The Convex Hull is Computable!

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Abstract

Despite a huge number of algorithms and articles published on robustness issues relating to the convex hull of a finite number of points in $\mathbb{R}^d$, the question of computability of the convex hull, important as it is, has never been addressed in the literature. In this paper, we use the domain-theoretic computable solid modeling framework to show that the convex hull of a finite number of computable points in $\mathbb{R}^d$ is indeed computable.

1 Introduction

Despite a huge number of algorithms and articles published on robustness issues relating to the convex hull of a finite number of points in $\mathbb{R}^d$, the question of computability of the convex hull, important as it is, has never been addressed in the literature [3, 1, 2, 4, 7, 8].

We use the domain-theoretic computable solid modeling framework introduced in [6] to show that the convex hull of a finite number of computable points in $\mathbb{R}^d$ is indeed computable, i.e. there exist two computable sequences of open rational polyhedra approximating, respectively, the interior and the exterior of the convex hull. Furthermore, one can compute two rational polyhedra, one in the interior and one in the exterior of the convex hull which are close to it with respect to either the Hausdorff metric or the Lebesgue measure with any given degree of accuracy.

2 Solid Modeling

Solid modeling and computational geometry are based on classical topology and geometry in which the basic predicates and operations, such as membership, subset inclusion, union and intersection, are not continuous and therefore not computable. A sound computational framework for solids modeling using domain theory has been introduced by Edalat and Lieutier in [6]. In this section, we give the formal definitions of a number of notations in computable solid modeling used in this paper. (For more details see [6].)

Definition 2.1 *The solid domain* $(\mathbf{S}X, \sqsubseteq)$ *of a topological space* X *is the set of ordered pairs* (A, B) *of disjoint open subsets of* X *endowed with the information order:* $(A_1, B_1) \sqsubseteq (A_2, B_2) \iff A_1 \subseteq A_2$ *and* $B_1 \subseteq B_2$.

Solid domain is a mathematical model for representing rigid solids. An element (A, B) of $\mathbf{S}X$ is called a *partial solid*: A and B are intended to capture, respectively, the interior and the exterior (interior of the complement) of a solid object, possibly, at some finite stage of computation.

In this paper we are considering the solid domain $\mathbf{S}\mathbb{R}^d$. In order to endow $\mathbf{S}\mathbb{R}^d$ with an effective structure, two different countable bases that are recursively equivalent have been introduced in [6]. To show the computability of the convex hull algorithm we use the partial rational polyhedra as the solid domain basis.

A *rational d-simplex* in $\mathbb{R}^d$ is the convex hull of $d+1$ points with rational coordinates that do not lie on the same hyper-plane. An *open rational polyhedron* is the interior of a finite union of rational d-simplexes. Starting with an effective enumeration of the rational d-simplexes, one can obtain an effective enumeration $(P_i)_{i\in\omega}$ of the set of open rational polyhedra with $P_i = \emptyset$ iff $i = 0$. The relations $cl(P_i) \subseteq P_j$ is decidable. Rational polyhedra are closed under the binary intersection and the regularized binary union. These operations are computable as they rely only on rational arithmetic and comparison of rational numbers.

A *partial open rational polyhedron* is a pair of disjoint open rational polyhedra. From the effective enumeration $(P_i)_{i\in\omega}$ of open rational polyhedra, one can obtain an effective enumeration $(\mathbb{P}_i)_{i\in\omega}$ of the partial open rational polyhedra.

The domain-theoretic notion of computability has the essential weakness of lacking a quantitative measure for the rate of convergence of basis elements to a computable element. This shortcoming has been redressed by enriching the domain-theoretic notion of computability with an additional requirement which allows a quantitative degree of approximation.

Definition 2.2 *A computable partial solid* (A, B) *is* μ-computable *if* $\mu(A)$ *and* $\mu(B)$ *are both computable real numbers.*

Definition 2.3 *A computable partial solid* (A, B) *is* Hausdorff computable *if there is a total recursive function* f *such that:*

- $A = \cup_{i\in\omega} O_{\alpha(f(i))}$ *with* $d_H(\overline{A}, \overline{O}_{\alpha(f(i))}) < 2^{-i}$ *and* $d_H(A^c, O^c_{\alpha(f(i))}) < 2^{-i}$.
- $B = \cup_{i\in\omega} O_{\beta(f(i))}$ *with* $d_H(\overline{B}, \overline{O}_{\beta(f(i))}) < 2^{-i}$ *and* $d_H(B^c, O^c_{\beta(f(i))}) < 2^{-i}$.

3 Partial Convex Hull

Assume that N points in the plane $\mathbb{R}^d$ are given and we want to compute the convex hull of them. In computable analysis, each real number can be approximated with a nested effective sequence of rational intervals; therefore each point in $\mathbb{R}^d$ is approximated with a nested effective sequence of rational d-rectangles. This framework is also compatible with solid modeling in practice. For example, in CAD, because of uncertainties of the input data, each point in the plane has a threshold of accuracy and can therefore be considered as a rational rectangle.

To capture points in the plane, we consider $\mathbf{I}\mathbb{R}^2$, the domain of all non-empty compact rectangles $[a, b] \times [c, d]$ in the plane, with the whole plane as the bottom element, partially ordered by reverse inclusion.

Note that the domain $\mathbf{I}\mathbb{R}^2$ can be considered as a sub-domain of $\mathbf{S}\mathbb{R}^2$. A point p of the plane can be represented as a partial solid $(\varnothing, \mathbb{R}^2 \setminus \{p\})$. Similarly, a compact rectangle R in $\mathbf{I}\mathbb{R}^2$ is represented by the partial solid $(\varnothing, \mathbb{R}^2 \setminus R)$, and we refer to it as a partial point.

Given N d-rectangles in $\mathbb{R}^d$, N partial points, we define a partial solid in such a way that whenever the N d-rectangles are actually refined into smaller and smaller d-rectangles converging into N points, then the sequence of the corresponding partial solids converges to the convex hull of these N points. In other words, we define a continuous map $f : (\mathbf{I}\mathbb{R}^d)^N \to \mathbf{S}\mathbb{R}^d$ for computing the partial convex hull of N given d-rectangles.

Here we consider the 2-dimensional case, the whole discussion can be generalized in a same way to the d-dimensional. We define partial convex hull function as following:

$$\begin{array}{cccc} f: & (\mathbf{I}\mathbb{R}^2)^N & \to & \mathbf{S}\mathbb{R}^2 \\ & \overline{R} & \mapsto & (I, E) \end{array}$$

where $\overline{R} = (R_1, \ldots, R_N)$ represents an ordered list of rectangles in the plane $\mathbb{R}^2$. The open sets I and E stand for the interior and the exterior of the partial convex hull of $\overline{R}$. The exterior of the convex hull of $\overline{R}$ is the set of points of the plane that are surely not in the convex hull of the (ordinary) points and the interior is the set of points that are surely in the convex hull of the set of (ordinary points).

In order to give the formal definition, let C be the classical convex hull map taking a set of points to the convex hull of them, considered as a compact subset of the plane

$$\begin{array}{cccc} C: & (\mathbb{R}^2)^N & \to & \mathcal{C}\mathbb{R}^2 \\ & (x_1, \ldots, x_N) & \mapsto & \{\sum_{i=1}^N \lambda_i x_i \mid \sum_{i=1}^N \lambda_i = 1 \text{ with } \lambda_i \geq 0\} \end{array}$$

where $\mathcal{C}\mathbb{R}^2$ is the set of all non-empty compact sets with the Hausdorff metric. For a given ordered list of N d-rectangles $\overline{R}$ in $(\mathbf{I}\mathbb{R}^2)^N$ define

$$P(\overline{R}) = \{(p_1, \ldots, p_N) \mid p_j \in R_j \text{ for } j = 1, \ldots, N\},$$

to be the set of all possible N-tuples of points of the d-rectangles. We now put

$$(I,E) = \left((\bigcap_{p \in P(\overline{R})} C(p))^{\circ}, (\bigcup_{p \in P(\overline{R})} C(p))^{c} \right).$$

Lemma 3.1 *The map f is monotone.*

Proof. Consider two order lists of rectangles $\overline{R}_1$ and $\overline{R}_2$ in $(\mathbf{I}\mathbb{R}^2)^N$ such that $\overline{R}_1 \sqsubseteq \overline{R}_2$. Let $f(\overline{R}_1) = (I_1, E_1)$ and $f(\overline{R}_2) = (I_2, E_2)$ we have

$$f(\overline{R}_1) \sqsubseteq f(\overline{R}_2) \Leftrightarrow I_1 \subseteq I_2 \text{ and } E_1 \subseteq E_2.$$

¿From the definition, it follows that

$$\begin{aligned} \overline{R}_1 \sqsubseteq \overline{R}_2 &\Rightarrow P(\overline{R}_2) \subseteq P(\overline{R}_1) \\ &\Rightarrow \bigcap_{p \in P(\overline{R}_1)} C(p) \subseteq \bigcap_{q \in P(\overline{R}_2)} C(q) \\ &\Rightarrow I_1 \subseteq I_2. \end{aligned}$$

Similarly $E_1 \subseteq E_2$. □

An alternative way to compute the interior part of the partial convex hull, using the idea of the support function, introduced below.

Define $\mathcal{H}$ to be the set of all the open half-planes, then

$$C(X) = \bigcap \{H | H \in \mathcal{H}, X \subset H\}.$$

We showed that $I = (\cap_{p \in P(\overline{R})} C(p))^{\circ}$, now with the above notation we have:

$$\begin{aligned} \bigcap_{p \in P(\overline{R})} C(p) &= \bigcap_{p \in P(\overline{R})} \bigcap \{H | H \in \mathcal{H}, C(p) \subset H\} \\ &= \bigcap \{H | H \in \mathcal{H}, \exists p \in P(\overline{R}), C(p) \subset H\} \\ &= \bigcap \{H | H \in \mathcal{H}, \forall j = 1, \ldots, N : \exists p_j \in R_j, p_j \in H\} \end{aligned}$$

For each s in S^1 (S^1 is the unit circle, S^{d-1} the unit sphere in $\mathbb{R}^d$), we call $\mathcal{H}_s$ to the set of all the half-planes whose equation can be written as $s.x < b$. For any half-plane H, there exists a unique $s \in S^1$ such that $H \in \mathcal{H}_s$. We have then :

$$\cap_{p \in P(\overline{R})} C(p) = \bigcap_{s \in S^1} \bigcap \{H | H \in \mathcal{H}_s, \forall j = 1, \ldots, N : \exists p_j \in R_j, p_j \in H\}.$$

Consider the last intersection in the above Formula. Since the intersection of half-planes sharing the same direction s is itself a half-plane of direction s (in this case a closed one), therefore the equation of the intersection half-plane can be expressed as

$$s.x \geq M \text{ with } M = \max_j \min_{p_j \in R_j} s.p_j.$$

But the the minimum value of $s.p_j$ for all $p_j \in R_j$ is reached when p_j is one of the vertices of R_j. So in the above Formula one can restrict p_j to one of the vertices of R_j. Define $P^*(\overline{R})$ to be the set of all possible N-tuples of vertices of the rectangles R_j :

$$P^*(\overline{R}) = \{(v_1, \ldots,\ v_N) \mid v_j \text{ is a vertex of } R_j \text{ for } j = 1, \ldots, N\}.$$

This proves the following lemma.

Lemma 3.2 *We have* $I = (\cap_{v \in P^*(\overline{R})} C(v))^\circ$. □

Define $V : (\mathbf{I}\mathbb{R}^2)^N \to \mathbb{R}^{4N}$ with $V(\overline{R}) = (v_j^1, v_j^2, v_j^3, v_j^4)_{j=1}^N$, an N-tuple of 4-tuples, considered as a $4N$-tuple. Here $(v_j^1, v_j^2, v_j^3, v_j^4)_{j=1}^N$ are the vertices of the rectangle R_j starting from bottom left corner and going in anti clockwise direction. From the classical definition of the convex hull it follows that $\cup_{p \in P(\overline{R})} C(p) = C(V(\overline{R}))$, which proves the following lemma.

Lemma 3.3 *We have* $E = \left(C(V(\overline{R}))\right)^c$. □

We need the following two lemmas for the proof of the main proposition.

Lemma 3.4 *The convex hull map* $C : (\mathbb{R}^2)^N \to \mathcal{C}\mathbb{R}^2$*, is a continuous function with respect to the product topology on* $(\mathbb{R}^2)^N$ *and the Hausdorff metric on* $\mathcal{C}\mathbb{R}^2$*.*

Proof. First we show the uniform continuity of C when all but one of its arguments are fixed, that is : for any $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$d\left((x_1, x_2, \ldots, x_N), (y_1, x_2, \ldots, x_N)\right) < \delta$$

implies

$$d_H(C(x_1, x_2, \ldots, x_N), C(y_1, x_2, \ldots, x_N)) < \varepsilon.$$

In fact $\delta = \varepsilon$ gives the required condition. Let

$$\begin{aligned} A &= \{\lambda_1 x_1 + \sum_{i=2}^N \lambda_i x_i \mid \sum_{i=1}^N \lambda_i = 1 \text{ with } \lambda_i \geq 0\} \\ B &= \{\lambda_1 y_1 + \sum_{i=2}^N \lambda_i x_i \mid \sum_{i=1}^N \lambda_i = 1 \text{ with } \lambda_i \geq 0\}. \end{aligned}$$

Define $A_r = \{x \in \mathbb{R}^2 | \, d(x, a) \leq r \text{ for some } a \in A\}$ (the definition of B_r is similar). We have

$$\begin{aligned} d_H(C(x_1, x_2, \ldots, x_N), C(y_1, x_2, \ldots, x_N)) &= d_H(A, B) \\ &= \inf_r \{A \subseteq B_r \text{ and } B \subseteq A_r\}. \end{aligned}$$

Note that

$$d\left((x_1, x_2, \ldots, x_N), (y_1, x_2, \ldots, x_M)\right) = d(x_1, y_1).$$

Now assume that $d(x_1, y_1) < \varepsilon$. For each $a = \lambda_1 x_1 + \sum_{i=2}^{N} \lambda_i x_i$ in A, define $b \in B$ with $b = \lambda_1 y_1 + \sum_{i=2}^{N} \lambda_i x_i$. Then

$$\begin{aligned} d(a,b) &= |\lambda_1 x_1 + \sum_{i=2}^{N} \lambda_i x_i - \lambda_1 x_1 - \sum_{i=2}^{N} \lambda_i x_i| \\ &= \lambda_1 |x_1 - y_1| \leq \lambda_1 \varepsilon \leq \varepsilon. \end{aligned}$$

Thus $a \in B_\varepsilon$ which proves $A \subseteq B_\varepsilon$. In the same way, one can show that $B \subseteq A_\varepsilon$ and we get $d_H(A,B) < \varepsilon$.

Next, note that

$$\begin{aligned} d_H\left(C(x_1, x_2, \ldots, x_N), C(y_1, y_2, \ldots, y_N)\right) \leq\ & d_H\left(C(x_1, x_2, \ldots, x_N), C(y_1, x_2, \ldots, x_N)\right) + \\ & d_H\left(C(y_1, x_2, \ldots, x_N)), C(y_1, y_2, x_3, \ldots, x_N)\right) + \\ & \ldots + \\ & d_H\left(C(y_1, y_2, \ldots, y_{N-1}, x_N)), C(y_1, y_2, \ldots, y_M)\right) \ , \end{aligned}$$

from which the continuity of C follows from the above uniform continuity. □

Lemma 3.5 *The map* $- \cap - : \mathcal{C}\mathbb{R}^2 \times \mathcal{C}\mathbb{R}^2 \to \mathcal{C}\mathbb{R}^2$ *is continuous at all* $(A, B) \in \mathcal{C}\mathbb{R}^2 \times \mathcal{C}\mathbb{R}^2$ *such that* $A \cap B$ *is a regular non-empty closed set.* □

Corollary 3.6 *The map*

$$\begin{array}{cccc} C': & (\mathbf{I}\mathbb{R}^2)^N & \to & \mathcal{C}\mathbb{R}^2 \\ & (R_1, \ldots, R_N) & \mapsto & (\bigcap_{v \in P^*(\overline{R})} C(v))^\circ \end{array}$$

is continuous. □

Proposition 3.7 *The partial convex hull function* f *is Scott continuous.*

Proof. From Lemma 3.1, f is monotone. It remains to show that f also preserves lubs of directed sets. Since $(\mathbf{I}\mathbb{Q}^2)^N$ is a basis for continuous domain $(\mathbf{I}\mathbb{R}^2)^N$, it is sufficient to prove that f preserves lubs of increasing chains in $(\mathbf{I}\mathbb{Q}^2)^N$. Assume that $(\overline{Q}_i)_{i \in \omega}$ is a given increasing chain with $\bigsqcup_{i \in \omega} \overline{Q}_i = \overline{R}$.

By Lemmas 3.2 and 3.3 we have

$$\begin{aligned} f(\overline{R}) &= (I, E) = \left((\bigcap_{v \in P^*(\overline{R})} C(v))^\circ, (C(V(\overline{R})))^c \right) \\ \bigsqcup_{i \in \omega} f(\overline{Q}_i) &= (\cup_{i \in \omega} I_i, \cup_{i \in \omega} E_i) \quad \text{where} \\ I_i &= (\bigcap_{v \in P^*(\overline{Q}_i)} C(v))^\circ \ , \ E_i = (C(V(\overline{Q}_i)))^c \end{aligned}$$

Now from Lemma 3.4 and Corollary 3.6 it follows that $f(\overline{R}) = \bigsqcup_{i \in \omega} f(\overline{Q}_i)$. □

Corollary 3.8 *Let $p = (p_1, \ldots, p_N) \in (\mathbf{I}\mathbb{R}^2)^N$ be an N-tuple of non-collinear points in the plane $\mathbb{R}^2$, then $f(p) = (I, E)$ is a maximal partial solid in $\mathbf{S}\mathbb{R}^2$ with I equal to the interior and E equal to the complement of the convex hull of the points p_j.*

□

3.1 Computability

In the previous section we proved that $f : (\mathbf{I}\mathbb{R}^2)^N \to \mathbf{S}\mathbb{R}^2$ is a continuous function, It is easy to check that f is computable. Consider $(\mathbf{I}\mathbb{Q}^2)^N$ as a countable basis for $(\mathbf{I}\mathbb{R}^2)^N$ and $(\mathbb{P}_i)_{i\in\omega}$ (partial open rational polygon) as a countable basis for $\mathbf{S}\mathbb{R}^2$. They give effective structures for the ω-continuous domains $(\mathbf{I}\mathbb{R}^2)^N$ and $\mathbf{S}\mathbb{R}^2$. The continuous map f is now computable if the set

$$\{< m, n > | \mathbb{P}_m \ll f(\overline{Q}_n)\}$$

is r.e. Let $f(\overline{Q}_n) = (I, E)$ then I and E are two rational polygons in $\mathbb{R}^2$. In Section 3 we saw how to compute I_n and E_n. Therefore

$$\begin{aligned} \mathbb{P}_m \ll f(\overline{Q}_n) &\Leftrightarrow \mathbb{P}_m \ll (I_n, E_n) \\ &\Leftrightarrow (P_{\alpha(m)}, P_{\beta(m)}) \ll (P_k, P_l) \\ &\Leftrightarrow (cl(P_{\alpha(m)}) \subseteq P_k) \ \& cl(P_{\beta(m)}) \subseteq P_l) \end{aligned}$$

The last relation is recursive and the set $\{< m, n > | \mathbb{P}_m \ll f(\overline{Q}_n)\}$ is in fact a recursive set. This implies that f is a computable map.

In order to give a quantitative degree of approximation we will prove the Lebesgue and Hausdorff computability of f in the next two sections.

Remark If a rectangle $R \in \mathbf{I}\mathbb{R}^2$ is Lebesgue (Hausdorff) computable then its vertices are computable. If $(\overline{R}_n)_{n\in\omega}$ in $(\mathbf{I}\mathbb{R}^2)^N$ is a Lebesgue (Hausdorff) computable sequence then the sequence of $4N$-tuples, $\left(V(\overline{R}_n)\right)_{n\in\omega}$, is computable.

3.2 Lebesgue Computability

In this section we discuss the Lebesgue computability of $f : (\mathbf{I}\mathbb{R}^2)^N \to \mathbf{S}\mathbb{R}^2$.

Proposition 3.9 *The map $f : (\mathbf{I}\mathbb{R}^2)^N \to \mathbf{S}\mathbb{R}^2$ takes any N-tuple of μ-computable rectangles to a μ-computable partial solid in $\mathbf{S}\mathbb{R}^2$.*

Proof. Assume that $\overline{R} = (R_1, \ldots, R_N)$ is given and each R_j is μ-computable. Let $f(\overline{R}) = (I, E)$, we have

$$f(\overline{R}) = \bigsqcup_{i\in\omega} \{f(\overline{Q}_i) | \overline{Q}_i \ll \overline{R}\} = (\cup_{i\in\omega} I_i, \cup_{i\in\omega} E_i).$$

The partial solid (I, E) is μ-computable iff there exists a total recursive function $\rho : \mathbb{N} \to \mathbb{N}$ such that

$$\mu(I) - \mu(I_{\rho(i)}) < 2^{-i} \text{ and } \mu(E) - \mu(E_{\rho(i)}) < 2^{-i} \text{ see [6].}$$

First consider the exterior part. Note that $\mu(E) - \mu(E_i)$ is the area of the region between the boundaries of E and E_i (Figure 1.a). For a polygon $C \in \mathbb{R}^2$ let $l(C)$ be the length of the perimeter of C.

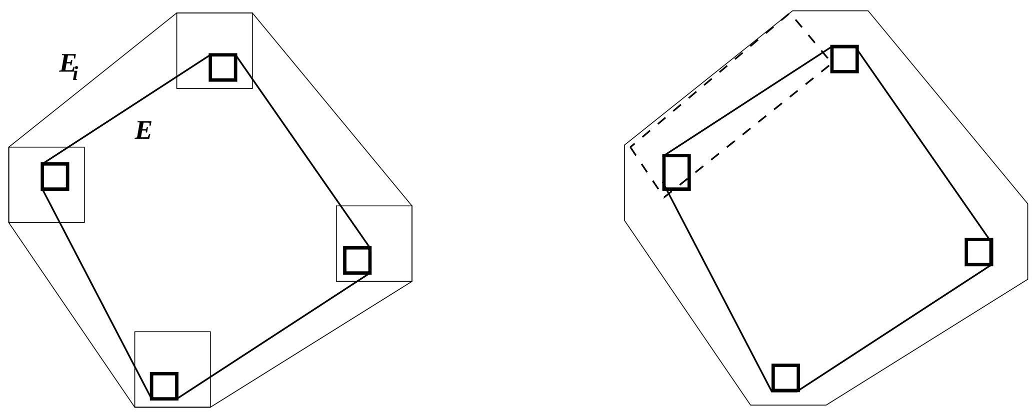

Figure 1: a , b.

We claim that the area of the region between the boundaries of E and E_i is less than $l(\partial(E_i))\ d_H(\overline{R}, \overline{Q}_i)$. For each edge e of $\partial(E_i)$, consider the rectangle with one edge e and perpendicular edge of length $d_H(\overline{R}, \overline{Q}_i)$ on the same side of e which E_i lies. These rectangles cover the region between the boundaries of E and E_i (Figure 1.b).

Since $(\overline{Q}_i)_{i\in\omega}$ is increasing, therefore $\forall i \in \mathbb{N} : l(\partial(E_i)) \leq l(\partial(E_1))$ and in general we have

$$\mu(E) - \mu(E_i) \leq l(\partial(E_1)) d_H(\overline{R}, \overline{Q}_i).$$

The same relation is true for the inner part: $\mu(I) - \mu(I_i)$ is the area of the region between the boundaries of I and I_i (Figure 2). The area of the region between the boundaries of I and I_i is less than $l(\partial(I_i)) d_H(\overline{R}, \overline{Q}_i)$.

For each i we have $I_i \subseteq I \subseteq E^c \subseteq E_1^c$, therefore

$$\forall i \in \mathbb{N} : l(\partial(I_i)) \leq l(\partial(E_1)).$$

Now we construct the function $\rho : \mathbb{N} \to \mathbb{N}$ as follows. For a given $i \in \mathbb{N}$ if $l(\partial(E_1)) d_H(\overline{R}, \overline{Q}_1) < 2^{-i}$ then take $\rho(i) = 1$, otherwise check the next approximation $\overline{Q}_2$ and repeat. Note that $d_H(\overline{R}, \overline{Q}_i)$ is computable. Since $(\overline{Q}_i)_{i\in\omega}$ is increasing, we have $d_H(\overline{R}, \overline{Q}_{j+1}) \leq d_H(\overline{R}, \overline{Q}_j)$ which implies

$$\forall j \in \mathbb{N} : l(\partial(E_1)) d_H(\overline{R}, \overline{Q}_{j+1}) \leq l(\partial(E_1)) d_H(\overline{R}, \overline{Q}_j).$$

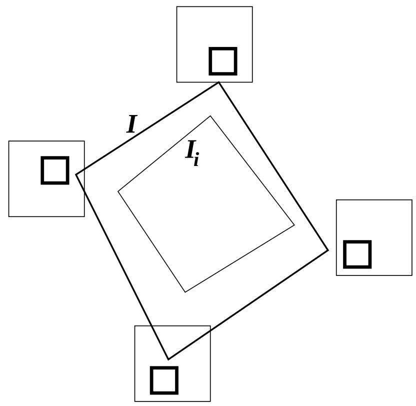

Figure 2:

This shows that after m finite stages we reach $\overline{Q}_m$ such that

$$l(\partial(E_1))d_H(\overline{R},\overline{Q}_m) < 2^{-i}.$$

Then we take $\rho(i) = m$. From the above discussion we have

$$\mu(I) - \mu(I_{\rho(i)}) < 2^{-i} \text{ and } \mu(E) - \mu(E_{\rho(i)}) < 2^{-i}.$$

□

We can extend the proof of the previous proposition to show that f is μ-computable. The computable function f is μ-computable if it takes any μ-computable sequence of N-tuples of rectangles in $(\mathbf{I}\mathbb{R}^2)^N$ to a μ-computable sequence of partial solids in $\mathbf{S}\mathbb{R}^2$. In the following, we show that this is true for a bounded sequence as input.

Proposition 3.10 *Assume that $(\overline{R}_k)_{k\in\omega}$ is a μ-computable sequence of ordered list of rectangles in $(\mathbf{I}[-d,d]^2)^N$. Then $(f(\overline{R}_k))_{k\in\omega}$ is a μ-computable sequence of partial solids in $\mathbf{S}[-d,d]^2$.*

Proof. For each $\overline{R}_k = (R_{k1},\ldots,R_{kN})$ take $f(\overline{R}_k) = (I_k, E_k)$. We have

$$f(\overline{R}_k) = \bigsqcup_{i\in\omega} f(\overline{Q}_{ki}) = (\cup_{i\in\omega} I_{ki}, \cup_{i\in\omega} E_{ki}),$$

where $(\overline{Q}_{ki})_{i\in\omega}$ is an increasing chain of ordered list of rectangles way-below $\overline{R}_k$ with lub $\overline{R}_k$. The sequence of partial solids $((I_k, E_k))_{k\in\omega}$ is μ-computable, by Lemma 6.3 in [6], if there exists a total recursive function $a : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that

$$\mu(I_k) - \mu(I_{a(i,k)}) < 2^{-i} \text{ and } \mu(E_k) - \mu(E_{a(i,k)}) < 2^{-i}.$$

We have $\overline{R}_k \in (\mathbf{I}[-d,d]^2)^N$ for $k \in \omega$, therefore there exists a rectangle M such that $\cup_{j=1}^N R_{jk} \subseteq M$ for $k \in \omega$ and

$$I_k \subseteq M \text{ and } E_k^c \subseteq M.$$

Now as in proof of Proposition 3.9, one can show that

$$\begin{aligned} \forall k &: \mu(E_k) - \mu(E_{ki}) \leq l(\partial(M)) d_H(\overline{R}_k, \overline{Q}_{ki}) \\ \forall k &: \mu(I_k) - \mu(I_{ki}) \leq l(\partial(M)) d_H(\overline{R}_k, \mathcal{Q}_{ki}). \end{aligned}$$

We define $a(i,k)$ to be the first integer j such that $l(\partial(M)) d_H(\overline{R}_k, \overline{Q}_{kj}) < 2^{-i}$. □

3.3 Hausdorff Computability

In this section we discuss the Hausdorff computability of f.

Lemma 3.11 *Assume that two ordered list of rectangles $\overline{R}_1, \overline{R}_2 \in (\mathbf{I}[-d,d]^2)^N$ are given such that $\overline{R}_1 \sqsubseteq \overline{R}_2$. Let $E_1 = (C(V(\overline{R}_1)))^c$ and $E_2 = (C(V(\overline{R}_2)))^c$ then $d_H(\overline{E}_1, \overline{E}_2)$ and $d_H(E_1^c, E_2^c)$ are less than $d_H(\overline{R}_1, \overline{R}_2)$.*

Proof. From $\overline{R}_1 \sqsubseteq \overline{R}_2$ it follows that $E_2 \subseteq E_1$. Assume that $d_H(\overline{R}_1, \overline{R}_2) = \epsilon$. In order to show that $d_H(\overline{E}_1, \overline{E}_2) \leq \epsilon$, it is enough to prove the following,

$$\forall x \in \partial(E_1) \exists y \in E_2 : d(x,y) \leq \epsilon.$$

Consider x in $\partial(E_1)$, then x lies on some edge e_{1i} and therefore

$$x = \lambda p_{1i} + (1-\lambda) q_{1i} \text{ for } 0 < \lambda < 1,$$

where p_{1i} and q_{1i} are vertices of the edge e_{1i}. Note that $d_H(\overline{R}_1, \overline{R}_2)$ is equal to the maximum distance between the corresponding vertices of the corresponding rectangles, therefore

$$\exists p'_{1i}, q'_{1i} \in E_2 : d(p_{1i}, p'_{1i}) \leq \epsilon \text{ and } d(q_{1i}, q'_{1i}) \leq \epsilon.$$

Define

$$y = \lambda p'_{1i} + (1-\lambda) q'_{1i}.$$

Since E_2 is a convex set we have $y \in E_2$. Furthermore

$$d(x,y) = \lambda d(p_{1i}, p'_{1i}) + (1-\lambda) d(q_{1i}, q'_{1i}) \leq \epsilon.$$

The other relation is proved in a similar way. □

Proposition 3.12 *The map $f : (\mathbf{I}[-d,d]^2)^N \to \mathbf{S}[-d,d]^2$ takes any N-tuple of Hausdorff computable rectangles to a Hausdorff computable partial solid.*

Proof. Assume that $\overline{R} = (R_1, \ldots, R_N)$ is an N-tuple of Hausdorff computable rectangles R_j. Let $f(\overline{R}) = (I, E)$, we have

$$f(\overline{R}) = \bigsqcup_{i\in\omega}\{f(\overline{Q}_i)|\overline{Q}_i \ll \overline{R}\} = (\cup_{i\in\omega} I_i, \cup_{i\in\omega} E_i).$$

The partial solid (I, E) is Hausdorff computable iff there exists a total recursive function $\rho : \mathbb{N} \to \mathbb{N}$ such that

$$\begin{aligned} d_H(\overline{I}, \overline{I}_{\rho(i)}) &< 2^{-i} \text{ and } d_H(I^c, I^c_{\rho(i)}) < 2^{-i} \\ d_H(\overline{E}, \overline{E}_{\rho(i)}) &< 2^{-i} \text{ and } d_H(E^c, E^c_{\rho(i)}) < 2^{-i}. \end{aligned}$$

Denote the vertices of polygon I with v_j, $(j = 1, \ldots, |V(I)|)$ and vertices of I_i with v_{ij}, $(j = 1, \ldots, |V(I_i)|)$. Then from convexity of polygons I and I_i it follows that

$$d_H(\overline{I}, \overline{I}_i) \& d_H(I^c, I_i^c) \le \max_k \min_j d(v_k, v_{ij}).$$

Now we construct the function $\rho : \mathbb{N} \to \mathbb{N}$ as follows. For a given integer number i, consider $\overline{Q}_i$ such that $d_H(\overline{R}, \overline{Q}_i) < 2^{-i}$ then compute the upper bound of $d_H(\overline{I}, \overline{I}_i)$ as described in above. If it was less than 2^{-i} take $\rho(i) = i$, otherwise check the next approximation $\overline{Q}_{i+1}$ and repeat. Note that since $\overline{I}_i \subseteq \overline{I}_{i+1} \subseteq \overline{I}$ and they are all convex polygons, we have

$$d_H(\overline{I}, \overline{I}_{i+1}) \le d_H(\overline{I}, \overline{I}_i).$$

Therefore after m finite stages we reach $\overline{Q}_m$ such that

$$d_H(\overline{I}, \overline{I}_m) < 2^{-i},$$

and we take $\rho(i) = m$.

Furthermore from Lemma 3.11 we have

$$d_H(E^c, E^c_{\rho(i)}) \le d_H(\overline{R}, \overline{Q}_{\rho(i)}) < 2^{-i},$$

which finishes the proof. □

In a similar way to the μ-computability case, we can extend the proof of the previous proposition to show that f is Hausdorff computable. The computable function f is Hausdorff computable if it takes any Hausdorff computable sequence of N-tuples of rectangles in $(\mathbf{I}\mathbb{R}^2)^N$ to a Hausdorff computable sequence of partial solids in $\mathbf{S}\mathbb{R}^2$. In the following, we show that this is true for a bounded sequence as input.

Proposition 3.13 *Assume that $(\overline{R}_k)_{k\in\omega}$ is a Hausdorff computable sequence of ordered list of rectangles in $(\mathbf{I}[-d, d]^2)^N$. Then $(f(\overline{R}_k))_{k\in\omega}$ is a Hausdorff computable sequence of partial solids in $\mathbf{S}[-d, d]^2$.*

Proof. For each $\overline{R}_k = (R_{k1}, \ldots, R_{kN})$ take $f(\overline{R}_k) = (I_k, E_k)$. We have

$$f(\overline{R}_k) = \bigsqcup_{i\in\omega} f(\overline{Q}_{ki}) = (\cup_{i\in\omega} I_{ki}, \cup_{i\in\omega} E_{ki}),$$

where $(\overline{Q}_{ki})_{i\in\omega}$ is an increasing chain of ordered list of rectangles way-below $\overline{R}_k$ with lub $\overline{R}_k$. The sequence of partial solids $((I_k, E_k))_{k\in\omega}$ is Hausdorff computable, by Lemma ?? in [6], if there exist total recursive functions $a, b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\begin{aligned} d_H(\overline{I}_k, \overline{I}_{a(i,k)}) &< 2^{-i} \text{ and } d_H(I_k^c, I_{b(i,k)}^c) < 2^{-i}, \\ d_H(\overline{E}_k, \overline{E}_{b(i,k)}) &< 2^{-i} \text{ and } d_H(E_k^c, E_{b(i,k)}^c) < 2^{-i}. \end{aligned}$$

We have $\overline{R}_k \in (\mathbf{I}[-d, d]^2)^N$ for $k \in \omega$, therefore there exists a rectangle M such that $\cup_{j=1}^N R_{jk} \subseteq M$ for $k \in \omega$ and

$$I_k \subseteq M \text{ and } E_k^c \subseteq M.$$

We define $a(i, k)$ to be the first integer j such that $d_H(\overline{I}_k, \overline{I}_{kj}) < 2^{-i}$ and $b(i, k)$ to be the first integer j such that $d_H(\overline{E}_k, \overline{E}_{kj}) < 2^{-i}$. □

3.4 Boundary Rectangle

An element of a set of points in the plain is called a *boundary point* if it is a vertex of the convex hull of these points. Now consider the question: is the point p a boundary point ? Using floating point computation we cannot give the correct answer to this question, but in the computable solid modeling we can answer it with a computable boolean function. Each point in the computable solid modeling is approximated with a rectangle. A rectangle R among a finite number of planar rectangles is a *boundary rectangle*, if the exterior part of the partial convex hull of any refinement of these rectangles will always contain a point of R. Based on this observation we define the following boolean function :

$$\begin{array}{rccc} b: & (\mathbf{I}\mathbb{R}^d) \times (\mathbf{I}\mathbb{R}^d)^{N-1} & \longrightarrow & \{\mathbf{tt}, \mathbf{ff}\}_\perp \\ & (R_1, (R_2, \ldots, R_N)) & \mapsto & \begin{cases} \mathbf{tt} & cl(R_1) \subseteq E \\ \mathbf{ff} & cl(R_1) \subseteq I \\ \perp & \text{otherwise} \end{cases} \end{array}$$

where $cl(R_1)$ denotes the closure of R_1 and (I, E) is the partial convex hull of the rectangles $R_2, \ldots, R_N$. It is clear that b is a well defined map.

Lemma 3.14 *The map b is monotone.*

Proof. A function with two variables is monotone, if it is monotone for each variable when the other one is fixed. Therefore the proof has two steps as follows.

i) Assume that $\overline{R} = (R_1, (R_2, \ldots, R_N))$ and $\overline{R}' = (R'_1, (R_2, \ldots, R_N))$ in $(\mathbf{I}\mathbb{R}^2) \times (\mathbf{I}\mathbb{R}^2)^{N-1}$ are given such that $\overline{R} \sqsubseteq \overline{R}'$, i.e. $R_1 \supseteq R'_1$. Let $(I, E) = f(R_2, \ldots, R_N)$.

$$\begin{aligned} b(R_1, (R_2, \ldots, R_N)) = \mathbf{tt} \quad &\Rightarrow \quad cl(R_1) \subseteq E \\ &\Rightarrow \quad cl(R'_1) \subseteq cl(R_1) \subseteq E \\ &\Rightarrow \quad b(R'_1, (R_2, \ldots, R_N)) = \mathbf{tt} \end{aligned}$$

$$\begin{aligned} b(R_1, (R_2, \ldots, R_N)) = \mathbf{ff} \quad &\Rightarrow \quad cl(R_1) \subseteq I \\ &\Rightarrow \quad cl(R'_1) \subseteq cl(R_1) \subseteq I \\ &\Rightarrow \quad b(R'_1, (R_2, \ldots, R_N)) = \mathbf{ff} \end{aligned}$$

Thus $b((R_1, (R_2, \ldots, R_N))) \sqsubseteq b((R'_1, (R_2, \ldots, R_N)))$.

ii) Assume that $\overline{R} = (R_1, (R_2, \ldots, R_N))$ and $\overline{R}' = (R_1, (R'_2, \ldots, R'_N))$ in $(\mathbf{I}\mathbb{R}^2) \times (\mathbf{I}\mathbb{R}^2)^{N-1}$ are given such that $\overline{R} \sqsubseteq \overline{R}'$. Let $(I, E) = f(R_2, \ldots, R_N)$ and $(I', E') = f(R'_2, \ldots, R'_N)$, since $\overline{R} \sqsubseteq \overline{R}'$ we have $I \subseteq I'$ and $E \subseteq E'$.

$$\begin{aligned} b(R_1, (R_2, \ldots, R_N)) = \mathbf{tt} \quad &\Rightarrow \quad cl(R_1) \subseteq E \\ &\Rightarrow \quad cl(R_1) \subseteq E \subseteq E' \\ &\Rightarrow \quad b(R_1, (R'_2, \ldots, R'_N)) = \mathbf{tt} \end{aligned}$$

$$\begin{aligned} b(R_1, (R_2, \ldots, R_N)) = \mathbf{ff} \quad &\Rightarrow \quad cl(R_1) \subseteq I \\ &\Rightarrow \quad cl(R_1) \subseteq I \subseteq I' \\ &\Rightarrow \quad b(R_1, (R'_2, \ldots, R'_N)) = \mathbf{ff} \end{aligned}$$

Thus $b((R_1, (R_2, \ldots, R_N))) \sqsubseteq b((R_1, (R'_2, \ldots, R'_N)))$. □

Proposition 3.15 *The map b is continuous.*

Proof. From Lemma 3.14, b is monotone. In following we show that it also preserves lubs of directed sets. Assume that $(\overline{Q}_i)_{i\in\omega}$ is a given increasing chain in $(\mathbf{I}\mathbb{Q}^2) \times (\mathbf{I}\mathbb{Q}^2)^{N-1}$ with lub $\overline{R} = (R_1, (R_2, \ldots, R_N))$. We show $b(\overline{R}) = \bigsqcup_{i\in\omega} b(\overline{Q}_i)$. One side of the equality is clear

$$\forall i : \overline{Q}_i \sqsubseteq \overline{R} \Rightarrow \forall i : b(\overline{Q}_i) \sqsubseteq b(\overline{R}) \Rightarrow \bigsqcup_{i\in\omega} b(\overline{Q}_i) \sqsubseteq b(\overline{R}).$$

For the other side, take $(I, E) = f(R_2, \ldots, R_N)$ and $(I_i, E_i) = f(R_{i2}, \ldots, R_{iN})$. Assume that $b(\overline{R}) = \mathbf{tt}$ thus $cl(R_1) \subseteq E$. Since

$$\cap_{i\in\omega} R_{i1} = R_1 \text{ and } \cup_{i\in\omega} E_i = E,$$

it follows from the compactness of R_1 and R_{i1} that there exists j such that $cl(R_{j1}) \subseteq E_j$. Therefore $b(\overline{Q}_j) = \mathbf{tt}$ which implies $\bigsqcup_{i\in\omega} b(\overline{Q}_i) = \mathbf{tt}$. A similar proof holds when $b(\overline{R}) = \mathbf{ff}$. This shows that $b(\overline{R}) \sqsubseteq \bigsqcup_{i\in\omega} b(\overline{Q}_i)$. □

Proposition 3.16 *The map b is computable.*

Proof. Let $(I, E) = f(Q_2, \ldots, Q_N)$, we have to show that the relations

$$\begin{aligned} b(Q_1, (Q_2, \ldots, Q_N)) &= \mathbf{tt} \\ b(Q_1, (Q_2, \ldots, Q_N)) &= \mathbf{ff} \end{aligned}$$

are both r.e. The first reduces to $cl(Q_1) \subseteq E$ and the second to $cl(Q_1) \subseteq I$. Since both side of the relations are rational polygons, it is in fact decidable by assumption. □

4 Algorithms for Computing the Partial Convex Hull

In this section we give two algorithms for computing 2-dimensional partial convex hull of a given set of rectangles.

Exterior. From Lemma 3.3 we have

$$E = \left(C(V(\overline{Q}))\right)^c.$$

Therefore, in order to compute the exterior part of the rational rectangles $\overline{Q}$, consider all the vertices of all rectangles and then simply use one of the existing algorithms for computing the convex hull. Since we are dealing with rational numbers this will give us the exact result. Using the fact that the complexity of any convex hull algorithm is at least $O(n \log(n))$, for N given rectangles we can compute the exterior part in $O(N \log(N))$.

Interior. In the section 3 we showed that

$$I = \bigcap_{s \in S^1} \bigcap \{H | H \in \mathcal{H}_s, \forall j = 1, \ldots, N : \exists v_j \in V(Q_j), v_j \in H\}.$$

Define $h_s = \bigcap \{H | H \in \mathcal{H}_s, \forall j = 1, \ldots, N : \exists v_j \in V(Q_j), v_j \in H\}$. For a given s it is easy to compute h_s. The equation of h_s is $s.x - b \leq 0$ where $b = \max_i \min_{p_i \in V(Q_i)} s \cdot p_i$. This expression shows that b is a continuous function of s.

Lemma 4.1 *Let s_j be the direction of the the j^{th} face of I and x_j a point in the interior of the j^{th} face. Then x_j belongs to the boundary of h_{s_j}.*

Proof. The equation of h_{s_j} is $s_j \cdot x - b(s_j) \leq 0$. Assume that x_j is not on the boundary, that is $s_j \cdot x_j - b(s_j) = \epsilon < 0$. Then from the continuity of b as a function of s, there is a neighborhood of s_j in which $s \cdot x_j - b(s) \leq \epsilon/2 < 0$. Define this neighborhood by $A(s_j, s) < \alpha$.

Since x_j is in the interior of the j^{th} face, there exist two other points y_j and z_j on the j^{th} face each one lies in one side of x_j, moreover : $x_j = (y_j + z_j)/2$. For any s, both $s \cdot y_j - b(s)$ and $s \cdot z_j - b(s)$ must be non positive (since h_s contains I_Q). Note that $(y_j - z_j)$ is orthogonal to s_j. As soon as s is outside the neighborhood then $A(s_j, s) \geq \alpha$ and $|s \cdot (y_j - z_j)| \geq ||y_j - z_j|| \sin(\alpha)$. From

$$|(s \cdot y_j - b(s)) - (s \cdot z_j - b(s))| \geq ||y_j - z_j|| \sin(\alpha)$$

and the fact that

$$s \cdot y_j - b(s) = (s \cdot y_j - b(s)) + (s \cdot z_j - b(s))/2,$$

we get

$$s_j \cdot x_j - b(s_j) \le -1/2||y_j - z_j|| \sin(\alpha).$$

Therefore, there is strictly negative constant β, such that for each s (inside or outside the neighborhood of s_j) $s \cdot x_j - b(s)$ is less than β, this means that the ball centered in x_j of radius β is in I_Q and we get the contradiction. □

Since x_j in the interior of the j^{th} face belongs to the boundary of h_{s_j} where h_{s_j} is the half-plane supporting the j^{th} face, therefore $I = \cap_j h_{s_j}$. From Lemma 3.2, I is the intersection of the 4^N polygons and its faces are parts of the faces of these polygons. Although there are 4^N polygons but the total number of the faces is at most $16N(N-1)/2$ (there are $N(N-1)/2$ pairs of Q_i and Q_j and each one gives 16 possible faces).

Based on the above argument for computing the interior part of the partial convex hull of the rectangles Q_j one can use the following algorithm. For each of the $16N(N-1)/2$ possible lines, supporting the faces, check which ones are the boundary of one of the h_{s_j}. From the definition, a line is boundary if it contains one vertex of each rectangle all in the same side, and this cost $O(N)$ operations. Finally, intersection of the boundary line is the the interior part of the partial convex hull, and this cost $O(N \log(N))$. Therefore we get an algorithm of $O(N^3)$. In $(R)^d$ the same algorithm would give a complexity of $O(N^{(d+1)})$.

We describe in the rest of the section how to compute an approximation of I in $O(N \log(N))$.

For a given N-tuple of rectangles $\overline{Q}$, let $\overline{B} = (Q_1, \ldots, Q_M)$ be the M-tuple ($M \le N$) of those rectangles which contribute at least one vertex to the boundary of the exterior part of the partial convex hull of $\overline{Q}$. Define

$$P^*(\overline{B}) = \{(v_1, \ldots,\ v_M) | v_j \text{ is a vertex of } Q_j \in \overline{B} \text{ for } j = 1, \ldots, M\},$$

and

$$I_b = (\cap_{v \in V^*(\overline{B})} C(v))^\circ.$$

Lemma 4.2 *We have $I_b \subseteq I$.*

Proof. Note that each $v \in P^*(\overline{B})$ is a sub-tuple of some $w \in P^*(\overline{Q})$ (i.e. each components of v is a component of w). Therefore

$$\forall w \in P^*(\overline{Q}) \exists v \in P^*(\overline{B}) : C(v) \subseteq C(w).$$

Take $x \in I_b$ then $\forall v \in P^*(\overline{B}) : x \in C(v)$. Using the above property we get $\forall w \in P^*(\overline{Q}) : x \in C(w)$ which implies $x \in I$. □

After computing E, we can define an order for the vertices of the boundary of E. Consider the left-most vertex as the origin and move in clockwise direction until the last vertex. Now for any line between two vertices we can speak of the left and the right side according to the order of the vertices.

For any two consecutive rectangles Q_i and Q_{i+1} in $\overline{B}$ define L_i to be the line between two vertices of Q_i and Q_{i+1} such that the rectangles Q_i and Q_{i+1} lie completely to the left of L_i. The right hand side of L_i defines the half-plane H_i.

Lemma 4.3 *We have:* $I_b = \bigcap_{i=1,\ldots,M} H_i$.

Proof. One side of the equality is trivial,

$$\forall v \in P^*(\overline{B}) : \bigcap_{i=1,\ldots,M} H_i \subseteq C(v) \Rightarrow \bigcap_{i=1,\ldots,M} H_i \subseteq I_b.$$

Now consider $x \in I_b$. There exists i such that x lies to the right of L_i. This implies that $x \in \bigcap\limits_{i=1,\ldots,M} H_i$. □

Based on the above lemma we can compute I_b as an approximation of I and in fact it is enough to compute $\bigcap_{i=1,\ldots,M} H_i$. For each pair of rectangles in $\overline{B}$ one can compute in a constant time the half plane H_i (there are at most N of them). Then I_b (intersection of half planes) is computed in $O(N\log(N))$.

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