x): \alpha \notin \mathcal{C}_{mcc}$. By Proposition 4.11, $V(x): \neg\alpha \in \mathcal{C}_{mcc}$. So $[\neg\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\neg\alpha]^*(V(x))$.

- **(Ax3):** $\forall x([\alpha \vee \beta]^*(x) \equiv ([\alpha]^*(x) \vee [\beta]^*(x))$
By the truth tables of $\equiv$ and $\vee$, it is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha\vee\beta]^*(V(x))$ if and only if $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$.

  <u>'Only If' half</u>
  We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha\vee\beta]^*(V(x))$. Then, by Definition 4.8, $[\alpha \vee \beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. So $V(x): \alpha \vee \beta \in \mathcal{C}_{mcc}$. By Proposition 4.14, $V(x): \alpha \in \mathcal{C}_{mcc}$ or $V(x): \beta \in \mathcal{C}_{mcc}$. So $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ or $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$.

  <u>'If' half</u>
  We assume that
  $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$. Then either $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ or $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. So either $V(x): \alpha \in \mathcal{C}_{mcc}$ or $V(x): \beta \in \mathcal{C}_{mcc}$. Then, by Proposition 4.14, $V(x): \alpha \vee \beta \in \mathcal{C}_{mcc}$. So $[\alpha \vee \beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha \vee \beta]^*(V(x))$.

- **(Ax4):** $\forall x([\alpha \rightarrow \beta]^*(x) \equiv ([\alpha]^*(x) \rightarrow [\beta]^*(x))$
By the truth tables of $\equiv$ and $\rightarrow$, it is sufficient to prove that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha\rightarrow\beta]^*(V(x))$ if and only if,
  if $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$ then
  $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$.

  <u>'Only if' half</u>
  We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha\rightarrow\beta]^*(V(x))$ and that

$\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(V(x))$. Then $[\alpha \to \beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ and $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Then $V(x) : \alpha \to \beta \in \mathcal{C}_{mcc}$ and $V(x) : \alpha \in \mathcal{C}_{mcc}$. By Proposition 4.16, $V(x) : \beta \in \mathcal{C}_{mcc}$ and then $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$.

'If' half
We assume that
$\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} \neg[\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\beta]^*(V(x))$.
By satisfiability of $(Ax2)$, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\neg\alpha]^*(V(x))$. Then, by Definition 4.8 $[\neg\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ or $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Then $V(x) : \neg\alpha \in \mathcal{C}_{mcc}$ or $V(x) : \beta \in \mathcal{C}_{mcc}$. Therefore, by Proposition 4.15, $V(x) : \alpha \to \beta \in \mathcal{C}_{mcc}$.
So $[\alpha \to \beta]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha \to \beta]^*(V(x))$.

- **(Ax5):** $\forall x([\Diamond\alpha]^*(x) \to (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x))))$.
By the truth tables of $\to$ and $\wedge$, it is sufficient to prove that
if $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\Diamond\alpha]^*(V(x))$ then
$\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), f_\alpha(V(x)))$[21] and
$\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(f_\alpha(V(x)))$.
We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\Diamond\alpha]^*(V(x))$. Then, by Definition 4.8 $[\Diamond\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$ and then $V(x) : \Diamond\alpha \in \mathcal{C}_{mcc}$.
By Proposition 4.17, $f_\alpha(V(x)) : \alpha \in \mathcal{C}_{mcc}$ and $R(V(x), f_\alpha(V(x))) \in \mathcal{C}_{mcc}$. So $R(V(x), f_\alpha(V(x))) \in FOT(\mathcal{C}_{mcc})$ and $[\alpha]^*(f_\alpha(V(x))) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and Remark 4.2,
$\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} R(V(x), f_\alpha(V(x)))$ and $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(f_\alpha(V(x)))$.

- **(Ax6):** $\forall x(\exists y(R(x, y) \wedge [\alpha]^*(y)) \to [\Diamond\alpha]^*(x))$.
We consider the equivalent formula
$$\forall x \forall y(R(x, y) \wedge [\alpha]^*(y)) \to [\Diamond\alpha]^*(x)).$$
By the truth tables of $\to$ and $\wedge$, it is sufficient to prove that if $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} R(V(x), U(y))$ and
$\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\alpha]^*(U(y))$ then
$\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\Diamond\alpha]^*(V(x))$.
We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} R(V(x), U(y))$ and that $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\alpha]^*(U(y))$. Then, by Definition 4.8,

[21]The definition of the interpretation function $I(\mathcal{C}_{mcc})$ of a maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc})$, allows us to write $f_\alpha(V(x))$ instead of $\| f_\alpha \|_{I(\mathcal{C}_{mcc})} (V(x))$

$R(V(x), U(y)) \in FOT(\mathcal{C}_{mcc})$ and $[\alpha]^*(U(y)) \in FOT(\mathcal{C}_{mcc})$. Then $R(V(x), U(y)) \in \mathcal{C}_{mcc}$ and $U(y) : \alpha \in \mathcal{C}_{mcc}$. Then by Proposition 4.19, $V(x) : \Diamond\alpha \in \mathcal{C}_{mcc}$. Then $[\Diamond\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\Diamond\alpha]^*(V(x))$.

- **(Ax7):** $\forall x(R(x, box_\alpha(x)) \to [\alpha]^*(box_\alpha(x))) \to [\Box\alpha]^*(x))$.
  By the truth table of $\to$, it is sufficient to prove that
  if $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} \neg R(V(x), box_\alpha(V(x)))$[22] or
  $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\alpha]^*(box_\alpha(V(x)))$, then
  $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\Box\alpha]^*(V(x))$.
  We assume that either $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} \neg R(V(x), box_\alpha(V(x)))$ or $\mathcal{M}_m, V \models_{FOL} [\alpha]^*(box_\alpha(V(x)))$. Then, by Definition 4.8, either $\neg R(V(x), box_\alpha(V(x))) \in FOT(\mathcal{C}_{mcc})$ or $[\alpha]^*(box_\alpha(V(x))) \in FOT(\mathcal{C}_{mcc})$. Then either $\neg R(V(x), box_\alpha(V(x))) \in \mathcal{C}_{mcc}$ or $box_\alpha(V(x)) : \alpha \in \mathcal{C}_{mcc}$. Then, by Proposition 4.20, $V(x) : \Box\alpha \in \mathcal{C}_{mcc}$. Then $[\Box\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V \models_{FOL} [\Box\alpha]^*(V(x))$.

- **(Ax8):** $\forall x([\Box\alpha]^*(x) \to (\forall y(R(x, y) \to [\alpha]^*(y)))$.
  We consider the equivalent formula
  $$\forall x \forall y(([\Box\alpha]^*(x) \wedge (R(x, y)) \to [\alpha]^*(y))$$
  By the truth table of $\to$, it is sufficient to prove that
  if $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\Box\alpha]^*(V(x))$ and
  $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} R(V(x), U(y))$, then
  $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\alpha]^*(U(y))$.
  We assume that $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\Box\alpha]^*(V(x))$ and that $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} R(V(x), U(y))$. Then, by Definition 4.8, $[\Box\alpha]^*(V(x)) \in FOT(\mathcal{C}_{mcc})$, and $R(V(x), U(y)) \in FOT(\mathcal{C}_{mcc})$. Then $V(x) : \Box\alpha \in \mathcal{C}_{mcc}$ and $R(V(x), U(y)) \in \mathcal{C}_{mcc}$. Then, by Proposition 4.18, $U(y) : \alpha \in \mathcal{C}_{mcc}$ and then $[\alpha]^*(U(y)) \in FOT(\mathcal{C}_{mcc})$. Hence by Definition 4.8 and Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}), V, U \models_{FOL} [\alpha]^*(U(y))$.

■

For each MLDS S we have proved that, given a consistent configuration $\mathcal{C}$ of S, there exists a maximal consistent configuration $\mathcal{C}_{mcc}$ such that $\mathcal{C} \subseteq \mathcal{C}_{mcc}$. We have

[22]The definition of the interpretation function $I(\mathcal{C}_{mcc})$ of a maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc})$, allows us to write $box_\alpha(V(x))$ instead of $\| box_\alpha \|_{I(\mathcal{C}_{mcc})} (V(x))$

then constructed a maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc})$ and proved that, $\mathcal{M}(\mathcal{C}_{mcc})$ is a model of the extended algebra $\mathcal{A}^+$ associated with S. Therefore we have proved that the maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc})$ is a semantic structure of S. We are now going to prove that the semantic structure $\mathcal{M}(\mathcal{C}_{mcc})$ satisfies the maximal consistent configuration $\mathcal{C}_{mcc}$. This last step is called *Model Existence Lemma.*

Lemma 4.2 (Model Existence Lemma)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration. Then for any π (where π is a declarative unit or an R–literal) of S, $\mathcal{M}(\mathcal{C}_{mcc}) \models_S \pi$ if $\pi \in \mathcal{C}_{mcc}$, and $\mathcal{M}(\mathcal{C}_{mcc}) \not\models_S \pi$ if $\pi \notin \mathcal{C}_{mcc}$.

Proof:
There are two cases to consider.

1. π is a declarative unit of the form $\lambda : \alpha$.
   If $\lambda : \alpha \in \mathcal{C}_{mcc}$, then $[\alpha]^*(\lambda) \in FOT(\mathcal{C}_{mcc})$. Then by Definition 4.8 $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} [\alpha]^*(\lambda)$. Hence, by Definition 3.4, $\mathcal{M}(\mathcal{C}_{mcc}) \models_S \lambda : \alpha$. If $\lambda : \alpha \notin \mathcal{C}_{mcc}$ then $[\alpha]^*(\lambda) \notin FOT(\mathcal{C}_{mcc})$. Then, by Definition 4.8 $\lambda \notin \| [\alpha]^* \|_{I(\mathcal{C}_{mcc})}$, and so by Remark 4.2, $\mathcal{M}(\mathcal{C}_{mcc}) \not\models_{FOL} [\alpha]^*(\lambda)$. Hence, by Definition 3.4, $\mathcal{M}(\mathcal{C}_{mcc}) \not\models_S \lambda : \alpha$.
2. π is an R–literal.
   Then π can be either equal to $R(\lambda_i, \lambda_j)$, or equal to $\neg R(\lambda_i, \lambda_j)$, where λ_i, λ_j are ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$.
   - Suppose that π is of the form $R(\lambda_i, \lambda_j)$. If $R(\lambda_i, \lambda_j) \in \mathcal{C}_{mcc}$, then $R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{mcc})$. So by Definition 4.8 $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} R(\lambda_i, \lambda_j)$. Hence, by Definition 3.5, $\mathcal{M}(\mathcal{C}_{mcc}) \models_S R(\lambda_i, \lambda_j)$. If $R(\lambda_i, \lambda_j) \notin \mathcal{C}_{mcc}$, then $R(\lambda_i, \lambda_j) \notin FOT(\mathcal{C}_{mcc})$. Then, by Definition 4.8 $\mathcal{M}(\mathcal{C}_{mcc}) \not\models_{FOL} R(\lambda_i, \lambda_j)$. Hence, by Definition 3.5, $\mathcal{M}(\mathcal{C}_{mcc}) \not\models_S R(\lambda_i, \lambda_j)$.
   - Suppose now that π is of the form $\neg R(\lambda_i, \lambda_j)$. If $\neg R(\lambda_i, \lambda_j) \in \mathcal{C}_{mcc}$ then by Proposition 4.12 $R(\lambda_i, \lambda_j) \notin \mathcal{C}_{mcc}$. Then $R(\lambda_i, \lambda_j) \notin FOT(\mathcal{C}_{mcc})$. So by Definition 4.8 $\mathcal{M}(\mathcal{C}_{mcc}) \not\models_{FOL} R(\lambda_i, \lambda_j)$ and then $\mathcal{M}(\mathcal{C}_{mcc}) \models_{FOL} \neg R(\lambda_i, \lambda_j)$. Hence by Definition 3.5 $\mathcal{M}(\mathcal{C}_{mcc}) \models_S \neg R(\lambda_i, \lambda_j)$. ■

Corollary 4.1
Let S be one of the normal MLDS's K, T, $K4$, KB, $S4$, $S5$, D, $D4$ and DB. Let $\mathcal{C}$ be a consistent configuration of S. Then $\mathcal{M}(MCC(\mathcal{C}))$ satisfies $\mathcal{C}$.

Proof:
By Definition 4.6 and Proposition 4.8, there exists a maximal consistent configuration $MCC(\mathcal{C})$ such that $\mathcal{C} \subseteq MCC(\mathcal{C})$. By Theorem 4.3, Proposition 4.27, Proposition 4.29, Proposition 4.30, Proposition 4.26 and Definition 3.3, the associated maximal interpretation

$\mathcal{M}(MCC(\mathcal{C}))$ is a semantic structure of S. Moreover by Lemma 4.2 and Definition 3.6, $\mathcal{M}(MCC(\mathcal{C}))$ satisfies the configuration $MCC(\mathcal{C})$. Hence $\mathcal{M}(MCC(\mathcal{C}))$ satisfies $\mathcal{C}$. ∎

We now prove the completeness theorem.

Theorem 4.4 (Completeness)
Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. Let $\mathcal{A}^+$ be the associated extended algebra. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of S such that the configuration difference $\mathcal{C}' - \mathcal{C}$ is finite.

$$\text{If } \mathcal{C} \models_S \mathcal{C}' \text{ then } \mathcal{C} \vdash_S \mathcal{C}'.$$

Proof:
The proof is by contrapositive. Assume that $\mathcal{C} \not\vdash_S \mathcal{C}'$. Then by Theorem 4.1 there exists a $\pi \in \mathcal{C}' - \mathcal{C}$, where π is a declarative unit or an R–literal, such that $\mathcal{C} \not\vdash_S \pi$. Then by Proposition 4.7, $\mathcal{C}+[\neg\pi]$ is a consistent configuration. By Corollary 4.1, $\mathcal{M}_m = \mathcal{M}(MCC(\mathcal{C}+[\neg\pi]))$ satisfies the configuration $\mathcal{C}+[\neg\pi]$. So by Definition 3.6, $\mathcal{M}_m \models_S \mathcal{C}$ and $\mathcal{M}_m \models_S \neg\pi$. There are two cases to consider.

1. π is an R–literal. By Definition 3.5, $\mathcal{M}_m \models_{FOL} \neg\pi$. So by satisfiability condition of first–order logic, $\mathcal{M}_m \not\models_{FOL} \pi$. Then by Definition 3.3 and Definition 3.6, $\mathcal{A}^+, FOT(\mathcal{C}) \not\models_{FOL} \pi$. Hence by Definition 3.8, $\mathcal{C} \not\models_S \mathcal{C}'$.
2. π is a declarative unit of the form $\lambda : \alpha$. By Definition 3.4, $\mathcal{M}_m \models_{FOL} FOT(\neg\pi)$, where $FOT(\neg\pi) = [\neg\alpha]^*(\lambda)$. So $\mathcal{M}_m \models_{FOL} [\neg\alpha]^*(\lambda)$. Then by Theorem 4.3, $\mathcal{M}_m \models_{FOL} \neg[\alpha]^*(\lambda)$.
So $\mathcal{M}_m \not\models_{FOL} [\alpha]^*(\lambda)$. Then by Definition 3.3 and Definition 3.6, $\mathcal{A}^+, FOT(\mathcal{C}) \not\models_{FOL} \pi$. Hence by Definition 3.8, $\mathcal{C} \not\models_S \mathcal{C}'$.

∎

Discussion

In this section, we have shown that the MLDS developed in Section 2 is complete with respect to the semantics defined in Section 3. The proof is simpler than the traditional Henkin style proof of completeness for propositional modal logic. The latter requires the construction of maximal consistent sets as *subordinate sets* (see for example [Hughes-68]). This construction involves first defining the 'initial world's maximal consistent set', then defining its subordinate (or accessible worlds') maximal consistent sets and repeating the same process for each one

of them until no more subordinate maximal consistent sets can be constructed. This process varies for each type of modal logic, since the definition of a subordinate set depends on the properties of the accessibility relation (see for example [Hughes-68]).
In our case, there is no need for subordinate maximal consistent sets. The construction of a maximal consistent configuration requires only consistently adding all the declarative units and the R–literals of a MLDL to an initial configuration. It does not require a without imposing any distinction between initial maximal consistent set and subordinate sets. This simplification is due to the fact that in a declarative unit the association of modal formulae with possible worlds is explicitly specified. Moreover the natural deduction rules for modal operators refer explicitly to accessible worlds and to the formulae that hold in accessible worlds (e.g. $\diamond$–Elimination rule).

The construction of a maximal consistent configuration (Definition 4.6) is unique for type of normal modal logic. However no variations in the method of proof are needed because of different properties of the accessibility relations. Again, this simplification is due to the modularity of a MLDS. The properties of the accessibility relation are set aside in a labelling algebra $\mathcal{A}$ which affects only the R–literals of a maximal consistent configuration. The consistency of the R–literals with respect to $\mathcal{A}$ is already guaranteed by the consistency checking included in the construction of a maximal consistent configuration.

In addition, it should be noted that the condition of a finite configuration difference, $\mathcal{C}' - \mathcal{C}$, introduced in the statement of the completeness theorem, is only a technical restriction. It depends on the fact that the notion of derivability has been defined on all configurations, $\mathcal{C} \vdash_S \mathcal{C}'$ (i.e. Definition 2.3). It is possible to give this definition in a slightly different way to eliminate such condition.

Finally, notice that the MLDS defined in Section 2 has the properties of *finiteness* and *compactness*. These properties are proved in Theorem 4.2 and Proposition 4.6 respectively.

5 Correspondence Theorem

In a MLDS, a modal theory is expressed as a configuration, i.e. a 'structure' of several 'local actual worlds' to which sets of modal formulae are associated. In this section we substantiate the claim that the notion of a configuration is strictly more general then the notion of a theory within traditional modal logic. We show this by proving the following two results. First, we show that there exists a correspondence between a MLDS and any sound and complete proof system for traditional modal logic, whenever certain restrictions are imposed on the initial configuration of a MLDS derivation. Second, we show that this correspondence fails if no restriction is imposed. As far as the first result is concerned, the restriction consists of identifying a particular constant symbol in $\mathcal{L}_L$, say W_0, and allowing only initial configurations of the form $\mathcal{C}_i = \langle \mathcal{D}_i, \mathcal{F}_i \rangle$ where $\mathcal{D}_i = \emptyset$ (i.e. no R–literal belongs to $\mathcal{C}_i$), and for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda \neq W_0$, $\mathcal{F}_i(\lambda) = \emptyset$. With this restriction, the only initial assumptions (if any) are modal formulae associated with the label W_0. We show that this corresponds to a traditional feature of existing proof systems for modal logic. This is that a theory is considered to be a collection of formulae holding at only one possible world, called the initial (or actual) world. More precisely, we show that any declarative unit of the form $W_0 : \alpha$ can be derived from an initial configuration of the form $\mathcal{C}_i$ if and only if its formula α is derivable, within a sound and complete axiomatic system for modal logic, from the set of modal formulae that appear in $\mathcal{C}_i$. Before going into more details about this correspondence theorem, we summarise some basic notions about axiomatic systems for modal logic, to which we will refer later in this section.

Background notions

We consider here the definition of axiomatic systems for modal logics given in [Hughes-68]. An *axiomatic system* for a modal logic S, where S is one of the normal modal logics K, T,$K4$,$S4$,$S5$,D, is defined by an *axiomatic basis* and a *derivability relation*. Its semantics is the traditional Kripke possible worlds semantics, in which a model is defined as a tuple $\langle W, <, h \rangle$, where W is a non–empty set of possible worlds, $<$ is a binary relation on W (i.e. the accessibility relation) and h is a valuation function which maps each propositional letter of the modal language into a subset of W (i.e. the set of possible worlds where the propositional letter is true). For basic notions of validity, not validity, satisfiability and not satisfiability within this semantics, the reader is refered to [Hughes-68].

We consider here axiomatic systems which are sound and complete with respect to the Kripke semantics. An *axiomatic basis* is a set of *axiom schemas* (i.e. a selected set of wffs which are valid within the Kripke semantics) and a set of *transformation rules* (i.e. a collection of inference rules which generate a new single formula from a set Φ of arbitrary formulae together with instantiations of axiom schemas).

A proof is defined as a finite sequence of applications of the transformation rules, from some initial set Φ of formulae, where each new formula generated is added to the set Φ. Φ is called an *assumption set.* The length of a proof is equal to the number of applications of the transformation rules. We say that a formula α is derivable from an initial assumption set Φ (possible empty), written $\Phi \vdash \alpha$, if there exists a proof of α from Φ. If $\Phi = \emptyset$ and $\emptyset \vdash \alpha$, then α is either an instantiation of an axiom schema or a wff derived from applications of transformation rules using only axiom schemas. In this case, written also as $\vdash \alpha$, α is a *theorem* and, because of the soundness and completeness of the system, it is semantically valid (i.e. it is true in all the possible worlds of any model of S). Moreover, if α is an instantiation of an axiom schema and $\vdash \alpha$, then its proofs have length equal to 0. If $\Phi \neq \emptyset$ and there exists a proof of a formula β from Φ, then β is said to be derivable from Φ, written $\Phi \vdash \beta$. Because of the soundness and completeness of the system, if $\Phi \vdash \beta$ then β is *semantically entailed* from the initial set of assumptions Φ, written as $\Phi \models \beta$.

In the definition of semantic entailment of modal logic, initial assumptions are often distinguished into *local assumptions* and *global assumptions* [Fitting-90]. The former are formulae true in some possible words of a given Kripke model (i.e 'assumed truths'), and the latter are formulae true in all the possible worlds of a given Kripke model (i.e. 'logical truths'). However, as will be shown later, this distinction is not relevant for the discussion which follows.

In the rest of this section we will consider both a MLDS with its semantics, and an axiomatic system for modal logic with Kripke semantics. It is therefore important to speficy the notation for the axiomatic system in order to avoid any confusion between the two systems. For a MLDS we still adopt all the notations introduced so far in the report.

Notation 5.1
We will use the symbol S to denote one of the normal modal logics K,T,$K4$,$S4$, $S5$,D. The axiomatic system for a modal logic S is denoted by S_{Ax} and its derivability relation by $\vdash_{S_{Ax}}$. A Kripke model is denoted by M_{Ax} and its possible worlds by m_i, for $i \geq 0$. Given a modal formula α, we denote with $M_{Ax}, m \models_{Ax} \alpha$ that α is true at the possible world m in the model M_{Ax}.

$\triangleleft$

We introduce now an axiomatic system for each normal modal logic S, which is defined upon the Hilbert's axiomatisation of propositional calculus [Hilbert-27]. The choice of this system is dictated by the fact that most of its axiom schemas have a straightforward connection with natural deduction rules and therefore it will be easier to prove their correspondence with the MLDS rules. We consider first the basic MLDS K and the axiomatic system K_{Ax}, and we prove a *Simple Correspondence* lemma between K and K_{Ax}. Then, since the other MLDS S and axiomatic systems S_{Ax} are given by the systems K and K_{Ax}, extended respectively with few extra axioms and axiom schemas, the simple correspondence property for these systems will be proved as corollaries of the case K. Finally, we will prove a *Strong Correspondence* theorem for any of the normal modal logic S.

Definition 5.1 (Axiomatic basis for modal logic K)
Let $\mathcal{L}_M$ be a propositional modal language and let α, β and γ be wffs of $\mathcal{L}_M$. An *axiomatic basis for a modal logic* K in the language $\mathcal{L}_M$, written K_{Ax}, is defined as the following set of axiom schemas and inference rules.

Axioms about $\rightarrow$

$\alpha \rightarrow (\beta \rightarrow \alpha)$ [A1]
$(\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta)$ [A2]
$(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$ [A3]
$(\beta \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$ [A4]

Axioms about $\wedge$ and $\vee$

$\alpha \wedge \beta \rightarrow \alpha$ [A5]
$\alpha \wedge \beta \rightarrow \beta$ [A6]
$\alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$ [A7]
$\alpha \rightarrow \alpha \vee \beta$ [A8]
$\beta \rightarrow \alpha \vee \beta$ [A9]
$((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)$ [A10]

Axioms about $\neg$

$(\alpha \rightarrow \beta \wedge \neg\beta) \rightarrow \neg\alpha$ [A11]
$\neg\neg\alpha \rightarrow \alpha$ [A12]

Axioms about $\Box$

$\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ [K]
$\Box\alpha \rightarrow \neg\Diamond\neg\alpha$ [A13]
$\neg\Diamond\neg\alpha \rightarrow \Box\alpha$ [A14]

Axioms about $\Diamond$

$\Diamond\alpha \rightarrow \neg\Box\neg\alpha$ [$A15$]
$\neg\Box\neg\alpha \rightarrow \Diamond\alpha$ [A16]

Inference rules.

$$\frac{\alpha, \quad \alpha \rightarrow \beta}{\beta}$$ [MP]

$$\frac{\vdash_{K_{Ax}} \alpha}{\Box\alpha}$$ [NEC]

$\Box$

It is important to observe that, in the above definition, the antecedents of the inference rule [MP] can also be arbitrary formulae and in this case the consequence β will not be a theorem. In contrast, the antecedent of the inference rule [NEC] can only be a theorem. This is because we should be allowed to infered a formula $\Box\alpha$ only under the condition that α is true in all the possible worlds. Obviously, the formula $\Box\alpha$ will also be a theorem.

Definition 5.2 (Axiomatic basis for modal logic T)
Let $\mathcal{L}_M$ be a propositional modal language and let K_{Ax} be an axiomatic basis for a modal logic K in the language $\mathcal{L}_M$. Let α be a modal formula of $\mathcal{L}_M$. Then an *axiomatic basis for a modal logic* T in the language $\mathcal{L}_M$, written T_{Ax}, is the set of axiom schemas $K_{Ax} \cup \{[T]\}$, where $[T]$ is the axiom schema $\Box\alpha \to \alpha$.

$\Box$

Definition 5.3 (Axiomatic basis for modal logic $K4$)
Let $\mathcal{L}_M$ be a propositional modal language and let K_{Ax} be an axiomatic basis for the modal logic K in the language $\mathcal{L}_M$. Let α be a modal formula of $\mathcal{L}_M$. Then an *axiomatic basis for a modal logic* $K4$ in the language $\mathcal{L}_M$, written $K4_{Ax}$, is the set of axiom schemas $K_{Ax} \cup \{[4]\}$, where [4] is the axiom schema $\Box\alpha \to \Box\Box\alpha$.

$\Box$

Definition 5.4 (Axiomatic basis for modal logic D)
Let $\mathcal{L}_M$ be a propositional modal language and let K_{Ax} be an axiomatic basis for a modal logic K in the language $\mathcal{L}_M$. Let α be a modal formula of $\mathcal{L}_M$. Then an *axiomatic basis for a modal logic* D in the language $\mathcal{L}_M$, written D_{Ax}, is the set of axiom schemas $K_{Ax} \cup \{[D]\}$, where $[D]$ is the axiom schema $\Box\alpha \to \Diamond\alpha$.

$\Box$

Definition 5.5 (Axiomatic basis for modal logic $S4$)
Let $\mathcal{L}_M$ be a propositional modal language and let K_{Ax} be an axiomatic basis for the modal logic K in the language $\mathcal{L}_M$. Let α be a modal formula of $\mathcal{L}_M$. Then an *axiomatic basis for a modal logic* $S4$ in the language $\mathcal{L}_M$, written $S4_{Ax}$, is the set of axiom schemas $K_{Ax} \cup \{[T], [4]\}$, where $[T]$ and [4] are respectively the axiom schemas $\Box\alpha \to \alpha$ and $\Box\alpha \to \Box\Box\alpha$.

$\Box$

Definition 5.6 (Axiomatic basis for modal logic $S5$)
Let $\mathcal{L}_M$ be a propositional modal language and let K_{Ax} be an axiomatic basis for the modal logic K in the language $\mathcal{L}_M$. Let α be a modal formula of $\mathcal{L}_M$. Then an *axiomatic basis for a modal logic* $S5$ in the language $\mathcal{L}_M$, written $S5_{Ax}$, is the set of axiom schemas $K_{Ax} \cup \{[T], [4], [B]\}$, where $[T]$, [4] and $[B]$ are respectively the axiom schemas $\Box\alpha \to \alpha$, $\Box\alpha \to \Box\Box\alpha$ and $\alpha \to \Box\Diamond\alpha$.

$\Box$

Definition 5.7 (Axiomatic system for a modal logic S)
Let $\mathcal{L}_M$ be a propositional modal language, let S_{Ax} be an axiomatic basis for a modal logic S in the language $\mathcal{L}_M$ and let $\vdash_{S_{Ax}}$ be the derivability relation defined upon S_{Ax}. The tuple $\langle S_{Ax}, \vdash_{S_{Ax}}\rangle$ is an axiomatic system for S.

$\square$

The axiomatic systems $\langle S_{Ax}, \vdash_{S_{Ax}}\rangle$ are sound and complete with respect to their Kripke semantics [Hughes-68]. In other words, for any modal formula α, $\vdash_{S_{Ax}} \alpha$ if and only if $\models_{Ax} \alpha$. Analogously, for any set of assumptions $\{\alpha_1, \ldots, \alpha_n\}$ and a formula β, $\alpha_1, \ldots, \alpha_n \vdash_{S_{Ax}} \beta$ if and only if $\alpha_1, \ldots, \alpha_n \models_{Ax} \beta$, with respect to the notion of semantic entailment $\models_{Ax}$ given in Definition 5.8

Lemma 5.1 (Simple Correspondece for modal logic K)
Let $K = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\langle K_{Ax}, \vdash_{K_{Ax}}\rangle$ be the axiomatic system for modal logic K. Let $\mathcal{C}_\emptyset = \langle \mathcal{D}_\emptyset, \mathcal{F}_\emptyset\rangle$, where $\mathcal{D}_\emptyset = \emptyset$ and for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_\emptyset(\lambda) = \emptyset$. Let α be a formula of $\mathcal{L}_M$. Then for all ground terms $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

$$\mathcal{C}_\emptyset \vdash_K \lambda : \alpha \quad \text{if and only if} \quad \vdash_{K_{Ax}} \alpha$$

Proof:
'Only if' half:
We prove the contrapositive statement. We show that given a formula $\alpha \in \mathcal{L}_M$, if $\not\vdash_{K_{Ax}} \alpha$ then there exists a ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$ such that $\mathcal{C}_\emptyset \not\vdash_K \lambda : \alpha$. Since the two derivability relations $\vdash_{K_{Ax}}$ and $\vdash_K$ are both sound and complete with respect to their semantics[23], it is sufficient to prove that if $\not\models_{Ax} \alpha$ then there exists a ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$ such that $\mathcal{C}_\emptyset \not\models_K \lambda : \alpha$. Suppose that $\not\models_{Ax} \alpha$. Since $\mathcal{C}_\emptyset$ is an empty configuration, it is sufficient to show that there exists a semantic structure of K (i.e. a model of $\mathcal{A}^+$) which satisfies the declarative unit $\lambda : \neg\alpha$. By hypothesis and by Kripke semantic definition of validity, there exists a Kripke model which satisfies the formula $\neg\alpha$ (i.e. a countermodel of the formula α). Let $M_{Ax} = \langle W, <, h\rangle$ be such a model. Thus there exists some possible world $m \in W$ such that $M_{Ax}, m \models_{Ax} \neg\alpha$. We assume now a canonical well–ordering relation on the set W[24]. Let $Sat = \{m_i \mid m_i \in W, M_{Ax}, m_i \models_{Ax} \neg\alpha\}$ and let m' be the first element of Sat according to W's canonical well-ordering relation [25]. Then $M_{Ax}, m' \models_{Ax} \neg\alpha$. Let $U = W$ be a universe of discourse. Let I be an interpretation function on the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ defined as follows.

[23] $\vdash_{K_{Ax}}$ is complete with respect to the Kripke semantics and $\vdash_K$ is complete with respect to the semantics defined in Section 3

[24] The existence of a well-ordering relation is guarateed by Zermelo's *Well–Ordering Theorem*

[25] The existence of this first element is guarateed by the fact that Sat is a non empty subset of the well–ordered set W

- For each constant symbol W_i
  $\| W_i \|_I = m'$.
- For each function symbol of the form f_β
  $\| f_\beta \|_I = f_\beta : U \longrightarrow U$ such that for each $m \in W$
  - If $Sat_\beta(m) = \{\tilde{m} \mid \tilde{m} \in W, m < \tilde{m} \textit{ and } M_{Ax}, \tilde{m} \models_{Ax} \beta\}$ is a non–empty set, then $f_\beta(m) = \overline{m}$ where $\overline{m}$ is the first element of $Sat_\beta(m)$ with respect to W's canoncial well–ordering.
  - Otherwise $f_\beta(m) = m_0$, where m_0 is the first element of W according to its assumed canonical well-ordering.
- For each function symbol box_β
  $\| box_\beta \|_I = box_\beta : U \longrightarrow U$ such that for each $m \in W$
  - If for all $\tilde{m} \in W$ $m \not< \tilde{m}$, then $box_\beta(m) = m$
  - If for each $m_i \in Acc(m) = \{\tilde{m} \mid \tilde{m} \in W, m < \tilde{m}\}$, $M_{Ax}, m_i \models_{Ax} \beta$, then $box_\beta(m) = \overline{m}'$, where $\overline{m}'$ is the first element of $Acc(m)$ with respect to the assumed canonical well-ordering of W.
  - Otherwise $box_\beta(m) = \overline{m}''$, where $\overline{m}''$ is the first element of the non–empty set $Sat_{\neg\beta} = \{\tilde{m} \mid \tilde{m} \in W, m < \tilde{m} \textit{ and } M_{Ax}, \tilde{m} \models_{Ax} \neg\beta\}$, according to W's canonical well-ordering.
- For each monadic predicate $[\beta]^*$
  $\| [\beta]^* \|_I = \{m \mid m \in W, M_{Ax}, m \models_{Ax} \beta\}$
- For the binary predicate R
  $\| R \|_I = <$

We prove now that $\langle U, I \rangle$ is a MLDS semantic structure. By Definition 3.3, we need to prove that $\langle U, I \rangle$ is a classical model of the extended algebra $\mathcal{A}^+$. We observe that in a MLDS K the labelling algebra $\mathcal{A} = \{\}$. Therefore the extended algebra $\mathcal{A}^+$ is given only by the set of axiom schemas $(Ax1)$–$(Ax8)$. The proof is by cases on each of the axiom schemas $(Ax1)$–$(Ax8)$. Let β, γ be two wffs of $\mathcal{L}_M$.

- **(Ax1):** $\forall x([\beta \wedge \gamma]^*(x) \equiv ([\beta]^*(x) \wedge [\gamma]^*(x))$
  Let $m \in U$ be an arbitrary element. It is sufficient to prove that $m \in \| [\beta \wedge \gamma]^* \|_I$ if and only if $m \in \| [\beta]^* \|_I$ and $m \in \| [\gamma]^* \|_I$. This follows directly from the definition of $\| \beta \wedge \gamma \|_I$ and the Kripke semantic definition of satisfiability for $\wedge$–formulae.

  The argument for the axiom schemas $(Ax2)$–$(Ax4)$ is analogous to $(Ax1)$.
- **(Ax5):** $\forall x([\Diamond\beta]^*(x) \rightarrow (R(x, f_\beta(x)) \wedge [\beta]^*(f_\beta(x))))$.
  Let $m \in U$ be an arbitrary element. It is sufficient to prove

that if $m \in \| [\Diamond\beta]^* \|_I$ then the tuple $\langle m, f_\beta(m)\rangle$[26] $\in \| R \|_I$ and $f_\beta(m) \in \| [\beta]^* \|_I$. We assume that $m \in \| [\Diamond\beta]^* \|_I$. By definition of I, $M_{Ax}, m \models_{Ax} \Diamond\beta$. Then, by Kripke semantic definition of satisfiability, there exists some possible world $\tilde{m} \in W$ such that $m < \tilde{m}$ and $M_{Ax}, \tilde{m} \models_{Ax} \beta$. Therefore, by definition of I, the set $Sat_\beta(m) \neq \emptyset$ and $f_\beta(m) = \overline{m}$, where $\overline{m}$ is the first element of $Sat_\beta(m)$ according to W's canonical well–ordering. Then $m < f_\beta(m)$ and $M_{Ax}, f_\beta(m) \models_{Ax} \beta$. Hence by definition of I, the tuple $\langle m, f_\beta(m)\rangle \in \| R \|_I$ and $f_\beta(m) \in \| [\beta]^* \|_I$.

- **(Ax6):** $\forall x(\exists y(R(x,y) \wedge [\beta]^*(y)) \rightarrow [\Diamond\beta]^*(x))$.
  We consider the equivalent formula
  $$\forall x \forall y((R(x,y) \wedge [\beta]^*(y)) \rightarrow [\Diamond\beta]^*(x)).$$
  Let m, m' be two arbitrary elements of U. It is sufficient to prove that if the tuple $\langle m, m'\rangle \in \| R \|_I$ and $m' \in \| [\beta]^* \|_I$, then $m \in \| [\Diamond\beta]^* \|_I$. We assume that $\langle m, m'\rangle \in \| R \|_I$ and $m' \in \| [\beta]^* \|_I$. By definition of I, $m < m'$ and $M_{Ax}, m' \models_{Ax} \beta$. Then by Kripke semantic definition of satisfiability of the $\Diamond$ operator, $M_{Ax}, m \models_{Ax} \Diamond\beta$. Hence by definition of I, $m \in \| [\Diamond\beta]^* \|_I$.

- **(Ax7):** $\forall x((R(x, box_\beta(x)) \rightarrow [\beta]^*(box_\beta(x))) \rightarrow [\Box\beta]^*(x))$.
  Let m be an arbitrary element of U. It is sufficient to prove that if the tuple $\langle m, box_\beta(m)\rangle$[27] $\notin \| R \|_I$ or $box_\beta(m) \in \| [\beta]^* \|_I$, then $m \in \| [\Box\beta]^* \|_I$. Suppose first, that the tuple
  $\langle m, box_\beta(m)\rangle \notin \| R \|_I$. This implies, by definition of $\| box_\beta \|_I$, that for all $m' \in W$, $m \not< m'$. Then by Kripke semantic definition of satisfiability of the $\Box$ operator, $M_{Ax}, m \models_{Ax} \Box\beta$. Hence by definition of I, $m \in \| [\Box\beta]^* \|_I$. Suppose now that $box_\beta(m) \in \| [\beta]^* \|_I$. This yields, by definition of $\| box_\beta \|_I$, to one of the following two cases. (i) For all $m' \in W$, $m \not< m'$. Then by Kripke semantic definition of satisfiability of the $\Box$ operator, $M_{Ax}, m \models_{Ax} \Box\beta$. Hence by definition of I, $m \in \| [\Box\beta]^* \|_I$. (ii) For all $m' \in W$ such that $m < m'$, $M_{Ax}, m' \models_{Ax} \beta$. Then again by Kripke semantic definition of satisfiability of the $\Box$ operator, $M_{Ax}, m \models_{Ax} \Box\beta$. Hence by definition of I, $m \in \| [\Box\beta]^* \|_I$.

- **(Ax8):** $\forall x([\Box\beta]^*(x) \rightarrow (\forall y(R(x,y) \rightarrow [\beta]^*(y))))$
  We consider the equivalent formula
  $$\forall x \forall y(([\Box\beta]^*(x) \wedge R(x,y)) \rightarrow [\beta]^*(y))$$
  Let m,m' be two arbitrary elements of U such that $m \in \| [\Box\beta]^* \|_I$ and the tuple $\langle m, m'\rangle \in \| R \|_I$. Then by definition of I,

[26] The definition of the interpretation function I allows us to write $f_\beta(m)$ instead of $\| f_\beta \|_I\ (m)$

[27] The definition of the interpretation function I allows us to write $box_\beta(m)$ instead of $\| box_\beta \|_I\ (m)$

$M_{Ax}, m \models_{Ax} \Box\beta$ and $m < m'$. By Kripke semantic satisfiability definition of the $\Box$ operator, $M_{Ax}, m' \models_{Ax} \beta$. Hence by definition of I, $m' \in \| [\beta]^* \|_I$.

Therefore, the tuple $\langle U, I\rangle$ is a model of $\mathcal{A}^+$. Thus, since by hypothesis $M_{Ax}, m' \models \neg\alpha$ and, by definition of I, $\| W_i \|_I = m'$, where W_i are constant symbols of $Mon(\mathcal{L}_L, \mathcal{L}_M)$, there exists a ground term λ, $\lambda = W_i$ for some $i \geq 0$, such that $I \models_{FOL} [\neg\alpha]^*(\lambda)$. Hence, by definition of $\models_K$, $\mathcal{C}_\emptyset \models_K \lambda : \neg\alpha$.

<u>'If' half:</u>
We need to prove that if $\vdash_{K_{Ax}} \alpha$ then $\mathcal{C}_\emptyset \vdash_K \lambda : \alpha$. Suppose that $\vdash_{K_{Ax}} \alpha$ and that $\alpha_1, \ldots, \alpha_m$, where $m \geq 1$ and $\alpha_m = \alpha$, is the shortest proof of α with length $l \geq 0$. We prove that $\mathcal{C}_\emptyset \vdash_K \lambda : \alpha$ by using induction on l.

Base Case
The base case is when $l = 0$. Then α is an instantiation of one of the axiom schemas of K_{Ax}. We prove that $\mathcal{C}_\emptyset \vdash_K \lambda : \alpha$ by cases on each axiom schema of K_{Ax}.

[A1] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_\emptyset$. We prove that $C_0 \vdash_K \lambda : \alpha \rightarrow (\beta \rightarrow \alpha)$.

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}'_0\langle[\lambda:\alpha]\rangle$	(new assumption)
$\mathcal{C}''_0\langle[\lambda:\alpha],[\lambda:\beta]\rangle$	(new assumption)
$\mathcal{C}''_1\langle[\lambda:\beta],\lambda:\alpha\rangle$	(C–R)
$\mathcal{C}'_1\langle[\lambda:\alpha],\lambda:\beta\rightarrow\alpha\rangle$	($\rightarrow$ I)
$\mathcal{C}_1\langle\lambda:\alpha\rightarrow(\beta\rightarrow\alpha)\rangle$	($\rightarrow$ I)

It is easy to see that the other axioms about the $\rightarrow$ connective can be proved within the MLDS K, by a sequence of ($\rightarrow$ I) and ($\rightarrow$ E) rules. The groups of axioms [A5]–[A6] and [A8]–[A9] can be derived in K using the ($\wedge$ E) and the ($\vee$ I) rules, respectively. Analogously the axioms [A7] and [A10] can be proved in K by using respectively the ($\wedge$ I) and the ($\vee$ E) rules. Finally, the axiom [A11] can be derived in K by using the ($\neg$ I) rule and the axiom [A12] by using the ($\neg$ E) rule.

[K] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_\emptyset$. We prove that $C_0 \vdash_K \lambda : \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$.

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}'_0\langle[\lambda:\Box(\alpha\rightarrow\beta)]\rangle$	(new assumption)

$\mathcal{C}_0''\langle[\lambda:\Box(\alpha\to\beta)],[\lambda:\Box\alpha],\rangle$	(new assumption)
$\mathcal{C}_0'''\langle[R(\lambda, box_\alpha(\lambda))]\rangle$	(new assumption)
$\mathcal{C}_1'''\langle box_\alpha(\lambda):\alpha\to\beta\rangle$	($\Box$ E)
$\mathcal{C}_2'''\langle box_\alpha(\lambda):\alpha\to\beta, box_\alpha(\lambda):\alpha\rangle$	($\Box$ E)
$\mathcal{C}_3'''\langle box_\alpha(\lambda):\beta\rangle$	($\to$ E)
$\mathcal{C}_1''\langle[\lambda:\Box\alpha],\lambda:\Box\beta\rangle$	($\Box$ I)
$\mathcal{C}_1'\langle[\lambda:\Box(\alpha\to\beta)],\lambda:\Box\alpha\to\Box\beta\rangle$	($\to$ I)
$\mathcal{C}_1\langle\lambda:\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)\rangle$	($\to$ I)

[A13] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L,\mathcal{L}_M)$ and let $\mathcal{C}_0=\mathcal{C}_\emptyset$. We prove that $C_0\vdash_K \lambda:\Box\alpha\to\neg\Diamond\neg\alpha$.

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}_0'\langle[\lambda:\Box\alpha]\rangle$	(new assumption)
$\mathcal{C}_0''\langle[\lambda:\Box\alpha],[\lambda:\Diamond\neg\alpha]\rangle$	(new assumption)
$\mathcal{C}_1''\langle R(\lambda, f_\alpha(\lambda)), f_\alpha(\lambda):\neg\alpha\rangle$	($\Diamond$ E)
$\mathcal{C}_2''\langle f_\alpha(\lambda):\neg\alpha, f_\alpha(\lambda):\alpha\rangle$	($\Box$ E)
$\mathcal{C}_3''\langle f_\alpha(\lambda):\bot\rangle$	($\wedge$ I)
$\mathcal{C}_1'\langle[\lambda:\Box\alpha],\lambda:\neg\Diamond\neg\alpha\rangle$	($\neg$ I)
$\mathcal{C}_1\langle\lambda:\Box\alpha\to\neg\Diamond\neg\alpha\rangle$	($\to$ I)

[A14] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L,\mathcal{L}_M)$ and let $\mathcal{C}_0=\mathcal{C}_\emptyset$. We prove that $C_0\vdash_K \lambda:\neg\Diamond\neg\alpha\to\Box\alpha$.

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}_0'\langle[\lambda:\neg\Diamond\neg\alpha]\rangle$	(new assumption)
$\mathcal{C}_0''\langle[\lambda:\neg\Diamond\neg\alpha],[R(\lambda, box_\alpha(\lambda))]\rangle$	(new assumption)
$\mathcal{C}_0'''\langle[\lambda:\neg\Diamond\neg\alpha],[box_\alpha(\lambda):\neg\alpha]\rangle$	(new assumption)
$\mathcal{C}_1'''\langle[\lambda:\neg\Diamond\neg\alpha],\lambda:\Diamond\neg\alpha\rangle$	($\Diamond$ I)
$\mathcal{C}_2'''\langle\lambda:\bot\rangle$	($\wedge$ I)
$\mathcal{C}_1''\langle box_\alpha(\lambda):\neg\neg\alpha\rangle$	($\neg$ I)
$\mathcal{C}_2''\langle box_\alpha(\lambda):\alpha\rangle$	($\neg$ E)
$\mathcal{C}_1'\langle\lambda:\Box\alpha\rangle$	($\Box$ I)
$\mathcal{C}_1\langle\lambda:\neg\Diamond\neg\alpha\ \to\ \Box\alpha\rangle$	($\to$ I)

[A15] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L,\mathcal{L}_M)$ and let $\mathcal{C}_0=\mathcal{C}_\emptyset$. We prove that $C_0\vdash_K \lambda:\Diamond\alpha\to\neg\Box\neg\alpha$.

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}_0'\langle[\lambda:\Diamond\alpha]\rangle$	(new assumption)
$\mathcal{C}_0''\langle[\lambda:\Diamond\alpha],[\lambda:\Box\neg\alpha]\rangle$	(new assumption)
$\mathcal{C}_1''\langle f_\alpha(\lambda):\alpha, R(\lambda, f_\alpha(\lambda))\rangle$	($\Diamond$ E)
$\mathcal{C}_2''\langle f_\alpha(\lambda):\alpha, f_\alpha(\lambda):\neg\alpha\rangle$	($\Box$ E)
$\mathcal{C}_3''\langle\lambda:\bot\rangle$	($\wedge$ I)

$$\mathcal{C}_1'\langle \lambda : \neg\Box\neg\alpha \rangle \qquad (\neg\ \mathrm{I})$$
$$\mathcal{C}_1\langle \lambda : \Diamond\alpha \rightarrow \neg\Box\neg\alpha \rangle \qquad (\rightarrow \mathrm{I})$$

[A16] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_\emptyset$. We prove that $C_0 \vdash_K \lambda : \neg\Box\neg\alpha \rightarrow \Diamond\alpha$.

$$\mathcal{C}_0\langle\rangle$$
$$\mathcal{C}_0'\langle[\lambda : \neg\Box\neg\alpha]\rangle \qquad (\text{new assumption})$$
$$\mathcal{C}_0''\langle[\lambda : \neg\Box\neg\alpha], [\lambda : \neg\Diamond\alpha]\rangle \qquad (\text{new assumption})$$
$$\mathcal{C}_1''\langle[\lambda : \neg\Box\neg\alpha], \lambda : \Box\neg\alpha\rangle \qquad ([\text{A14}] \text{ and } \rightarrow \mathrm{E})$$
$$\mathcal{C}_2''\langle \lambda : \bot \rangle \qquad (\wedge\ \mathrm{I})$$
$$\mathcal{C}_1'\langle \lambda : \neg\neg\Diamond\alpha \rangle \qquad (\neg\ \mathrm{I})$$
$$\mathcal{C}_2'\langle \lambda : \Diamond\alpha \rangle \qquad (\neg\ \mathrm{E})$$
$$\mathcal{C}_1\langle \lambda : \neg\Box\neg\alpha \rightarrow \Diamond\alpha \rangle \qquad (\rightarrow \mathrm{I})$$

Inductive Step

Assume, by inductive hopthesis, that for any formula α' such that $\vdash_{K_{Ax}} \alpha'$ and such that there exists a proof $\alpha_1', \ldots, \alpha_m'$, where $\alpha_m' = \alpha'$, of lenght $l = n$, with $n > 0$, then for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}_\emptyset \vdash_K \lambda : \alpha'$.

Suppose now that $\vdash_{K_{Ax}} \alpha$ with a shortest proof $\alpha_1, \ldots, \alpha_m$, where $\alpha_m = \alpha$, of length $l = n+1$, with $n \geq 0$. We consider this proof to be composed of the first n–steps (i.e. $\alpha_1, \ldots, \alpha_i$, with $1 \leq i < m$) in which a formula β has been derived (i.e. $\beta \in \{\alpha_1, \ldots, \alpha_i\}$) and the last step in which the formula α is derived. Then $\vdash_{K_{Ax}} \beta$ accepts a proof of length strictly less than $n+1$. Therefore, by the inductive hypothesis, for an arbitrary ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}_\emptyset \vdash_K \lambda : \beta$. Thus, there exists a configuration $\mathcal{C}_\beta$ such that $\lambda : \beta \in \mathcal{C}_\beta$ and $\mathcal{C}_\emptyset \vdash_K \mathcal{C}_\beta$. Therefore, by the transitivity property of $\vdash_K$, it is sufficient to prove that $\mathcal{C}_\beta \vdash_K \lambda : \alpha$.

[MP] Suppose that the last step is given by the application of the [MP] rule. Then there exist two formulae α_k and $\alpha_k \rightarrow \alpha$ in the sequence $\alpha_1, \ldots, \alpha_{m-1}$, which are either instantiations of axiom schemas or derived formulae. In both the two cases, $\vdash_{K_{Ax}} \alpha_k$ and $\vdash_{K_{Ax}} \alpha_k \rightarrow \alpha$ accept proofs of length strictly less than $n+1$. Therefore, by the inductive hypothesis, $\mathcal{C}_\emptyset \vdash_K \lambda : \alpha_k$ and $\mathcal{C}_\emptyset \vdash_K \lambda : \alpha_k \rightarrow \alpha$. Since $\mathcal{C}_\emptyset \subset \mathcal{C}_\beta$, by the monotonicity property of $\vdash_K$, $\mathcal{C}_\beta \vdash_K \lambda : \alpha_k$. Therefore there exists a configuration $\mathcal{C}'$ such that $\lambda:\alpha_k \in \mathcal{C}'$ and $\mathcal{C}_\beta \vdash_K \mathcal{C}'$. Moreover, since $\mathcal{C}_\emptyset \subset \mathcal{C}'$, by monotonicity, $\mathcal{C}' \vdash_K \lambda:\alpha_k \rightarrow \alpha$. Therefore, there exists a configuration $\mathcal{C}''$ such that $\lambda:\alpha_k \rightarrow \alpha \in \mathcal{C}''$ and $\mathcal{C}' \vdash_K \mathcal{C}''$. Without loss of generality we can assume that $\lambda:\alpha_k \in \mathcal{C}''$ too. Then, $\mathcal{C}'' \vdash_K \lambda:\alpha$. Hence, by the transitivity property of $\vdash_K$, $\mathcal{C}_\beta \vdash_K \lambda:\alpha$.

[**Nec**] Suppose that the last step is given by the application of the [NEC] rule. Then there exists a formula α_k in the sequence $\alpha_1,\ldots,\alpha_{m-1}$ such that $\vdash_{K_{Ax}} \alpha_k$ and $\alpha = \Box\alpha_k$. Therefore, $\vdash_{K_{Ax}} \alpha_k$ accepts a proof of length less than $n+1$ and then by inductive hypothesis, $\mathcal{C}_\emptyset \vdash_K \lambda:\alpha_k$. Since λ is any arbitrary ground term of $Mon(\mathcal{L}_L,\mathcal{L}_M)$, it is also the case that $\mathcal{C}_\emptyset \vdash_K box_{\alpha_k}(\lambda):\alpha_k$. Since $\mathcal{C}_\emptyset \subset \mathcal{C}_\beta$, by the monotonicity property of $\vdash_K$, $\mathcal{C}_\beta \vdash_K box_{\alpha_k}(\lambda):\alpha_k$. Analogously, $\mathcal{C}_\beta + [R(\lambda, box_{\alpha_k}(\lambda))] \vdash_K box_{\alpha_k}(\lambda):\alpha_k$. Hence, by the definition of the ($\Box$ I) rule of MLDS, $\mathcal{C}_\beta \vdash_K \lambda:\Box\alpha_k$.

■

We show now that the above result holds for the other normal propositional modal logic S.

Corollary 5.1 (Simple Correspondence for modal logic T)
Let $T = \langle\langle\mathcal{L}_L,\mathcal{L}_M\rangle,\mathcal{A},\mathcal{R}\rangle$ be a MLDS and let $\langle T_{Ax},\vdash_{T_{Ax}}\rangle$ be an axiomatic system for modal logic T. Let $\mathcal{C}_\emptyset = \langle\mathcal{D}_\emptyset,\mathcal{F}_\emptyset\rangle$, where $\mathcal{D}_\emptyset = \emptyset$ and for any ground term $\lambda \in Mon(\mathcal{L}_L,\mathcal{L}_M)$, $\mathcal{F}_\emptyset = \emptyset$. Let α be a formula of $\mathcal{L}_M$. Then for all ground terms $\lambda \in Mon(\mathcal{L}_L,\mathcal{L}_M)$

$$\mathcal{C}_\emptyset \vdash_T \lambda:\alpha \quad \text{if and only if} \quad \vdash_{T_{Ax}} \alpha$$

Proof:
<u>'Only If' half:</u>
The argument is analogous to the 'Only If' half of Lemma 5.1, taking into account, however, the following observations. Firstly, we consider in this case a Kripke model $M_{Ax} = \langle W,<,h\rangle$ whose accessibility relation $<$ is reflexive. This means that for any possible world $m \in W$, $m < m$. Secondly, since in the MLDS T the labelling algebra $\mathcal{A} = \{\forall x R(x,x)\}$, and therefore the extended algebra $\mathcal{A}^+ = \mathcal{A} \cup \{(Ax1),\ldots,(Ax8)\}$, it is also necessary to prove, as extra case, that the MLDS semantic structure $\langle U,I\rangle$ satisfies the axiom $\forall x R(x,x)$. This follows directly by the definition of $\| R \|_I$ and the reflexivity property of the accessibility relation $<$.

<u>'If' half:</u>
The proof is similar to the 'If' half of Lemma 5.1. However, in the

base case it is also necessary to consider an extra case for the axiom schema $[T]$ whose proof is given below.

[T] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_\emptyset$. We show that $\mathcal{C}_0 \vdash_T \lambda : \Box\alpha \rightarrow \alpha$.

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\hline
\quad \mathcal{C}_0'\langle[\lambda : \Box\alpha]\rangle & \text{(new assumption)} \\
\hline
\quad \mathcal{C}_1'\langle\lambda : \Box\alpha, R(\lambda, \lambda)\rangle & \text{(R–A)} \\
\hline
\quad \mathcal{C}_2'\langle\lambda : \alpha\rangle & (\Box\ \text{E}) \\
\hline
\mathcal{C}_1\langle\lambda : \Box\alpha \rightarrow \alpha\rangle & (\rightarrow \text{I})
\end{array}
$$

■

Corollary 5.2 (Simple Correspondence for modal logic D)
Let $D = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\langle D_{Ax}, \vdash_{D_{Ax}}\rangle$ be an axiomatic system for modal logic D. Let $\mathcal{C}_\emptyset = \langle\mathcal{D}_\emptyset, \mathcal{F}_\emptyset\rangle$, where $\mathcal{D}_\emptyset = \emptyset$ and for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_\emptyset = \emptyset$. Let α be a formula of $\mathcal{L}_M$. Then for all ground terms $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

$$\mathcal{C}_\emptyset \vdash_D \lambda : \alpha \quad \text{if and only if} \quad \vdash_{D_{Ax}} \alpha$$

Proof:
<u>'Only If' half:</u>
The argument is analogous to the 'Only If' half of Lemma 5.1, taking into account, however, the following observations. Firstly, we consider in this case a Kripke model $M_{Ax} = \langle W, <, h\rangle$ whose accessibility relation $<$ respects the seriality property. This means that for any possible world $m \in W$, there exists a possible world m' such that $m < m'$. Secondly, since in the MLDS D the labelling algebra $\mathcal{A} = \{\forall x R(x, succ(x))\}$, and therefore the extended algebra $\mathcal{A}^+ = \mathcal{A} \cup \{(Ax1), \ldots, (Ax8)\}$, it is also necessary to prove, as extra case, that the MLDS semantic structure $\langle U, I\rangle$ satisfies the axiom $\forall x R(x, succ(x))$. This follows directly by the definition of $\| R \|_I$ and the seriality property of the accessibility relation $<$.

<u>'If' half:</u>
The proof is similar to the 'If' half of Lemma 5.1. However, in the base case it is also necessary to consider an extra case for the axiom schema $[D]$ whose proof is given below.

[D] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_\emptyset$. We show that $\mathcal{C}_0 \vdash_D \lambda : \Box\alpha \rightarrow \Diamond\alpha$.

$$\dfrac{\mathcal{C}_0\langle\rangle}{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{C}_0'\langle[\lambda:\Box\alpha]\rangle}{\mathcal{C}_1'\langle\lambda:\Box\alpha, R(\lambda, succ(\lambda))\rangle}}{\mathcal{C}_2'\langle succ(\lambda):\alpha\rangle}}{\mathcal{C}_3'\langle\lambda:\Diamond\alpha\rangle}}{\mathcal{C}_1\langle\lambda:\Box\alpha\rightarrow\Diamond\alpha\rangle}}$$

(new assumption)
(R–A)
($\Box$ E)
($\Diamond$ I)
($\rightarrow$ I)

■

Corollary 5.3 (Simple Correspondence for modal logic $K4$)
Let $K4 = \langle\langle\mathcal{L}_L,\mathcal{L}_M\rangle,\mathcal{A},\mathcal{R}\rangle$ be a MLDS and let $\langle K4_{Ax},\vdash_{K4_{Ax}}\rangle$ be an axiomatic system for modal logic $K4$. Let $\mathcal{C}_\emptyset = \langle\mathcal{D}_\emptyset,\mathcal{F}_\emptyset\rangle$, where $\mathcal{D}_\emptyset = \emptyset$ and for any ground term $\lambda \in Mon(\mathcal{L}_L,\mathcal{L}_M)$, $\mathcal{F}_\emptyset = \emptyset$. Let α be a formula of $\mathcal{L}_M$. Then for all ground terms $\lambda \in Mon(\mathcal{L}_L,\mathcal{L}_M)$

$$\mathcal{C}_\emptyset \vdash_{K4} \lambda:\alpha \quad \text{if and only if} \quad \vdash_{K4_{Ax}} \alpha$$

Proof:
'Only If' half:
The argument is analogous to the 'Only If' half of Lemma 5.1, taking into account, however, the following observations. Firstly, we consider in this case a Kripke model $M_{Ax} = \langle W,<,h\rangle$ whose accessibility relation $<$ is transitive. This means that for any three possible worlds $m_1, m_2, m_3 \in W$, if $m_1 < m_2$ and $m_2 < m_3$ then $m_1 < m_3$. Secondly, since in the MLDS $K4$ the labelling algebra is given by $\mathcal{A} = \{\forall x,y,z(R(x,y)\wedge R(y,z)) \rightarrow R(x,z)\}$, and therefore the extended algebra $\mathcal{A}^+ = \mathcal{A}\cup\{(Ax1),\ldots,(Ax8)\}$, it is also necessary to prove, as extra case, that the MLDS semantic structure $\langle U,I\rangle$ satisfies the axiom $\forall x,y,z(R(x,y)\wedge R(y,z))\rightarrow R(x,z)$. This follows directly by the definition of $\| R\|_I$ and the transitivity property of the accessibility relation $<$.

'If' half:
The proof is similar to the 'If' half of Lemma 5.1. However, in the base case it is also necessary to consider an extra case for the axiom schema [4] whose proof is given below.

[4] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L,\mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_\emptyset$. We show that $\mathcal{C}_0 \vdash_T \lambda:\Box\alpha\rightarrow\Box\Box\alpha$.

$$\dfrac{\mathcal{C}_0\langle\rangle}{\dfrac{\mathcal{C}_0'\langle[\lambda:\Box\alpha]\rangle}{}}$$

(new assumption)

$$
\begin{array}{lr}
\mathcal{C}_0''\langle[R(\lambda, box_{\Box\alpha}(\lambda))]\rangle & \text{(new assumption)} \\
\mathcal{C}_0'''\langle[R(box_{\Box\alpha}(\lambda), box_{\alpha}(box_{\Box\alpha}(\lambda)))]\rangle & \text{(new assumption)} \\
\mathcal{C}_1'''\langle R(\lambda, box_{\alpha}(box_{\Box\alpha}(\lambda)))\rangle & \text{(R–A)} \\
\mathcal{C}_2'''\langle box_{\alpha}(box_{\Box\alpha}(\lambda)) : \alpha\rangle & (\Box\ \text{E}) \\
\mathcal{C}_1''\langle box_{\Box\alpha}(\lambda) : \Box\alpha\rangle & (\Box\ \text{I}) \\
\mathcal{C}_1'\langle \lambda : \Box\Box\alpha\rangle & (\Box\ \text{I}) \\
\mathcal{C}_1\langle \lambda : \Box\alpha \rightarrow \Box\Box\alpha\rangle & (\rightarrow \text{I})
\end{array}
$$

■

Corollary 5.4 (Simple Correspondence for modal logic $S4$)
Let $S4 = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\langle S4_{Ax}, \vdash_{S4_{Ax}}\rangle$ be an axiomatic system for modal logic $S4$. Let $\mathcal{C}_\emptyset = \langle\mathcal{D}_\emptyset, \mathcal{F}_\emptyset\rangle$, where $\mathcal{D}_\emptyset = \emptyset$ and for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_\emptyset = \emptyset$. Let α be a formula of $\mathcal{L}_M$. Then for all ground terms $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

$$\mathcal{C}_\emptyset \vdash_{S4} \lambda : \alpha \quad \text{if and only if} \quad \vdash_{S4_{Ax}} \alpha$$

Proof:
We observe that the modal logic $S4$ is equivalent to the union of modal logics $K4$ and T. Therefore, the proof follows directly from Lemma 5.1, Corollary 5.1 and Corollary 5.3.

■

Corollary 5.5 (Simple Correspondence for modal logic $S5$)
Let $S5 = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\langle S5_{Ax}, \vdash_{S5_{Ax}}\rangle$ be an axiomatic system for modal logic $S5$. Let $\mathcal{C}_\emptyset = \langle\mathcal{D}_\emptyset, \mathcal{F}_\emptyset\rangle$, where $\mathcal{D}_\emptyset = \emptyset$ and for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_\emptyset = \emptyset$. Let α be a formula of $\mathcal{L}_M$. Then for all ground terms $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

$$\mathcal{C}_\emptyset \vdash_{S5} \lambda : \alpha \quad \text{if and only if} \quad \vdash_{S5_{Ax}} \alpha$$

Proof:
'Only If' half:
The argument is analogous to the 'Only If' half of Lemma 5.1, taking into account, however, the following observations. Firstly, we consider in this case a Kripke model $M_{Ax} = \langle W, <, h\rangle$ whose accessibility relation $<$ is riflexive, transitive and symmetric. This means that the extended algebra $\mathcal{A}^+$ of the MLDS $S5$ is given by the extended algebra $\mathcal{A}^+$ of a MLDS $S4$ together with the axiom $\forall x, y(R(x, y) \rightarrow R(y, x))$. Then, given the Corollary 5.4, it is still necessary to prove that the MLDS $S5$ semantic structure $\langle U, I\rangle$ satisfies

the axiom $\forall x, y(R(x,y) \rightarrow R(y,x))$. This follows directly from the definition of $\| R \|_I$ and from the simmetry property of the accessibility relation $<$.

<u>'If' half:</u>
The proof is similar to the 'If' half of Lemma 5.1. However, it is also necessary in the base case to consider an extra case for the axiom schema $[B]$, given below, as the cases for the axiom schemas $[T]$ and [4] are already guarateed by Corollary 5.4.

[B] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_\emptyset$. We show that $\mathcal{C}_0 \vdash_T \lambda : \alpha \rightarrow \Box\Diamond\alpha$.

$$\dfrac{\mathcal{C}_0\langle\rangle}{\dfrac{\dfrac{\mathcal{C}_0'\langle[\lambda:\alpha]\rangle \quad \text{(new assumption)}}{\dfrac{\dfrac{\dfrac{\mathcal{C}_0''\langle[R(\lambda, box_{\Diamond\alpha}(\lambda))]\rangle \quad \text{(new assumption)}}{\mathcal{C}_1''\langle R(box_{\Diamond\alpha}(\lambda), \lambda)\rangle \quad \text{(R–A)}}}{\mathcal{C}_2''\langle box_{\Diamond\alpha}(\lambda) : \Diamond\alpha\rangle \quad (\Diamond\ \text{I})}}{\mathcal{C}_1'\langle\lambda : \Box\Diamond\alpha\rangle \quad (\Box\ \text{I})}}}{\mathcal{C}_1\langle\lambda :\ \alpha : \rightarrow \Box\Diamond\alpha\rangle \quad (\rightarrow\ \text{I})}}$$

■

Corollary 5.6
Let $S = \langle\langle\mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\langle S_{Ax}, \vdash_{S_{Ax}}\rangle$ be the axiomatic system for modal logic S. Let $\mathcal{C}_\emptyset = \langle\mathcal{D}_\emptyset, \mathcal{F}_\emptyset\rangle$, where $\mathcal{D}_\emptyset = \emptyset$ and for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_\emptyset(\lambda) = \emptyset$. Let α be a formula of $\mathcal{L}_M$. Then for all ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

$$\mathcal{C}_\emptyset \models_S \lambda : \alpha \quad \text{if and only if} \quad \models_{Ax} \alpha$$

Proof:
It follows directely from Lemma 5.1, Corollaries 5.1- 5.5, and from the respective soundness and completeness of $\vdash_S$ and $\vdash_{S_{Ax}}$ ■

Corollary 5.6 shows how the Kripke semantics definition of validity of a modal formula α (i.e. $\models_{Ax} \alpha$) can be reformulated in terms of MLDS semantics. Given a modal logic S and its Kripke semantics, a formula α is valid if and only if it holds in all the possible worlds of each model of S. Within a MLDS, this notion is equivalent to the condition that all the declarative units $\lambda : \alpha$, obtained from α by taking each ground term λ of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ are true in all the MLDS semantic structures (i.e in all the models of $\mathcal{A}^+$). This result conferms even more the intuition, pointed out in Section 1, that labels can be regarded as possible worlds.

Moreover, in the 'Only If' part of Lemma 5.1 we have also shown how, given a Kripke model M_{Ax}, it is possible to construct a MLDS semantic structure, $\langle U, I\rangle$ (e.g. a corresponding MLDS model) such that for any modal formula satisfied in M_{Ax}, there exists a corresponding declarative unit satisfied in $\langle U, I\rangle$. Therefore, $\langle U, I\rangle$ preserves the set of satisfied modal formulae. These results yield to the following formulation in terms of MLDS semantics of Kripke semantics notions of satisfiability and validity in a Kripke model.

A formula α is valid in a Kripke model M_{Ax} if and only if the declarative unit $\lambda : \alpha$ is satisfied in the corresponding MLDS model for all the labels λ of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Analogously, a formula α is satisfied in a Kripke model M_{Ax}, if and only if there exists a declarative unit $\lambda : \alpha$, for some label $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, which is satisfied in the corresponding MLDS model.
We extend now the above results to the notion of semantic entailment. We consider, in particular, the following definition of Kripke semantic entailment [Hughes-68].

Definition 5.8 (Entailment in Kripke semantics)
Let $\mathcal{L}_M$ be a propositional modal language, let $\langle S_{Ax}, \vdash_{S_{Ax}}\rangle$ be an axiomatic system for a modal logic S, let $\langle W, <\rangle$ be a Kripke frame for S and let C be the associated class of Kripke models $\langle W, <, h_i\rangle$. Let $\alpha_1, \ldots, \alpha_n, \beta$ be modal formulae in the language $\mathcal{L}_M$. Then, $\alpha_1, \ldots, \alpha_n$ semantically entail β, written $\alpha_1, \ldots, \alpha_n \models_{Ax} \beta$, if for each Kripke model $\langle W, <, h\rangle$ belonging to C, *β is true in all the possible worlds where $\alpha_1, \ldots, \alpha_n$ are true.*

$\square$

We observe that in [Fitting-90] the notion of semantic entailment is more general. It is defined in terms of global assumptions and local assumptions as follows. A formula β is said to be entailed from a set of global assumptions G and a set of local assumptions U if and only if for all the models where the formulae of G are valid, β is true in all the possible worlds where all the formulae of U are true. However, we are only interested in global assumptions given by the instantiations of the axiom schemas of a modal logic S. In this case given a modal logic S, the class of models validating G is the entire class of models which characterizes S. Under this assumption, Fitting's notion of semantic entailment is equivalent to the one given above, where the specification of G has been omitted for simplicity and where $\alpha_1, \ldots, \alpha_n$ are local assumptions. We also recall that each normal modal logic S satisfies the *deduction theorem* for local assumptions (see *Local Deduction Theorem* in [Fitting-90]). We will use this result in the following theorem.

Theorem 5.1 (Strong Correspondence for a normal modal logic S)
Let $N \in \{K, T, K4, KB, S4, S5, D, D4, DB\}$ be a normal propositional modal

logic. Let $\langle N_{Ax}, \vdash_{N_{Ax}} \rangle$ be an axiomatic system for N which is sound and complete with respect to Kripke semantics. Let N–MLDS be the corresponding MLDS. Let $\Psi = \{\alpha_1, \ldots, \alpha_n\}$ $(n \geq 0)$ be an arbitrary set of wffs of N, and let $\mathcal{C}$ be the configuration of N–MLDS consisting only of all declarative units of the form $W_0 : \alpha_i$, $\alpha_i \in \Psi$, and containing no R–literals. Then for any formula γ of N,

$$\mathcal{C} \vdash_N W_0 : \gamma \quad \text{if and only if} \quad \Psi \vdash_{N_{Ax}} \gamma$$

Proof:
The proof is by induction on n.

Base Case
The base case is when $n = 0$. Then the theorem holds by Lemma 5.1 and Corollaries 5.1– 5.5.

Inductive Step
Assume, by inductive hypothesis, that for any configuration $\tilde{\mathcal{C}}\langle W_0 : \alpha'_1, \ldots, W_n : \alpha'_m \rangle$, with $0 \leq m$, and $\tilde{\Psi} = \{\alpha'_1, \ldots, \alpha'_m\}$, then for any formula γ', $\tilde{\mathcal{C}} \vdash_N W_0 : \gamma'$ if and only if $\tilde{\Psi} \vdash_{N_{Ax}} \gamma'$.

Suppose that $n = m + 1$, so $n > 0$. Let $\mathcal{C}'$ be the configuration $\mathcal{C}'\langle W_0 : \alpha_1, \ldots, W_0 : \alpha_{n-1} \rangle$ and let Ψ' be the set $\{\alpha_1, \ldots, \alpha_{n-1}\}$. We rewrite $\mathcal{C} = \mathcal{C}' \cup \{W_0 : \alpha_n\}$ and $\Psi = \Psi' \cup \{\alpha_n\}$. Since the two derivability relations $\vdash_N$ and $\vdash_{N_{Ax}}$ are both sound and complete with respect to their semantics, it is sufficient to prove that $\mathcal{C}' \cup \{W_0 : \alpha_n\} \models_N W_0 : \gamma$ if and only if $\Psi' \cup \{\alpha_n\} \models_{Ax} \gamma$. We assume that $\mathcal{C}' \cup \{W_0 : \alpha_n\} \models_N W_0 : \gamma$. By definition of $\models_N$, this is equivalent to $\mathcal{A}^+, FOT(\mathcal{C}'), [\alpha_n]^*(W_0) \models_{FOL} [\gamma]^*(W_0)$. By the deduction theorem of first–order logic, this is equivalent to $\mathcal{A}^+, FOT(\mathcal{C}') \models_{FOL} [\alpha_n]^*(W_0) \rightarrow [\gamma]^*(W_0)$. By axiom schema $(Ax4)$ of $\mathcal{A}^+$, we can equivalently write $\mathcal{A}^+, FOT(\mathcal{C}') \models_{FOL} [\alpha_n \rightarrow \gamma]^*(W_0)$. By definition of $\models_N$, this is equivalent to $\mathcal{C}' \models_N W_0 : \alpha_n \rightarrow \gamma$. By the inductive hypothesis, this is if and only if $\Psi' \models_{Ax} \alpha_n \rightarrow \gamma$ which is equivalent to $\Psi' \cup \{\alpha_n\} \models_{Ax} \gamma$ by the local deduction theorem of modal logic N.

■

The above theorem shows an equivalence between a MLDS S and an axiomatic system S_{Ax}, given the restriction on the MLDS that the only initial assumptions (if any) are modal formulae associated with a particular constant symbol W_0. However, a MLDS allows also for a more general form of initial assumptions – declarative units in the initial configuration may have different labels. Moreover, information may be explicitly given about the accessibility relation. In an axiomatic system for modal logic, local assumptions are formulae that must hold in the same possible world. Therefore, it is not possible in general to represent an initial configuration of a MLDS in terms of a set of initial assumptions within an axiomatic system. Hence, the notion of a MLDS is strictly more general than the conventional notion of a normal propositional modal logic with Kripke semantics. We expect the greater expressivity MLDS's provide to be an advantage in several areas of application.

6 Related work

As pointed out in the introduction of this report, a MLDS is an 'hybrid' formalization of modal logic, based on implicit and explicit formalisms. Therefore, it is interesting to compare our natural deduction rules with other proof systems for propositional modal logic that belong to these two categories. Before going into detailed comparisons, it is important to underline a major difference between a MLDS and proof systems of the first category. A MLDS allows us to consider more then one local actual world within a modal theory (e.g. a configuration). This feature will also be incorporated in predicate MLDS. In contrast, proof systems of the first category refer to a modal theory as a set of formulae holding at only one possible world, 'the initial world'. In this respect, a MLDS is more general then any of these systems. Therefore, in the following comparisons we will restrict our attention to initial configurations containing only one label, say W_0. W_0, is interpreted as the initial world. Under this restriction, we will assume that a MLDS corresponds to a traditional formalization of modal logic. We intend to prove this correspondence formally in future work.

6.1 Fitting's natural deduction system

Fitting's natural deduction system [Fitting-83] is the main reference for proof theory in natural deduction for modal logic. It is based on the implicit formalization: a modal language is defined as extention of a classical language and its semantics is a Kripke possible worlds semantics. Considering the propositional case, the inference rules for classical connectives are defined in the same style as [Prawitz-65]. They include introduction and elimination rules for each of the logical connectives. Some of these rely on the notion of 'defining and closing new boxes' to respectively introduce and discharge new assumptions in a proof. This mechanism requires extra rules, called *reiteration rules* which specify how formulae can be 'reiterated' from the outside to the inside of a box. The proof within a 'box' is considered to be a *subordinate derivation* in the sense that it provides a condition for the overlying inference rule to be applied. The same principle is applied to the inference rules for modal operators, following the idea of Fitch's *strict subordinate derivations* [Fitch-52]. Fitting defines a particular type of box called a *strict box*. The inference rules for modal operators are defined as rules for 'creating' and 'closing' strict boxes[28]. A strict box can represent either a 'general accessible world derivation' or a 'particular accessible world derivation'. However, only one of these interpretations can be used within one proof. In the first case the natural deduction system is said to be in A–style and in the second case in I–style. Finally, extra rules for reiterating formulae from the outside to the inside of a strict box are defined. These are called *strict iteration rules*, and are based on some 'special rules' which formalize the properties of the accessibility relation for the different modal logics.

As far as classical connectives are concerned, in a MLDS the notion of 'boxes' is replaced by explicit definitions of subordinate derivations within the particular inference rules that require them as conditions (e.g. $\rightarrow$–Elimination). In this way, no reiteration rules are needed. Concerning modal operators, a MLDS includes introduction and elimination rules for both $\Box$ and $\Diamond$. These rules allow us to refer to accessible worlds explicitly and to infer formulae associated with them within the same proof, without introducing strict subordinate derivations (e.g $\Diamond$–Elimination rule). This is due to the definition of a declarative unit which explicitly associates labels with modal formulae. Therefore Fitting's strict iteration rules are also unnecessary in a MLDS. In addition, the distiction between I-style and A-style proofs is eliminated, as are the special rules for the different modal logics. In fact, the latter are replaced by the the single R–assertion rule, which allows us to infer R–literals (i.e. relations between labels) according to the particular labelling algebra $\mathcal{A}$[29]. In this way, the application of the same modal rules, together with the inferred R–literals, leads to the same results that are given by several special rules in the Fitting's system.

A comparison with the predicate case will be part of future work. We can already point out that the existing natural deduction proof systems for predicate modal logic do not cover the case of varying domains, which will be the main feature of a predicate MLDS. Fitting has presented only a tableaux proof system for this general semantics of predicate modal logics, and has conjectured [Fitting-83] that similar techniques could be applied in a natural deduction style proof system. We believe that the predicate extention of the MLDS presented in this report will add substance to this conjecture.

6.2 Fitting's prefixed tableaux system

As said in the above section, a particular tableaux proof system has been developed in [Fitting-83], both for the propositional and predicate case, in order to deal with a varying domain semantics. This work and the MLDS paradigm seem

[28]They are not expressed as introduction and elimination rules

[29]The labelling algebra depends on the particular modal logic

to have in common the idea of adding explicit declarations of possible worlds within a proof. Therefore, despite their different methodologies (tableaux and natural deduction), it is interesting to compare them. We consider in this report only the propositional case. Future comparisons between a predicate prefixed tableaux system and a predicate MLDS will provide a more complete picture of their respective advantages and disadvantages.

Fitting's prefixed tableaux system is defined upon an expanded modal language in which modal formulae are *prefixed*. A prefix is a 'name' for possible worlds and it facilitates the construction of alternate tableaux at the occurrences of modal operators. These prefixes are defined as sequences of characters and are somewhat analogous to labels. A relation '*accessible from*' on prefixes is also defined according to the type of modal logic. In the simple case of a modal logic K, a prefix is accessible from another if the latter is 'included' in the former. For example the prefix $1, 1$ is accessible from the prefix 1. The tableau rules for modal operators define the accessible prefixes and the formulae associated with them. Because of the different definitions of the relation 'accessible from' on prefixes, side conditions are specified next to each modal rule. These side conditions define the type of infered prefixes according to the particular modal logic. The disadvantage of Fitting's prefixed tableaux system is that it does not provide a logical way of handling prefixes and relations between them, even though it allows for syntactical representation of possible worlds. In contrast, a MLDS allows for the logical specification of possible worlds and accessibility relations between them. This simplifies the natural deduction rules for modal operators. They no longer require side conditions and they can be applied in a consistent way in any proof, independently of the particular modal logic.

6.3 Remarks on [Benevides-90]

It is also interesting to compare the work developed in this report with Benevides's proof [Benevides-90]. Both are concerned with extentional techniques to deal with modal operators (i.e. explicit representation of possible worlds) within a natural deduction paradigm. However the motivation behind Benevides's work seems somewhat different from that here. We have shown, with a MLDS, that it is possible to express possible worlds and accessibility relations syntactically in a separate theory set aside from modal logic. In this way, the *intentional* feature of modal operators[30] is explicitly represented. As a consequence, it is possible in a MLDS to define introduction and elimination natural deduction rules for modal operators in the same way as classical connectives, without stating any particular side conditions. Therefore, within a MLDS theory (i.e. configuration), modal

[30]For example, given a formula of the form $\Diamond A$, its interpretation is intentional in the sense that it relays on the existence of an accessible world where A holds.

operators and classical connectives are considered to be object level operators.

In contrast, Benevides provides a constructive natural deduction system for propositional modal logic in which modal operators are presented as 'higher order' connectives, fundamently different from 'object level' classical connectives. The term 'constructive' reflects the intuition which is behind Benevides's work. A modal system can be considered as a set of sets of formulae where one is the 'initial set' and the others are sets generated from the initial one as 'accessible sets of formulae', by the application of modal rules[31]. Each set s_i has an associated assertion sign, $\vdash_{s_i}$. '$\vdash_{s_i} A$' can be read as 'the proposition A is asserted to be in the set s_i'. Introduction and elimination rules are defined only for the modal operator $\Box$. However a more extended formalization of this system has been presented subsequently in [Benevides-91] where introduction and elimination rules for $\Diamond$ operator are also included. The 'indexes' for the different sets of formulae are constructed using a function symbol $f_\Box$ (and also a function symbol $f_\Diamond$ in [Benevides-91]) applied repeatedly to the singleton set $\{s_0\}$ (initial world). The inference rules for the $\Box$ operator are defined on different 'assertion signs'. For example, a $\Box$–Elimination rule is as follows.

$$\frac{\vdash_s \Box A}{\vdash_{f_\Box(s)} A}$$

This proof system includes an extra set of inference rules for each type of modal logic. From this point of view, a MLDS is more general. The syntactical representation of the accessibility relation between labels and the definition of the R–Assertion rule make a MLDS system suitable for any type of modal logic. It does not require any further addition of natural deduction rules.

6.4 Remarks on [Ohlbach-91]

So far, we have compared the work developed in this report with proof systems based on implicit formalizations of modal logic. However, given the 'hybrid' feature of our work, it is also interesting to investigate the relationship between a MLDS and proof systems based on translation methods. A first translation method from modal logic to classical logic was defined by Moore [Moore-80]. According to this translation a formula $\Box P$ is translated into $\forall x(R(0,x) \rightarrow P'(x))$, where 0 represents the 'initial world' and P' is identical to P but with an additional 'world context argument'. Ohlbach [Ohlbach-91] defined a similar translation method called *relational translation* for both propositional and predicate modal logic. Ohlbach's main concern, in both this work and other work based on different translation methods, is to provide an efficient automated theorem prover

[31]The higher level of modal operators is represented by the fact that their rules define how to generate 'accessible sets of formulae'

for modal logic. The idea is to develop a general framework for translating logical formulae from one logic into another for which efficient theorem provers already exist. In this way, modal formulae can be translated into predicate logic formulae in order to exploit the well established and efficient classical predicate logic theorem provers (e.g. those based on resolution). In a relational translation the modal logic's possible worlds structure is explicitly formalized by introducing a distinguished binary predicate symbol, R, to represent the accessibility relation. For example, formulae of the form $\Box P$ are translated into $\forall x(R(0,x) \rightarrow P^{'}(x))$ and formulae of the form $\Diamond P$ are translated into $\exists x(R(0,x) \wedge P^{'}(0))$, in the case where P is a propositional letter. For more complex formulae, the translation function is applied recursively. Therefore, given a modal formula with n modal operators, its first order translation defines a first order formula with at least 2^n literals. Hence an exponential number of first order clauses is generated. This is one of the main disadvantages of this type of translation method.

In a MLDS, the idea of formalizing the accessibility relation explicitly and introducing possible world arguments is also used, althought the possible world argument is represented as an associated label instead of a proper argument. The main difference is that a MLDS also includes a modal logic syntax together with a first order syntax for the accessibility relation. In this way, a complete translation into first order logic is not required before applying a proof system. This seems to be a promising solution to the inefficiency problem of Ohlbach's relational translation. Consider the following example to clarify this point, which is of a modal theory composed of the following two formulae, in a modal logic K.

$$\begin{array}{ll} \Diamond\Diamond\Box[(\Box p \wedge q) \wedge s] & \text{(a)} \\ \Box\neg\Diamond\Box[(\Box p \wedge q) \wedge s] & \text{(b)} \end{array}$$

The two formulae are inconsistent, so that it is possible to derive falsity, $\perp$. The derivation of $\perp$, in the case of a MLDS, is very straight–forward, and it is given by the following proof. For simplisity, the two formulae are rewritten as (a) $\Diamond\alpha$ and (b) $\Box\neg\alpha$, where α is equal to $\Diamond\Box[(\Box p \wedge q) \wedge s]$.

$$\begin{array}{ll} \mathcal{C}\langle W_0 : \Diamond\alpha, W_0 : \Box\neg\alpha\rangle & \Diamond\text{–E} \\ \hline \mathcal{C}_1\langle f_\alpha(W_0) : \alpha, R(W_0, f_\alpha(W_0)), \Box\neg\alpha\rangle & \Box\text{–E} \\ \hline \mathcal{C}_2\langle f_\alpha(W_0) : \alpha, f_\alpha(W_0) : \neg\alpha\rangle & \wedge\text{–I} \\ \hline \mathcal{C}_3\langle f_\alpha(W_0) : \perp\rangle & \end{array}$$

In Ohlbach's first order translation the proof of $\perp$ is much more complex. The two formulae are first translated into first order clauses and then an empty clause is proved from the resulting first order theory. We only show the translation into clauses here. This gives an idea of the dimension and complexity of the proof. The formulae (a) and (b) generate respectively the following sets of clauses:

(a) $R(0, a)$
$R(a, b)$
$\neg R(b, x) \vee S(x)$
$\neg R(b, x) \vee Q(x)$
$\neg R(b, x) \vee \neg R(x, y) \vee P(y)$

(b) $\neg R(0, x) \vee \neg R(x, y) \vee R(y, f(x, y))$
$\neg R(0, x) \vee \neg R(x, y) \neg S(f(x, y))$
$\neg R(0, x) \vee \neg R(x, y) \vee \neg Q(f(x, y))$
$\neg R(0, x) \vee \neg R(x, y) \vee R(y, g(x, y))$
$\neg R(0, x) \vee \neg R(x, y) \vee \neg P(g(x, y))$

Hence, even for a small modal theory, the translation is complex. The disadvantage is that each resolution on a predicate different from R has to be accompanied by a chain of resolutions with the R–literals.

Ohlbach has himself pointed out the inefficiency of the relational translation and has defined [Ohlbach-91] an optimaized translation method of modal logic into classical logic. It is mainly based on a functional definition of the accessibility relation. But this method still has the disadvantage of requiring a special unification algorithm whose complexity is still exponential. We will not compare a MLDS with this second translation method here since the comparison would be less direct.

7 Future work

For the purpose of the PhD thesis, this work will be extended as follows.

7.1 A predicate MLDS

We have pointed out in the introduction of this report that a main motivation for a MLDS is the possibility of defining a sound and complete proof system for predicate modal logic with respect to a semantics of *varying domains*. So the extention of the current work to the predicate case will be an essential part of the future work. In a varying domain semantics (see [Fitting-83]), different universes of discourse are associated with different possible worlds, even though the possible worlds might be in relation to each other. Therefore, a corresponding proof system needs a way of expressing the existence of different elements in different possible worlds. The explicit declarations of possible worlds (i.e. labels) in a MLDS will facilitate the definition of such proof systems. Before describing the way in which a MLDS might be extended to the predicate case, we present an example of an inference rule for quantifiers and we illustrate how a predicate MLDS might be able to distinguish between formulae like $\exists x \Diamond A(x)$ and $\Diamond \exists x A(x)$ proof theoretically.

A simple $\exists$–Elimination rule for a MLDS might be as follows:

$$\frac{\mathcal{C}\langle \lambda : \exists x A(x) \rangle}{\mathcal{C}\langle \lambda : A(c(\lambda)), \lambda : c(\lambda) \rangle}$$

If we apply this rule to the two formulae (i) $\Diamond\exists x A(x)$ and (ii) $\exists x \Diamond A(x)$, we can generate different proofs, as it is shown below.

(i)

$$\cfrac{\cfrac{\mathcal{C}\langle \lambda : \Diamond\exists x A(x) \rangle}{\mathcal{C}'\langle f_{\exists x A(x)}(\lambda) : \exists x A(x), R(\lambda, f_{\exists x A(x)}(\lambda)) \rangle}\ \Diamond\text{-E}}{\mathcal{C}''\langle f_{\exists x A(x)}(\lambda) : A(c(f_{\exists x A(x)}(\lambda))), f_{\exists x A(x)}(\lambda) : c(f_{\exists x A(x)}(\lambda)) \rangle}\ \exists\text{–E}$$

(ii)

$$\cfrac{\cfrac{\mathcal{C}\langle \lambda : \exists x \Diamond A(x) \rangle}{\mathcal{C}'\langle \lambda : \Diamond A(c(\lambda)), \lambda : c(\lambda) \rangle}\ \exists\text{-E}}{\mathcal{C}''\langle f_{A(c(\lambda))}(\lambda) : A(c(\lambda)), R(\lambda, f_{A(c(\lambda))}(\lambda)) \rangle}\ \Diamond\text{–E}$$

The main idea is to define a predicate modal language $\mathcal{L}_M$ in which constant symbols are replaced by unary function symbols with labels as their arguments. In this way, we could express the link between constants and labels (or possible worlds). As shown in the above examples, an $\exists$–Elimination rule will introduce 'new unary function symbols' instead of the traditional 'skolem constants' (see [Fitting-83]) and $n+1$–arity function symbols, where one argument is given by the label, instead of n–arity 'skolem functions' (see [Fitting-83]), in the case where the existential quantifier depends on n universally quantified variable. As a consequence, the application of $\exists$–Elimination rule will generate different instantiation for an existential variable according to the label (or 'possible world') in which it is applied.

This idea has been informally suggested by Gabbay [Gabbay-92a].

A MLDS for predicate case will include elimination and introduction rules for existential and universal quantifiers as well as some additional rules, called *visa rules* [Gabbay-92a], which will specify the conditions under which 'constant symbols' can be transfered from one label to another. Via these rules, it will be possible to represent in a MLDS modal logics based either on 'constant domains' or on 'increasing domains', so that comparisons with other existing proof systems will be possible.

In the light of what has been said above, we list the steps that we will be following in order to extend a MLDS to the predicate case.

Step 1
The language $\mathcal{L}_M$ will be defined as a predicate modal language, where unary function symbols will be introduced instead of constant symbols. Moreover it will 'share' the set of ground terms of the labelling language $\mathcal{L}_L$[32]

Step 2
Introduction and elimination rules for existential and universal quantifiers will be defined, in the same style as the natural deduction rules presented in Section 2 of this report. Two extra rules (Visa rules) might be introduced, to deal with the association of terms of $\mathcal{L}_M$ with labels. They might possibly be defined as follows.

$$\frac{\mathcal{C}\langle \lambda_1 : c(\lambda'), R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_2 : c(\lambda')\rangle} \qquad \frac{\mathcal{C}\langle \lambda_2 : c(\lambda'), R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_1 : c(\lambda')\rangle}$$

Step 3
The extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ will be defined as two–sorted first order logic, one sort for the ground terms of the language $\mathcal{L}_L$ and the other for the ground terms of $\mathcal{L}_M$. Extra predicates will be included in order to express the association between terms of $\mathcal{L}_M$ and labels of $\mathcal{L}_L$. To do so, we will follow Ohlbach's idea [Ohlbach-91] of using a two–sorted predicate $Exist$. Particular attention will be given to the relation between the monadic predicates of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and the arguments of their associated modal formulae. Finally, new axiom schemas will be introduced in the extended algebra $\mathcal{A}^+$ in order to formalize the semantic properties of varying domains. $\mathcal{A}^+$ will be a two–sorted first order theory.

Step 4
A new semantic structure will be defined for the two–sorted extended algebra $\mathcal{A}^+$.

Step 5
The soundness theorem will be proved, following the same style presented in Section 3, with respect to the new extended algebra and the new definition of a semantic structure.

[32] This is due to the fact that the unary function symbols of $\mathcal{L}_M$ will have labels of the language $\mathcal{L}_L$, as arguments.

Step 6
The completeness theorem will also be proved following the method defined in Section 4 of this report. Modifications will be made on some of the main definitions and theorems, such as the construction of a maximal consistent configuration and the construction of a maximal interpretation. Further properties of a maximal consistent configuration with respect to quantifiers will be proved.

Step 7
We will also specify the conditions under which a predicate MLDS will correspond to a constant domain modal logic. We will show that the typical 'Barcan formula' $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$ [Hughes-68] could then be proved in a MLDS. Analogously we will define the conditions under which a predicate MLDS will correspond to an increasing domain modal logic and we will show, in this case, that the 'converse Barcan formula' $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$ can also be proved.

7.2 Applications

The work presented in this report and its extention to the predicate case suggest many other avenues of investigation. In fact, it would be interesting to make applicative case studies, in which *states* and *states transitions* are basic phenomena, within a MLDS framework.

The correspondence between MLDS and a traditional formalization of modal logic will guarantee that the existing modal logic extentions such as belief logic, action logic can also be formalized within the framework of a MLDS. Therefore it would be interesting to investigate, for example, the formalization of systems based on the treatment of knowledge and belief as a MLDS. In fact, for these type of systems the idea is important that besides the real world, other worlds or 'frames of mind' have to be taken into account where different facts may hold. An adequate MLDS formalization for such systems will require the introduction of different sorts of accessibility relations and the investigation of how they might interact each other. This might lead to multi–modal labelled deductive systems.

However, we would be particulally interested in defining problems or applications that can be formalized via the propositional MLDS as it has been defined in this report, in which the use of configurations as 'structures of possible worlds' could directly be exploited. Any comments or suggestions would be greatly appreciated!

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