Algebraic Models and Complete Proof Calculi for Classical BI

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Abstract. We consider the classical (propositional) version, CBI, of O'Hearn and Pym's logic of bunched implications (BI) from a model- and proof-theoretic perspective. We make two main contributions in this paper. Firstly, we present a class of algebraic models for CBI which permit the full range of classical multiplicative connectives to be modelled. Our models can be seen as generalisations of Abelian groups, and include several computationally interesting models as concrete instances. Secondly, we give a display calculus proof system for CBI that is an instance of Belnap's general display logic — hence cut-eliminating — and demonstrate this system to be sound and complete with respect to validity in our models. To achieve the latter, we first define a simple extension of the usual sequent calculus for BI by axioms that directly capture properties of our models, and show this extension to be sound and complete (though not cut-eliminating). Soundness and completeness of our display calculus then follows by establishing faithful translations between the display calculus and this sequent calculus.

1 Introduction

The *logic of bunched implications* (BI), due to O'Hearn and Pym [10], is a substructural logic suitable for reasoning about various domains that incorporate a notion of *resource* [9]. Its best-known application to computer science is in *separation logic*, a Hoare logic for reasoning about imperative, pointer-manipulating programs, which essentially is obtained by considering a particular model of BI based on heaps [13]. Semantically, BI arises by considering cartesian doubly closed categories (in contrast to linear logic which usually is based upon two closed categories). This viewpoint gives rise to the following propositional connectives for BI:

Additive:	$\top$	$\bot$	$\wedge$	$\vee$	$\rightarrow$
Multiplicative:	$\top^*$		$*$		$-\!\!*$

From the aforementioned categorical perspective, this presentation of BI is necessarily intuitionistic. By instead using the algebraic semantics of BI, in which the multiplicatives are modelled by partial commutative monoids, the additive connectives can be interpreted either classically or intuitionistically according to

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preference. When the additives are interpreted classically (e.g. using a boolean algebra) the resulting logic is often called *boolean* BI. In this paper, we consider the extension of boolean BI to *classical* BI, in which both the additives and the multiplicatives are treated classically. Specifically, classical BI includes the multiplicative analogues of additive falsity, negation and disjunction, which are "missing" in BI. We give an algebraic semantics for classical BI, and present a proof system, based on Belnap's *display logic* [1], that is sound and complete with respect to this semantics and that satisfies cut-elimination.

In BI, the presence of the two implications ($\rightarrow$ and $-\!\!*$) gives rise to two context-forming operations, ';' and ',', which correspond to the conjunctions $\wedge$ and $*$ at the meta-level, as exemplified by the sequent calculus right-introduction rules for the implications:

$$\frac{\Gamma; F_1 \vdash F_2}{\Gamma \vdash F_1 \rightarrow F_2}\ (\rightarrow\mathrm{R}) \qquad \frac{\Gamma, F_1 \vdash F_2}{\Gamma \vdash F_1 -\!\!* F_2}\ (-\!\!*\mathrm{R})$$

Accordingly, the contexts Γ on the left-hand side of the sequents in the rules above are not sets or lists, as in standard sequent systems, but rather *bunches*: trees whose leaves are formulas and whose internal nodes are either ';' or ',', denoting additive and multiplicative combination respectively. The crucial difference between the two operations is that weakening and contraction are possible for ';' but not for ','. Since BI is an intuitionistic logic, bunches arise only on the left-hand side of sequents, with a single formula on the right. For classical BI, the natural approach from a proof-theoretic perspective is to consider a full two-sided sequent calculus in which ';' and ',' in bunches on the right of sequents correspond to the two disjunctions at the meta-level. Unfortunately, there is no known formulation of such a sequent calculus that admits cut-elimination (see [4, 11] for some discussion of the difficulties).

In Section 2 we present a class of algebraic models for classical BI which provide the necessary structure to model the full range of multiplicative connectives. Our models are obtained by imposing extra conditions on the usual partial commutative monoid models of (Boolean) BI, the main such condition being the presence of a natural involution operation. In fact, our models include all Abelian groups as special instances. We consider a range of natural examples from mathematics and computer science.

In Section 3 we extend the usual sequent calculus for boolean BI with axioms that capture the behaviour of the involution in our models. Using standard techniques, we show that this extended proof system, LBI^+, is sound and complete with respect to validity in our models. However, this proof system does not contain primitive introduction rules for every connective of classical BI, nor does it obey cut elimination. In Section 4 we present the syntax of full classical BI and the interpretations of its multiplicative connectives in our algebraic models. The interpretations of multiplicative falsity, negation and disjunction are similar to those employed by relevant logicians (see e.g. [12, 6]) and are justified by the resulting semantic equivalences between formulas. For example, under our interpretation $F -\!\!* G$ is semantically equivalent to $\sim\!F \otimes G$, where $\sim$ is multiplicative negation and $\otimes$ is multiplicative disjunction. We then give a proof system for classical BI, which is not a sequent calculus but rather a *display calculus* based on Belnap's display logic [1]. Display logic is a generalised Gentzen-style system that can be instantiated to a wide class of logics simply by choosing families of connectives and the structural rules governing those families. The power of display logic comes from its generic structural principles, which are sufficient to guarantee certain desirable proof-theoretic properties, more or less independently of the particular choice of connective families and structural rules employed. In particular, our display calculus for classical BI obeys cut-elimination.

In Section 5 we present the proof of our main technical result: the soundness and completeness of our display calculus, $\mathrm{DL}_{\mathrm{BI}}$, with respect to validity in our algebraic models. This is achieved by proving admissibility of $\mathrm{DL}_{\mathrm{BI}}$ in our sequent calculus LBI^+ under a suitable embedding, and vice versa. Finally, in Section 6, we conclude and identify the main directions for future work.

2 A class of algebraic models for classical BI

In this section we present our algebraic models for classical BI, which are based on the partial commutative monoids used to model (the multiplicative part of) ordinary BI [10]. Our models generalise these monoids in the sense that we consider the monoid operation to be a relation over the carrier set rather than a function, as is needed for our completeness argument in Section 3. If we disregard relationality, our models are special cases of these monoids, containing structure that is not required to be present in BI-models: an involution operation on elements and a distinguished element[1] ∞ that characterises the result of combining an element with its involution. In particular, our models include as instances all Abelian groups, which can be seen by taking $\circ$ to be a total function and ∞ to be the identity element of the monoid.

In the following, note that we write $\mathrm{Pow}(X)$ for the powerset of a set X.

Definition 2.1 (Classical BI-model). A *classical* BI-*model* is given by a tuple $\langle R, \circ, e, -, \infty\rangle$, where $\circ : R \times R \to \mathrm{Pow}(R)$, $e \in R$, $- : R \to \mathrm{Pow}(R)$, and $\infty \subseteq R$ such that:

1. $\circ$ is commutative and associative, with $x \circ e = \{x\}$
2. $-x = \{y \in R \mid \exists z.\, z \in x \circ y \cap \infty\}$
3. $--x = \{x\}$

We extend the domains of $-$ and $\circ$ to $\mathrm{Pow}(R)$ and $\mathrm{Pow}(R)\times\mathrm{Pow}(R)$ respectively by $-X =_{\mathrm{def}} \bigcup_{x\in X} -x$ and $X \circ Y =_{\mathrm{def}} \bigcup_{x\in X, y\in Y} x \circ y$.

Lemma 2.2. *Let $\langle R, \circ, e, -, \infty\rangle$ be a tuple with the same types as in Definition 2.1, and extend $-$ and $\circ$ to $Pow(R)$ and $Pow(R) \times Pow(R)$ respectively as in that definition. Then $\langle R, \circ, e, -, \infty\rangle$ is a classical* BI-*model iff the following hold for all $X, Y, Z \in Pow(R)$:*

[1] In fact, we shall technically allow ∞ to be a set of elements. However, the conditions defining our models force ∞ to be a singleton set. See Convention 2.4.

1. $X \circ Y = Y \circ X$ *and* $X \circ (Y \circ Z) = (X \circ Y) \circ Z$ *and* $\{e\} \circ X = X$
2. $-X = X \multimap\!\!\bullet\, \infty$
3. $--X = X$

where $X \multimap\!\!\bullet\, Y =_{def} \{z \in R \mid \exists x \in X, y \in Y.\, y \in x \circ z\}$.

Proof. ($\Rightarrow$) The required properties follow straightforwardly from the corresponding conditions on classical BI-models and the extension of $-$ and $\circ$ to sets of elements.
($\Leftarrow$) The conditions required for $\langle R, \circ, e, -, \infty\rangle$ to be a classical BI-model follow from taking X, Y, Z to be singleton sets in the given conditions and noting that $-\{x\} = -x$ and $\{x\} \circ \{y\} = x \circ y$ for any $x, y \in R$. □

Proposition 2.3. *If* $\langle R, \circ, e, -, \infty\rangle$ *is a classical* BI*-monoid then:*

1. $\forall x \in R.\ -x$ *is a singleton set;*
2. $-e = \infty$*;*
3. $\forall x \in R.\ x \circ -x \supseteq \infty$*;*
4. $\forall X \subseteq R.\ R \setminus (-X) = -(R \setminus X)$*;*

Proof. 1. By contradiction. If $-x = \emptyset$ then $--x = \bigcup_{y \in -x} -y = \emptyset$, which contradicts $--x = \{x\}$. If $x_1, x_2 \in -x$ with $x_1 \neq x_2$, then $-x_1 \cup -x_2 \subseteq --x$. Also, $-x_1 \neq -x_2$, otherwise we would have $\{x_1\} = --x_1 = --x_2 = \{x_2\}$ and thus $x_1 = x_2$. Since $-x_1$ and $-x_2$ have cardinality > 0 (see above), $--x$ must have cardinality > 1, which contradicts $--x = \{x\}$.

2. We have:
$$\begin{aligned} -e &= \{y \in R \mid \exists z.\, z \in e \circ y \cap \infty\} \\ &= \{y \in R \mid \exists z.\, z \in \{y\} \cap \infty\} \\ &= \{y \in R \mid y \in \infty\} \\ &= \infty \end{aligned}$$

3. Using part 1, let $x' \in R$ be the unique element such that $-x = \{x'\}$. Then $\{x'\} = \{y \in R \mid \exists z.\, z \in x \circ y \cap \infty\}$, so there exists $z \in R$ such that $z \in x \circ x' \cap \infty = x \circ -x \cap \infty$. By parts 1 and 2, $\infty = -e$ is a singleton set, so we must have $x \circ -x \supseteq \infty$ as required.

4. ($\subseteq$) Suppose $x \in R \setminus -X$, i.e. $x \notin -X = \bigcup_{y \in X} -y$, so $x \notin -y$ for any $y \in X$. Also, using part 1, we have $x \in --x = -\{z\} = -z$ for some z. It must be the case that $z \notin X$, so $x \in \bigcup_{z \notin X} -z = \bigcup_{z \in R \setminus X} -z = -(R \setminus X)$ as required.
($\supseteq$) Suppose $x \in -(R \setminus X)$, i.e. $x \in -y$ for some $y \notin X$. Note that we cannot have $x \in -z$ for some $z \in X$, otherwise by part 1 we have $-y = -z = \{x\}$ and thus $\{y\} = --y = --z = \{z\}$, so $y = z$, which is a contradiction. Thus $x \notin \bigcup_{z \in X} -z = -X$, i.e. $x \in R \setminus -X$ as required.

□

The first two parts of Proposition 2.3 justify the following convention.

Convention 2.4. Given a classical BI-model $\langle R, \circ, e, -, \infty\rangle$, for any $x \in R$ the notation $-x$ is henceforth to be understood as the unique element $z \in R$ such that $-x = \{z\}$. Similarly, ∞ is to be understood as the unique $z \in R$ such that $\infty = \{z\}$.

Proposition 2.5. *Let* $\langle R, \circ, e, -, \infty \rangle$ *be a classical* BI*-model. If* $\infty = e$ *and the cardinality of* $x \circ y$ *is* ≤ 1 *for all* $x, y \in R$ *then, if* $\circ$ *is understood as a partial function* $R \times R \rightharpoonup R$ *in the obvious way,* $\langle R, \circ, e, - \rangle$ *is an Abelian group.*

Proof. First note that by part 3 of Proposition 2.3 and the fact that $\circ$ is a partial function, we have $-x \circ x = \infty = e$. Now, to see that $x \circ y$ is defined for any $x, y \in R$, observe that $-x \circ (x \circ y) = (-x \circ x) \circ y = e \circ y = y$. Thus $-x \circ' (x \circ' y)$ is defined (and equal to y), which can only be the case if $x \circ' y$ is defined.

To see that $\langle R, \circ, e, - \rangle$ is an Abelian group, we first observe that $\langle R, \circ, e \rangle$ is already a partial commutative monoid by the conditions placed on $\circ$ in the definition of classical BI-model. Furthermore, $\circ$ is a total function by the above, and $-x$ is the unique inverse of x for any $x \in R$, since $-x \circ x = e$ and $y \circ x = e$ implies $-x = (y \circ x) \circ -x = y \circ (x \circ -x) = y$. □

2.1 Examples of classical BI-models

We now turn to some concrete examples of classical BI-models. Note that, in all of our examples, the monoid operation $\circ$ is a partial function rather than a relation (so that $x \circ y$ is either undefined or a model element).

Example 2.6 (Bit arithmetic). For any $n \in \mathbb{N}$, the tuple:

$$\langle \{0,1\}^n, \mathrm{XOR}, \{0\}^n, \mathrm{NOT}, \{1\}^n \rangle$$

is a classical BI-model. In this model, the resources are n-bit binary numbers, which can be combined and "inverted" using the usual logical operations XOR and NOT respectively. Accordingly, the resources e and ∞ are respectively the n-bit representations of 0 and $2^n - 1$.

The following example shows that, even when the monoid structure of a classical BI-model is fixed, the choice of ∞ is not unique in general.

Example 2.7 (Integer modulo arithmetic). Consider the monoid $\langle \mathbb{Z}_n, +_n, 0 \rangle$, where $\mathbb{Z}_n$ is the set of integers modulo n, and $+_n$ is addition modulo n. We can form a classical BI-model from this monoid by choosing, for any $m \in \mathbb{Z}_n$, $\infty =_{\mathrm{def}} m$ and $-k =_{\mathrm{def}} m -_n k$ (where $-_n$ is subtraction modulo n).

Example 2.8 (Syntactic models). Given an arbitrary monoid $\langle R, \circ, e \rangle$, we give a syntactic construction to generate a classical BI-model $\langle R', \circ', e', -', \infty' \rangle$. Consider the set T of terms given by the grammar:

$$t \in T ::= r \in R \mid \infty \mid t \cdot t \mid -t$$

and let $\approx$ be the least congruence such that: $r_1 \cdot r_2 \approx r$ when $r_1 \circ r_2 = r$; $t_1 \cdot t_2 \approx t_2 \cdot t_1$; $t_1 \cdot (t_2 \cdot t_3) \approx (t_1 \cdot t_2) \cdot t_3$; $- - t \approx t$; $t \cdot (-t) \approx \infty$, and $t_1 \approx -t_2$ whenever $t_1 \circ t_2 \approx \infty$. Write $T/\!\approx$ for the quotient of T by the relation $\approx$, and $[t]$ for the equivalence class of t. The required classical BI-model is obtained by defining $R' =_{\mathrm{def}} T/\!\approx$, $\circ'([t_1],[t_2]) =_{\mathrm{def}} [t_1 \circ t_2]$, $e' =_{\mathrm{def}} [e]$, $-'(t) =_{\mathrm{def}} [-t]$, and $\infty' =_{\mathrm{def}} [\infty]$.

Example 2.9 (Generalised heaps). A natural question is whether BI models used in separation logic are also classical BI-models. Consider the partial commutative monoid $\langle H, \circ, e \rangle$, where $H =_{\text{def}} \mathbb{Z}_{>0} \rightharpoonup \mathbb{Z}$ is the set of partial functions from positive integers to integers, $\circ$ is disjoint union of the graph of functions, and e is the function with empty domain. Unfortunately, no choice of ∞ gives rise to a classical BI-monoid. However, it is possible to embed the heap monoid into a more general structure $\langle H', \circ', e' \rangle$, where $H' =_{\text{def}} \text{Pow}(\mathbb{Z}_{>0} \times \mathbb{Z})$ is the set of relations instead of partial functions, $\circ$ is disjoint union, and e is the empty relation. A classical BI-model is then obtained by setting $\infty =_{\text{def}} \mathbb{Z}_{>0} \times \mathbb{Z}$, and $-r =_{\text{def}} (\mathbb{Z}_{>0} \times \mathbb{Z}) \setminus r$.

Example 2.10 (Heaps with fractional permissions). As a final example, we consider a heap monoid with fractional permissions [3] $\langle H_p, \circ_p, e_p \rangle$, where $H_p =_{\text{def}} \mathbb{Z}_{>0} \rightharpoonup \mathbb{Z} \times (0,1]$ consists of functions which in addition return a permission in the real interval $(0,1]$, and $\circ$ is defined on functions with overlapping domains using a partial composition function $\oplus : (\mathbb{Z} \times (0,1]) \times (\mathbb{Z} \times (0,1]) \rightharpoonup (\mathbb{Z} \times (0,1])$ such that $\oplus((v_1, p_1), (v_2, p_2))$ is defined if and only if $v_1 = v_2$ and $p_1 + p_2 \leq 1$, and returns $(v_1, p_1 + p_2)$. The unit e_p is again the function with empty domain. In analogy with our approach to ordinary heaps in the previous example, we define a more general structure $\langle H'_p, \circ'_p, e'_p \rangle$, where $H'_p =_{\text{def}} \mathbb{Z}_{>0} \times \mathbb{Z} \to [0,1]$ is the set of *total* functions, and $\circ'_p$ is defined point-wise using $+ : [0,1] \times [0,1] \rightharpoonup [0,1]$, which is ordinary addition restricted to be defined only when the result is ≤ 1. The function e'_p maps everything to 0. A classical BI-model is then obtained by setting ∞ as mapping everything to 1, and $-r =_{\text{def}} \{(l, v, 1-p) \mid (l, v, p) \in r\}$. Observe that, in this case, the general model is in a way simpler, and that the $-$ operation returns the complement of the permissions.

3 BI$^+$: a basis for classical BI

In this section we define a simple extension BI$^+$ of standard propositional BI (cf. [10, 11]), and show a sequent calculus system for BI$^+$ to be sound and complete with respect to our classical BI-models defined in Section 2.

We fix a set $\mathcal{V}$ of propositional variables. *Formulas* of BI$^+$ are given by the following grammar:

$$F ::= \; P \mid \top \mid \bot \mid F \wedge F \mid F \vee F \mid F \to F \mid \top^* \mid \bowtie \mid F * F \mid F \mathrel{-\!\!*} F$$

where P ranges over $\mathcal{V}$. These are exactly[2] the formulas of BI plus the new atomic formula $\bowtie$. We also use the following abbreviations:

$$\neg F =_{\text{def}} F \to \bot$$
$$-F =_{\text{def}} \neg(F \mathrel{-\!\!*} \neg \bowtie)$$

Now let $M = \langle R, \circ, e, -, \infty \rangle$ be a classical BI-model. An *environment for M* is a function $\rho : \mathcal{V} \to \text{Pow}(R)$ interpreting propositional variables as true or false

[2] However, note that we write $\top^*$ for the multiplicative unitary formula I of BI.

relative to elements of the model. *Satisfaction* of a BI$^+$-formula F in the model M under the environment ρ is then also defined relative to model elements, and is given by the relation $r \models F$, where $r \in R$, with the key clauses being those for the propositional variables and multiplicative connectives:

$$
\begin{array}{rcl}
r \models P & \Leftrightarrow & r \in \rho(P) \\
r \models \top^* & \Leftrightarrow & r = e \\
r \models \bowtie & \Leftrightarrow & r = \infty \\
r \models F_1 * F_2 & \Leftrightarrow & \exists r_1, r_2.\ r \in r_1 \circ r_2 \text{ and } M, r_1 \models F_1 \text{ and } M, r_2 \models F_2 \\
r \models F_1 \mathrel{-\!\!*} F_2 & \Leftrightarrow & \forall r', r''.\ r'' \in r \circ r' \text{ and } M, r' \models F_1 \text{ implies } M, r'' \models F_2
\end{array}
$$

The clauses for the additive connectives are defined in the standard way, i.e., independently of the model element r. Note that additive implication $\rightarrow$ is interpreted classically, which entails that $r \models \neg F$ iff $r \not\models F$.

Lemma 3.1. *Let $M = \langle R, \circ, e, -, \infty \rangle$ be a classical* BI*-model and let ρ be an environment for M. For any $r \in R$ and formula F we have $r \models -F$ iff $-r \models F$.*

Proof. We have by the definitions of $-F$ and of satisfaction:

$$
\begin{array}{rcl}
r \models -F & \Leftrightarrow & r \models \neg(F \mathrel{-\!\!*} \neg \bowtie) \\
 & \Leftrightarrow & r \not\models F \mathrel{-\!\!*} \neg \bowtie \\
 & \Leftrightarrow & \exists r', r''.\ r'' \in r \circ r' \text{ and } r' \models F \text{ but } r'' \not\models \neg \bowtie \\
 & \Leftrightarrow & \exists r', r''.\ r'' \in r \circ r' \text{ and } r' \models F \text{ and } r'' = \infty \\
 & \Leftrightarrow & \exists r'.\ \infty \in r \circ r' \text{ and } r' \models F \\
 & \Leftrightarrow & -r \models F
\end{array}
$$

Note that the final equivalence above is justified by the fact that $-r$ is the unique element of R satisfying $\infty \in r \circ -r$, which follows from Proposition 2.3.

As is standard in ordinary BI, we write *sequents* of the form $\Gamma \vdash F$, where F is a BI$^+$-formula and Γ is a *bunch*, given by the following grammar:

$$\Gamma ::= F \mid \Gamma; \Gamma \mid \Gamma, \Gamma$$

where F ranges over BI$^+$-formulas. Thus bunches are trees whose leaves are formulas and whose internal nodes are either ';' or ','. We write $\Gamma(\Delta)$ for a bunch of which Δ is a distinguished sub-bunch (i.e. subtree), and in such cases write $\Gamma(\Delta')$ for the bunch obtained by replacing Δ by the bunch Δ' in $\Gamma(\Delta)$. In analogy to the use of sets in ordinary sequent calculus, and as is again standard for BI, we consider bunches up to *coherent equivalence*:

Definition 3.2 (Coherent equivalence). $\equiv$ is the least relation on bunches satisfying commutative monoid equations for ';' and $\top$, and for ',' and $\top^*$, plus the rule of congruence: if $\Delta \equiv \Delta'$ then $\Gamma(\Delta) \equiv \Gamma(\Delta')$.

Definition 3.3 (Validity). For any bunch Γ we define a formula Φ_Γ by recursion on the structure of Γ as follows:

$$
\begin{array}{rcl}
\Phi_F & = & F \\
\Phi_{\Gamma_1;\Gamma_2} & = & \Phi_{\Gamma_1} \wedge \Phi_{\Gamma_2} \\
\Phi_{\Gamma_1,\Gamma_2} & = & \Phi_{\Gamma_1} * \Phi_{\Gamma_2}
\end{array}
$$

A sequent $\Gamma \vdash F$ is said to be *true* in a classical BI-model $\langle R, \circ, e, -, \infty \rangle$ if for any environment ρ and for all $r \in R$, $r \models \Phi_\Gamma$ implies $r \models F$. $\Gamma \vdash F$ is said to be *valid* if it is true in all classical BI-models.

We give the rules of a sequent calculus proof system LBI^+ for BI^+ in Figure 1. Its rules extend the rules of the usual sequent calculus for BI (cf. [11, 7]) with the double negation axiom needed for boolean BI, and two further axioms that directly reflect the fact that $-$ behaves as an involution in our models.

Structural rules:

$$\frac{}{F \vdash F}\,(\mathrm{Id}) \qquad \frac{\Gamma(\Delta) \vdash F}{\Gamma(\Delta; \Delta') \vdash F}\,(\mathrm{Weak}) \qquad \frac{\Gamma(\Delta; \Delta) \vdash F}{\Gamma(\Delta) \vdash F}\,(\mathrm{Contr})$$

$$\frac{\Gamma' \vdash F}{\Gamma \vdash F}\ \Gamma \equiv \Gamma'\ (\mathrm{Equiv}) \qquad \frac{\Delta \vdash G \quad \Gamma(G) \vdash F}{\Gamma(\Delta) \vdash F}\,(\mathrm{Cut})$$

Propositional rules:

$$\frac{}{\Gamma(\bot) \vdash F}\,(\bot\mathrm{L}) \qquad \frac{\Gamma(F_1) \vdash F \quad \Gamma(F_2) \vdash F}{\Gamma(F_1 \vee F_2) \vdash F}\,(\vee\mathrm{L}) \qquad \frac{\Gamma(F_1; F_2) \vdash F}{\Gamma(F_1 \wedge F_2) \vdash F}\,(\wedge\mathrm{L})$$

$$\frac{}{\Gamma \vdash \top}\,(\top\mathrm{R}) \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ i \in \{1,2\}\ (\vee R_i) \qquad \frac{\Gamma \vdash F_1 \quad \Gamma \vdash F_2}{\Gamma \vdash F_1 \wedge F_2}\,(\wedge\mathrm{R})$$

$$\frac{\Delta \vdash F_1 \quad \Gamma(F_2) \vdash F}{\Gamma(\Delta, F_1 \multimap F_2) \vdash F}\,(\multimap\mathrm{L}) \qquad \frac{\Delta \vdash F_1 \quad \Gamma(\Delta; F_2) \vdash F}{\Gamma(\Delta; F_1 \to F_2) \vdash F}\,(\to\mathrm{L}) \qquad \frac{\Gamma(F_1, F_2) \vdash F}{\Gamma(F_1 * F_2) \vdash F}\,(*\mathrm{L})$$

$$\frac{\Gamma, F_1 \vdash F_2}{\Gamma \vdash F_1 \multimap F_2}\,(\multimap\mathrm{R}) \qquad \frac{\Gamma; F_1 \vdash F_2}{\Gamma \vdash F_1 \to F_2}\,(\to\mathrm{R}) \qquad \frac{\Gamma \vdash F_1 \quad \Delta \vdash F_2}{\Gamma, \Delta \vdash F_1 * F_2}\,(*\mathrm{R})$$

BI^+ **axioms:**

$$\frac{}{\neg\neg F \vdash F}\,(\mathrm{DNE}) \qquad \frac{}{--F \vdash F}\,(\mathrm{DIE}) \qquad \frac{}{F \vdash --F}\,(\mathrm{DII})$$

Fig. 1. The proof rules of LBI^+.

Proposition 3.4. LBI^+ *is sound with respect to validity in classical* BI*-models.*

Proof. As usual, soundness follows from the fact that the proof rules of BI^+ preserve truth in classical BI-models. First, note that the rules of BI preserve truth in BI-models and thus in classical BI-models in particular. Thus it only remains to show that the BI^+ axioms are true in any classical BI-model. Soundness of the axiom (DNE) follows from the fact that additive implication is interpreted classically in BI^+, so that $r \models \neg F$ iff $r \not\models F$. For the axioms (DIE) and (DII), note that $r \models {-}{-}F$ iff ${-}{-}r \models F$ by Lemma 3.1. Soundness of these axioms then follows from the fact that ${-}{-}r = r$ in classical BI-models. □

3.1 Completeness of LBI^+

We now show completeness of LBI^+ with respect to classical BI-models by appealing to a general theorem of modal logic due to Sahlqvist. The result is an adaptation of the analogous completeness result for BI in [5].

We first define MBI^+ pre-models, which interpret the LBI^+ connectives as modalities.

Definition 3.5. *An* MBI^+ pre-model *is a tuple* $\langle R, \circ, {-\bullet}, e, -, \infty \rangle$*, where* $\circ : R \times R \to Pow(R)$*,* ${-\bullet} : R \times R \to Pow(R)$*,* $e \in R$*,* $- : R \to Pow(R)$*, and* $\infty \subseteq R$*.*

The satisfaction relation for BI^+-formulas in MBI^+ pre-models is defined exactly as the satisfaction relation given above for BI^+-formulas in classical BI-models, except that the clause for formulas of the form $F \mathrel{-\!\!*} G$ is replaced by the following one:

$$r \models F_1 \mathrel{-\!\!*} F_2 \Leftrightarrow \forall r', r''.\, r \in r' \mathrel{-\bullet} r'' \text{ and } M, r' \models F_1 \text{ implies } M, r'' \not\models F_2$$

Then, given any set AX of axioms, we define *AX-models* to be the MBI^+ pre-models in which every axiom in AX holds.

Definition 3.6 (Modal Logic Formulas). *Modal logic formulas F are defined by the grammar:*

$$F ::= \bot \mid P \mid F \wedge F \mid \neg F \mid \triangle(F_1, \ldots, F_n)$$

where P ranges over $\mathcal{V}$ and $\triangle$ ranges over the modalities $\{e, -, \circ, {-\bullet}, \infty\}$. We identify BI^+*-formulas and modal logic formulas, by implicitly applying the usual translation for additives, plus the abbreviations $\bowtie = \infty$, $\top^* = e$ and $F_1 \mathrel{-\!\!*} F_2 = \neg(F_1 \mathrel{-\bullet} \neg F_2)$.*

Definition 3.7 (Very Simple Sahlqvist Formulas). *A* very simple Sahlqvist antecedent *A is a formula given by the grammar:*

$$A ::= \top \mid \bot \mid P \mid A \wedge A \mid \triangle(A_1, \ldots, A_n)$$

where P ranges over $\mathcal{V}$ and $\triangle$ ranges over the modalities $\{e, -, \circ, {-\bullet}\}$. A very simple Sahlqvist formula *is a formula of the form $A \Rightarrow F^+$, where A is a very simple Sahlqvist antecedent and F^+ is a modal logic formula which is* positive *in that no propositional variable P in F^+ may occur inside the scope of an odd number of occurrences of $\neg$.*

Theorem 3.8 (Sahlqvist [2]). *For every axiom set AX consisting of very simple Sahlqvist formulas, the modal logic proof theory generated by AX is complete with respect to the class of AX-models.*

Definition 3.9 (BI$^+$**-Axioms).** *The axiom set AX_{BI^+} consists of the following formulas:*

1. $e \circ P \Rightarrow P$
2. $P \Rightarrow e \circ P$
3. $P \circ Q \Rightarrow Q \circ P$
4. $(P \circ Q) \circ R \Rightarrow P \circ (Q \circ R)$
5. $P \circ (Q \circ R) \Rightarrow (P \circ Q) \circ R$
6. $Q \wedge (R \circ P) \Rightarrow (R \wedge (P \multimapdot Q)) \circ \top$
7. $R \wedge (P \multimapdot Q) \Rightarrow \top \multimapdot (Q \wedge (R \circ P))$
8. $- - P \Rightarrow P$
9. $P \Rightarrow - - P$
10. $-P \Rightarrow P \multimapdot \infty$
11. $P \multimapdot \infty \Rightarrow -P$

We write $\mathrm{LAX}_{\mathrm{BI}^+}$ for the modal logic proof theory generated by the AX_{BI^+} axioms.

Corollary 3.10. *LAX_{BI^+} is complete with respect to the class of AX_{BI^+} models.*

The following two propositions extend analogous results in [5]. Note that Θ_Γ denotes the BI$^+$-formula constructed from a bunch Γ by Definition 3.3.

Proposition 3.11. *$\Gamma \vdash F$ is derivable in* LBI$^+$ *iff $\Theta_\Gamma \rightarrow F$ is derivable in* $\mathrm{LAX}_{\mathrm{BI}^+}$.

Proposition 3.12. *$\Gamma \vdash F$ is valid with respect to classical BI-models iff $\Theta_\Gamma \rightarrow F$ is valid with respect to AX_{BI^+}-models.*

The specific properties of $-$ and ∞ given by the AX_{BI^+} axioms are consequences of Lemma 2.2.

Theorem 3.13. LBI$^+$ *is complete with respect to validity in classical* BI*-models.*

Proof. If $\Gamma \vdash F$ is valid with respect to classical BI-models then, by Proposition 3.12 $\Theta_\Gamma \rightarrow F$ is valid with respect to AX_{BI^+}-models and thus provable in LAX_{BI^+} by Corollary 3.10. By Proposition 3.11, $\Gamma \vdash F$ is then provable in LBI$^+$ as required. □

4 Classical BI and display logic

In this section we define CBI, the fully classical version of (propositional) BI featuring additive and multiplicative versions of all the usual propositional connectives (cf. [11]), and show how it can be interpreted in our classical BI-models. Then, we define a display calculus proof system for CBI, based on Belnap's *display logic* [1], which is a generalised Gentzen-style system in which many substructural logics can be encoded. Our display calculus satisfies cut-elimination by

Belnap's generalised cut-elimination theorem, and is sound and complete with respect to our classical BI-models.

Formulas of CBI are given by the following grammar:

$$\begin{array}{rl} F ::= & P \mid \top \mid \bot \mid \neg F \mid F \wedge F \mid F \vee F \mid F \rightarrow F \mid \\ & \top^* \mid \bot^* \mid {\sim} F \mid F * F \mid F \otimes F \mid F \mathrel{-\!\!*} F \end{array}$$

where P ranges over $\mathcal{V}$. We remark that CBI-formulas extend BI-formulas but not BI$^+$-formulas, since $\bowtie$ is not a CBI-formula. Given a classical BI-model $M = \langle R, \circ, e, -, \infty \rangle$ and an environment ρ for M, *satisfaction* of a CBI-formula in M under ρ then extends the definition of satisfaction of a BI$^+$ formula (cf. Section 3) as follows:

$$\begin{array}{rcl} r \models \neg F & \Leftrightarrow & r \not\models F \\ r \models \bot^* & \Leftrightarrow & r \neq \infty \\ r \models {\sim} F & \Leftrightarrow & -r \not\models F \\ r \models F_1 \otimes F_2 & \Leftrightarrow & \forall r_1, r_2, r.\ -r \in r_1 \circ r_2 \text{ implies } -r_1 \models F_1 \text{ or } -r_2 \models F_2 \end{array}$$

Perhaps surprisingly, multiplicative falsity $\bot^*$ and negation ${\sim}F$ are not interpreted in the same way as the BI$^+$-formulas $\bowtie$ and $-F$ respectively, but rather as $\neg \bowtie$ and $\neg - F$. The reason for the presence of the additive negation "inside" these multiplicative connectives is to ensure that the expected semantic equivalences hold between formulas. For example, ${\sim}F$ and $F \mathrel{-\!\!*} \bot^*$ are semantically equivalent, whereas the BI$^+$-formulas $-F$ and $F \mathrel{-\!\!*} \bowtie$ are not. As expected, multiplicative disjunction is interpreted as the de Morgan dual of $*$ with respect to multiplicative negation.

4.1 DL$_{\mathrm{BI}}$: a display calculus for CBI

We now turn to DL$_{\mathrm{BI}}$, our display calculus proof system for CBI. DL$_{\mathrm{BI}}$ can be seen as a particular instantiation of Belnap's generalised display logic to CBI.

Rather than using sequents built from bunches and formulas as in LBI$^+$, the proof judgements of our display calculus, called consecutions, are built from structures which generalise bunches.

Definition 4.1 (Structure / Consecution). A DL$_{\mathrm{BI}}$-*structure* X is an object constructed according to the following grammar:

$$X ::= F \mid \emptyset \mid \sharp X \mid X ; X \mid \varnothing \mid \flat X \mid X , X$$

where F ranges over CBI-formulas.

If X and Y are structures then $X \vdash Y$ is said to be a *consecution.*

We remark that if an LBI$^+$ sequent contains no occurrences of the formula $\bowtie$ then it is a special case of a DL$_{\mathrm{BI}}$ consecution.

We can divide the structural and logical connectives of DL$_{\mathrm{BI}}$ into an additive family and a multiplicative family, as illustrated in Figure 2. In Belnap's display logic, an arbitrary number of families of connectives may be involved; the

	Structural connectives			**Formula connectives**					
Additive family:	$\emptyset$	$\sharp$	;	$\top$	$\bot$	$\neg$	$\wedge$	$\vee$	$\rightarrow$
Multiplicative family:	$\varnothing$	$\flat$	,	$\top^*$	$\bot^*$	$\sim$	$*$	$\otimes$	$-\!\!*$
Arity:	0	1	2	0	0	1	2	2	2

Fig. 2. The connective families of $\mathrm{DL_{BI}}$.

structural connectives are prescribed for each family while the logical connectives may be chosen from a given set. Then, for each family, certain bidirectional rules called *display postulates*, involving only the structural connectives of the family, are prescribed by display logic. The logical introduction rules for formulas are similarly prescribed, leaving only the structural rules governing the family to be freely chosen.

We give the display postulates for $\mathrm{DL_{BI}}$ in Figure 3. These are the instantiations to our connective families of Belnap's original postulates [1], though other formulations are possible (see e.g. [8]). Note that we write a rule with a double line to indicate that it is invertible, i.e., that it may be also be applied if the premise is swapped with the conclusion. A figure with three consecutions separated by two double lines is used to abbreviate two invertible rules in the obvious way. Two consecutions are said to be *display-equivalent* if there is a derivation of one from the other using only the display postulates.

Additive family:

$$\frac{X;Y \vdash Z}{\overline{X \vdash \sharp Y; Z}}\ (\mathrm{AD1}) \qquad \begin{array}{c} X \vdash Y;Z \\ \hline\hline X;\sharp Y \vdash Z \\ \hline\hline X \vdash Z;Y \end{array}\ \begin{array}{l}(\mathrm{AD2a})\\(\mathrm{AD2b})\end{array} \qquad \begin{array}{c} X \vdash Y \\ \hline\hline \sharp Y \vdash \sharp X \\ \hline\hline \sharp\sharp X \vdash Y \end{array}\ \begin{array}{l}(\mathrm{AD3a})\\(\mathrm{AD3b})\end{array}$$

Multiplicative family:

$$\frac{X,Y \vdash Z}{\overline{X \vdash \flat Y, Z}}\ (\mathrm{MD1}) \qquad \begin{array}{c} X \vdash Y,Z \\ \hline\hline X,\flat Y \vdash Z \\ \hline\hline X \vdash Z,Y \end{array}\ \begin{array}{l}(\mathrm{MD2a})\\(\mathrm{MD2b})\end{array} \qquad \begin{array}{c} X \vdash Y \\ \hline\hline \flat Y \vdash \flat X \\ \hline\hline \flat\flat X \vdash Y \end{array}\ \begin{array}{l}(\mathrm{MD3a})\\(\mathrm{MD3b})\end{array}$$

Fig. 3. The display postulates for $\mathrm{DL_{BI}}$.

The following definition is used to set up the fundamental *display property* of display logic.

Definition 4.2 (Antecedent part / consequent part). A structure W is said to be a *part* of another structure Z if W is a substructure of Z (in the obvious sense). W is said to be a *positive part* of Z if W occurs inside an even number of occurrences of $\sharp$ and $\flat$ in Z, and a *negative part* of Z otherwise.

Now let $S = X \vdash Y$ be a consecution. W is said to be an *antecedent part* of S if it is a positive part of X or a negative part of Y. W is said to be a *consequent part* of S if it is a negative part of X or a positive part of Y.

Theorem 4.3 (Display theorem (Belnap [1])). *Let S be a consecution. Then for any antecedent part W of S there exists a consecution S' that is display-equivalent to S and such that $S' = W \vdash Z$ for some Z. Similarly, for any consequent part W of S there exists a consecution S' that is display-equivalent to S and such that $S' = Z \vdash W$ for some Z.*

We note that the display theorem holds even when connectives from different families occur in the same consecution.

Example 4.4. Consider the consecution $\flat(X, \sharp Y) \vdash Z; \flat W$. Then Y, which is an antecedent part of this consecution, can be displayed as follows:

$$\dfrac{\flat(X, \sharp Y) \vdash Z; \flat W}{\flat(Z; \flat W) \vdash \flat\flat(X, \sharp Y)}\ (\text{MD3a})$$

$$\dfrac{\flat(Z; \flat W) \vdash \flat\flat(X, \sharp Y)}{\flat\flat\flat(Z; \flat W) \vdash \flat\flat(X, \sharp Y)}\ (\text{MD3a,b})$$

$$\dfrac{\flat\flat\flat(Z; \flat W) \vdash \flat\flat(X, \sharp Y)}{\flat(X, \sharp Y) \vdash \flat\flat(Z; \flat W)}\ (\text{MD3a})$$

$$\dfrac{\flat(X, \sharp Y) \vdash \flat\flat(Z; \flat W)}{\flat(Z; \flat W) \vdash X, \sharp Y}\ (\text{MD3a})$$

$$\dfrac{\flat(Z; \flat W) \vdash X, \sharp Y}{\flat(Z; \flat W), \flat X \vdash \sharp Y}\ (\text{MD2b})$$

$$\dfrac{\flat(Z; \flat W), \flat X \vdash \sharp Y}{\sharp\sharp Y \vdash \sharp(\flat(Z; \flat W), \flat X)}\ (\text{AD3a})$$

$$\dfrac{\sharp\sharp Y \vdash \sharp(\flat(Z; \flat W), \flat X)}{Y \vdash \sharp(\flat(Z; \flat W), \flat X)}\ (\text{AD3a,b})$$

The logical rules for $\mathrm{DL_{BI}}$, given in Figure 4, follow the familiar division between left and right introduction rules (plus the standard rule for identity, and a cut rule). Again, these are the instantiations of the standard display logic rules to the connective families we consider for classical BI. The structural rules of $\mathrm{DL_{BI}}$ are given in Figure 5. These implement the coherent equivalence equations for bunches (cf. Definition 3.2) on both sides of consecutions, plus weakening and contraction on both sides for ';'.

Proposition 4.5. *$F \vdash F$ is $\mathrm{DL_{BI}}$-provable for all CBI-formulas F.*

Proof. By structural induction on F. □

Theorem 4.6 (Cut-elimination (Belnap [1])). *If a consecution $X \vdash Y$ is provable in $\mathrm{DL_{BI}}$ then it is also provable without the use of the rule (Cut).*

Proof. By inspection, our proof rules satisfy the 8 conditions shown by Belnap in [1] to be sufficient for cut-elimination to hold. See appendix A.1 for details. □

Identity rules:

$$\frac{}{P \vdash P}\,(\text{Id}) \qquad \frac{X \vdash F \quad F \vdash Y}{X \vdash Y}\,(\text{Cut})$$

Additive family:

$$\frac{\emptyset \vdash X}{\top \vdash X}\,(\top\text{L}) \qquad \frac{}{\bot \vdash \emptyset}\,(\bot\text{L}) \qquad \frac{\sharp F \vdash X}{\neg F \vdash X}\,(\neg\text{L})$$

$$\frac{}{\emptyset \vdash \top}\,(\top\text{R}) \qquad \frac{X \vdash \emptyset}{X \vdash \bot}\,(\bot\text{R}) \qquad \frac{X \vdash \sharp F}{X \vdash \neg F}\,(\neg\text{R})$$

$$\frac{F ; G \vdash X}{F \wedge G \vdash X}\,(\wedge\text{L}) \qquad \frac{F \vdash X \quad G \vdash Y}{F \vee G \vdash X ; Y}\,(\vee\text{L}) \qquad \frac{X \vdash F \quad G \vdash Y}{F \rightarrow G \vdash \sharp X ; Y}\,(\rightarrow\text{L})$$

$$\frac{X \vdash F \quad Y \vdash G}{X ; Y \vdash F \wedge G}\,(\wedge\text{R}) \qquad \frac{X \vdash F ; G}{X \vdash F \vee G}\,(\vee\text{R}) \qquad \frac{X ; F \vdash G}{X \vdash F \rightarrow G}\,(\rightarrow\text{R})$$

Multiplicative family:

$$\frac{\varnothing \vdash X}{\top^* \vdash X}\,(\top^*\text{L}) \qquad \frac{}{\bot^* \vdash \varnothing}\,(\bot^*\text{L}) \qquad \frac{\flat F \vdash X}{\sim F \vdash X}\,(\sim\text{L})$$

$$\frac{}{\varnothing \vdash \top^*}\,(\top^*\text{R}) \qquad \frac{X \vdash \varnothing}{X \vdash \bot^*}\,(\bot^*\text{R}) \qquad \frac{X \vdash \flat F}{X \vdash \sim F}\,(\sim\text{R})$$

$$\frac{F , G \vdash X}{F * G \vdash X}\,(*\text{L}) \qquad \frac{F \vdash X \quad G \vdash Y}{F \otimes G \vdash X , Y}\,(\otimes\text{L}) \qquad \frac{X \vdash F \quad G \vdash Y}{F \mathrel{-\!\!*} G \vdash \flat X , Y}\,(\mathrel{-\!\!*}\text{L})$$

$$\frac{X \vdash F \quad Y \vdash G}{X , Y \vdash F * G}\,(*\text{R}) \qquad \frac{X \vdash F , G}{X \vdash F \otimes G}\,(\otimes\text{R}) \qquad \frac{X , F \vdash G}{X \vdash F \mathrel{-\!\!*} G}\,(\mathrel{-\!\!*}\text{R})$$

Fig. 4. Logical rules for $\mathrm{DL}_{\mathrm{BI}}$. Note that F, G range over CBI-formulas while P ranges over propositional variables in $\mathcal{V}$.

We remark that, although cut-free proofs in $\mathrm{DL}_{\mathrm{BI}}$ enjoy the subformula property, cut-free proof search in the calculus is still highly non-deterministic due to the presence of the display postulates and structural rules. In Figure 6 we give a sample cut-free proof of the consecution $\sim\neg F \vdash \neg\sim F$, which illustrates this point.

Additive family:

$$\frac{W;(X;Y) \vdash Z}{(W;X);Y \vdash Z}\text{(AAL)} \qquad \frac{X;Y \vdash Z}{Y;X \vdash Z}\text{(ACL)} \qquad \frac{\emptyset;X \vdash Y}{X \vdash Y}\text{(AIL)}$$

$$\frac{W \vdash (X;Y);Z}{W \vdash X;(Y;Z)}\text{(AAR)} \qquad \frac{X \vdash Y;Z}{X \vdash Z;Y}\text{(ACR)} \qquad \frac{X \vdash Y;\emptyset}{X \vdash Y}\text{(AIR)}$$

$$\frac{X \vdash Z}{X;Y \vdash Z}\text{(WkL)} \qquad \frac{X \vdash Z}{X \vdash Y;Z}\text{(WkR)} \qquad \frac{X;X \vdash Z}{X \vdash Z}\text{(CtrL)} \qquad \frac{X \vdash Z;Z}{X \vdash Z}\text{(CtrR)}$$

Multiplicative family:

$$\frac{W,(X,Y) \vdash Z}{(W,X),Y \vdash Z}\text{(MAL)} \qquad \frac{X,Y \vdash Z}{Y,X \vdash Z}\text{(MCL)} \qquad \frac{\varnothing,X \vdash Y}{X \vdash Y}\text{(MIL)}$$

$$\frac{W \vdash (X,Y),Z}{W \vdash X,(Y,Z)}\text{(MAR)} \qquad \frac{X \vdash Y,Z}{X \vdash Z,Y}\text{(MCR)} \qquad \frac{X \vdash Y,\varnothing}{X \vdash Y}\text{(MIR)}$$

Fig. 5. Structural rules for $\mathrm{DL_{BI}}$.

Definition 4.7 (Validity in $\mathrm{DL_{BI}}$). For any structure X we define two formulas Ψ_X and Υ_X by mutual recursion on the structure of X as follows:

$$\begin{array}{rlrl}
\Psi_F &= F & \Upsilon_F &= F \\
\Psi_\emptyset &= \top & \Upsilon_\emptyset &= \bot \\
\Psi_{\sharp X} &= \neg\Upsilon_X & \Upsilon_{\sharp X} &= \neg\Psi_X \\
\Psi_{X_1;X_2} &= \Psi_{X_1} \wedge \Psi_{X_2} & \Upsilon_{X_1;X_2} &= \Upsilon_{X_1} \vee \Upsilon_{X_2} \\
\Psi_\varnothing &= \top^* & \Upsilon_\varnothing &= \bot^* \\
\Psi_{\flat X} &= {\sim}\Upsilon_X & \Upsilon_{\flat X} &= {\sim}\Psi_X \\
\Psi_{X_1,X_2} &= \Psi_{X_1} * \Psi_{X_2} & \Upsilon_{X_1,X_2} &= \Upsilon_{X_1} \otimes \Upsilon_{X_2}
\end{array}$$

A consecution $X \vdash Y$ is said to be *true* in a classical BI-model $\langle R, \circ, e, -, \infty\rangle$ if for any environment ρ and for all $r \in R$, $r \models \Psi_X$ implies $r \models \Upsilon_X$. $X \vdash Y$ is said to be *valid* if it is true in all classical BI-models.

We remark that, when $X \vdash Y$ is an LBI^+ sequent, the notion of validity given in Definition 4.7 coincides with the usual notion of validity in BI^+ (Definition 3.3).

We end this section by stating our main technical results concerning $\mathrm{DL_{BI}}$.

Theorem 4.8. $\mathrm{DL_{BI}}$ *is sound with respect to validity in classical* BI-*models.*

Theorem 4.9. $\mathrm{DL_{BI}}$ *is complete with respect to validity in classical* BI-*models.*

We give the proofs of Theorems 4.8 and 4.9 in Section 5.

$$\dfrac{}{F \vdash F}\ (\text{Id})$$
$$\dfrac{F \vdash F}{\sharp F \vdash \sharp F}\ (\text{D}\equiv)$$
$$\dfrac{\sharp F \vdash \sharp F}{\sharp F \vdash \neg F}\ (\neg\text{R})$$
$$\dfrac{\sharp F \vdash \neg F}{\flat\neg F \vdash \flat\sharp F}\ (\text{D}\equiv)$$
$$\dfrac{\flat\neg F \vdash \flat\sharp F}{\sim\neg F \vdash \flat\sharp F}\ (\sim\text{L})$$
$$\dfrac{\sim\neg F \vdash \flat\sharp F}{\sim\neg F;\ \sim F \vdash \flat\sharp F}\ (\text{WkL})$$
$$\dfrac{\sim\neg F;\ \sim F \vdash \flat\sharp F}{\sharp F \vdash \flat(\sim\neg F;\ \sim F)}\ (\text{D}\equiv)$$
$$\dfrac{\sharp F \vdash \flat(\sim\neg F;\ \sim F)}{\sharp\flat(\sim\neg F;\ \sim F) \vdash F}\ (\text{D}\equiv)$$
$$\dfrac{\sharp\flat(\sim\neg F;\ \sim F) \vdash F}{\flat F \vdash \flat\sharp\flat(\sim\neg F;\ \sim F)}\ (\text{D}\equiv)$$
$$\vdots$$
(contd. right)

$$\vdots$$
$$\dfrac{\flat F \vdash \flat\sharp\flat(\sim\neg F;\ \sim F)}{\sim F \vdash \flat\sharp\flat(\sim\neg F;\ \sim F)}\ (\sim\text{L})$$
$$\dfrac{\sim F \vdash \flat\sharp\flat(\sim\neg F;\ \sim F)}{\sim\neg F;\ \sim F \vdash \flat\sharp\flat(\sim\neg F;\ \sim F)}\ (\text{WkL})$$
$$\dfrac{\sim\neg F;\ \sim F \vdash \flat\sharp\flat(\sim\neg F;\ \sim F)}{\sharp\flat(\sim\neg F;\ \sim F) \vdash \flat(\sim\neg F;\ \sim F)}\ (\text{D}\equiv)$$
$$\dfrac{\sharp\flat(\sim\neg F;\ \sim F) \vdash \flat(\sim\neg F;\ \sim F)}{\flat\emptyset;\sharp\flat(\sim\neg F;\ \sim F) \vdash \flat(\sim\neg F;\ \sim F)}\ (\text{WkL})$$
$$\dfrac{\flat\emptyset;\sharp\flat(\sim\neg F;\ \sim F) \vdash \flat(\sim\neg F;\ \sim F)}{\flat\emptyset \vdash \flat(\sim\neg F;\ \sim F);\flat(\sim\neg F;\ \sim F)}\ (\text{D}\equiv)$$
$$\dfrac{\flat\emptyset \vdash \flat(\sim\neg F;\ \sim F);\flat(\sim\neg F;\ \sim F)}{\flat\emptyset \vdash \flat(\sim\neg F;\ \sim F)}\ (\text{CtrR})$$
$$\dfrac{\flat\emptyset \vdash \flat(\sim\neg F;\ \sim F)}{\sim\neg F;\ \sim F \vdash \emptyset}\ (\text{D}\equiv)$$
$$\dfrac{\sim\neg F;\ \sim F \vdash \emptyset}{\sim\neg F \vdash \sharp\sim F;\emptyset}\ (\text{D}\equiv)$$
$$\dfrac{\sim\neg F \vdash \sharp\sim F;\emptyset}{\sim\neg F \vdash \sharp\sim F}\ (\text{AIR})$$
$$\dfrac{\sim\neg F \vdash \sharp\sim F}{\sim\neg F \vdash \neg\sim F}\ (\neg\text{R})$$

Fig. 6. A cut-free DL_{BI} proof of $\sim\neg F \vdash \neg\sim F$.

5 Proofs of soundness and completeness of DL_{BI}

Definition 5.1 (Embedding of DL_{BI} in LBI^+). We define a function $\ulcorner - \urcorner$ from DL_{BI}-formulas to BI^+-formulas by recursion on the structure of DL_{BI}-formulas, as follows:

$$\begin{array}{rcll}
\ulcorner F \urcorner & = & F & \text{where } F \in \{P \mid P \in \mathcal{V}\} \cup \{\top, \bot, \top^*\} \\
\ulcorner F_1 \mathbin{?} F_2 \urcorner & = & \ulcorner F_1 \urcorner \mathbin{?} \ulcorner F_2 \urcorner & \text{where } ? \in \{\wedge, \vee, \rightarrow, *, -\!\!*\} \\
\ulcorner \neg F \urcorner & = & \neg \ulcorner F \urcorner & \\
\ulcorner \bot^* \urcorner & = & \neg \bowtie & \\
\ulcorner \sim F \urcorner & = & \neg - \ulcorner F \urcorner & \\
\ulcorner F_1 \otimes F_2 \urcorner & = & \ulcorner \sim(\sim F_1 * \sim F_2) \urcorner = \neg - (\neg - \ulcorner F_1 \urcorner * \neg - \ulcorner F_2 \urcorner) &
\end{array}$$

We extend $\ulcorner - \urcorner$ to a function from DL_{BI} consecutions to LBI^+ sequents by:

$$\ulcorner X \vdash Y \urcorner = \ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_Y \urcorner$$

where Ψ_- and Υ_- are the functions given in Definition 4.7. We call the function $\ulcorner - \urcorner$ the *embedding from* DL_{BI} *to* LBI^+.

Lemma 5.2. *A* DL_{BI} *consecution* $X \vdash Y$ *is valid iff* $\ulcorner X \vdash Y \urcorner$ *is valid.*

Proof. We first show by structural induction on CBI-formulas F that $r \models F$ iff $r \models \ulcorner F \urcorner$. The lemma then follows straightforwardly. See Appendix A.2 for details. □

We write $F \dashv\vdash G$ to mean that both $F \vdash G$ and $G \vdash F$ are derivable (in some proof system), and call $F \dashv\vdash G$ a *derivable equivalence* of the system.

Lemma 5.3. *The following are all derivable equivalences of* LBI$^+$*:*

1. $\neg - \neg - F \dashv\vdash F$
2. $\neg - (F * \neg - G) \dashv\vdash F \twoheadrightarrow G$
3. $F \twoheadrightarrow G \dashv\vdash \neg - G \twoheadrightarrow \neg - F$
4. $F \dashv\vdash \neg - (\neg - F * \neg - \neg \bowtie)$

The following lemma says that we can rewrite formulas in BI$^+$ sequents according to derivable equivalences without affecting LBI$^+$-derivability.

Lemma 5.4. *Write $F(G)$ for a formula F of which G is a distinguished subformula, and when $F(G)$ is understood write $F(G')$ for the formula obtained by replacing G by G' in F. (This is analogous to the notation for bunches.)*

Now suppose that $A \dashv\vdash B$ is a derivable equivalence of LBI$^+$ *(where A, B are* BI$^+$*-formulas). Then the following two proof rules are derivable in* LBI$^+$*:*

$$\frac{\Gamma(F(A)) \vdash C}{\Gamma(F(B)) \vdash C}\,(\dashv\vdash L) \qquad \frac{\Gamma \vdash F(A)}{\Gamma \vdash F(B)}\,(\dashv\vdash R)$$

Proof. By considering the following two instances of (Cut):

$$\frac{F(B) \vdash F(A) \quad \Gamma(F(A)) \vdash C}{\Gamma(F(B)) \vdash C}\,(\text{Cut}) \qquad \frac{\Gamma \vdash F(A) \quad F(A) \vdash F(B)}{\Gamma \vdash F(B)}\,(\text{Cut})$$

it suffices to prove that $F(A) \vdash F(B)$ is derivable in LBI$^+$, whence it follows by symmetry that $F(B) \vdash F(A)$ is also derivable. If $F(A) = A$ then this is immediate by assumption. Otherwise A is a (distinguished) strict subformula of F and we proceed by an easy structural induction on F. □

Proposition 5.5. *The proof rules of* DL$_{\text{BI}}$ *are admissible in* LBI$^+$ *under the embedding $\ulcorner - \urcorner$. That is, for any instance of a* DL$_{\text{BI}}$ *rule, say:*

$$\frac{\{X_i \vdash Y_i \mid 1 \leq i \leq j\}}{X \vdash Y}\; j \in \{0, 1, 2\}$$

if $\ulcorner X_i \vdash Y_i \urcorner$ is derivable for all $1 \leq i \leq j$ then so is $\ulcorner X \vdash Y \urcorner$.

Proof. We distinguish a case for each proof rule. Most of the cases are straightforward. The main interesting cases are the logical rules ($\otimes$L)and ($\twoheadrightarrow$L), the structural rule (MIR) and the display postulates for the multiplicative family. These can be derived in LBI$^+$ under the embedding $\ulcorner - \urcorner$ with the aid of the rewrite rules given by Lemma 5.4 in conjunction with the derivable equivalences of Lemma 5.3. See Appendix A.3 for details. □

We can now prove the soundness of $\mathrm{DL_{BI}}$ as follows.

Proof of Theorem 4.8. If $X \vdash Y$ is provable in $\mathrm{DL_{BI}}$ then $\ulcorner X \vdash Y \urcorner$ is provable in $\mathrm{DL_{BI}}$ by Proposition 5.5, and thus is valid by the soundness of LBI^+ (Proposition 3.4), so $X \vdash Y$ is valid by Lemma 5.2. □

Definition 5.6 (Embedding of LBI^+ in $\mathrm{DL_{BI}}$). We define a function $\llcorner - \lrcorner$ from BI^+ sequents to $\mathrm{DL_{BI}}$ consecutions by: $\llcorner \Gamma \vdash F \lrcorner$ is the consecution obtained by replacing every occurrence of the formula $\bowtie$ in $\Gamma \vdash F$ by the formula $\neg\bot^*$.

We remark that $\llcorner - \lrcorner$ can be defined recursively over BI^+ formulas and extended to LBI^+ sequents in a manner similar to that in Definition 5.1.

Lemma 5.7. *The following are all derivable equivalences of* $\mathrm{DL_{BI}}$*:*

1. $\neg\neg F \dashv\vdash F$
2. $\neg F \dashv\vdash F \rightarrow \bot$
3. $\sim\! F \dashv\vdash F \mathrel{-\!\!*} \bot^*$
4. $\neg \sim\! F \dashv\vdash \sim\!\neg F$
5. $F_1 \otimes F_2 \dashv\vdash \sim\!(\sim\! F_1 * \sim\! F_2)$

Proposition 5.8. *The proof rules of* LBI^+ *are admissible in* $\mathrm{DL_{BI}}$ *under the embedding* $\llcorner - \lrcorner$*. That is, for any instance of an* LBI^+ *rule, say:*

$$\frac{\{\Gamma_i \vdash F_i \mid 1 \leq i \leq j\}}{\Gamma \vdash F}\; j \in \{0, 1, 2\}$$

if $\llcorner \Gamma_i \vdash F_i \lrcorner$ *is derivable for all* $1 \leq i \leq j$ *then so is* $\llcorner \Gamma \vdash F \lrcorner$*.*

Proof. We distinguish a case for each proof rule of LBI^+. The main interesting cases are the rules that operate inside bunches. We observe that $\llcorner \Gamma \lrcorner$ is a structure for any bunch Γ and that, in particular, $\llcorner \Delta \lrcorner$ is always an antecedent part of $\llcorner \Gamma \lrcorner(\llcorner \Delta \lrcorner)$. By the display theorem (Theorem 4.3) we can display the sub-bunch on which the rule operates as the entire antecedent of a display-equivalent consecution. We can then apply the corresponding rule of $\mathrm{DL_{BI}}$ to this antecedent and then simply invert the display postulate steps used to display the antecedent to restore the original context. See Appendix A.4 for details. □

Lemma 5.9. *If* $\llcorner \ulcorner X \vdash Y \urcorner \lrcorner$ *is* $\mathrm{DL_{BI}}$*-provable then so is* $X \vdash Y$*.*

Proof. The proof proceeds in three stages. First, we show by induction on CBI-formulas F that $F \dashv\vdash \llcorner \ulcorner F \urcorner \lrcorner$ is $\mathrm{DL_{BI}}$-provable, making use of the derivable equivalences given by Lemma 5.7 in the non-trivial cases. Second, we show by induction on $\mathrm{DL_{BI}}$-structures X that $X \vdash \Psi_X$ and $\Upsilon_X \vdash X$ are $\mathrm{DL_{BI}}$-provable. Finally, we can construct a proof of $X \vdash Y$ using the given proof of $\llcorner \ulcorner X \vdash Y \urcorner \lrcorner = \llcorner \ulcorner \Psi_X \urcorner \lrcorner \vdash \llcorner \ulcorner \Upsilon_Y \urcorner \lrcorner$ using the first two stages together with (Cut). See Appendix A.5 for details. □

We can now prove completeness for $\mathrm{DL_{BI}}$ as follows.

Proof of Theorem 4.9. If $X \vdash Y$ is valid then so is $\ulcorner X \vdash Y \urcorner$ by Lemma 5.2, which is then LBI^+-provable by Theorem 3.13. By Proposition 5.8, $\llcorner \ulcorner X \vdash Y \urcorner \lrcorner$ is then provable in $\mathrm{DL_{BI}}$, whence $X \vdash Y$ is also $\mathrm{DL_{BI}}$-provable by Lemma 5.9. □

6 Conclusion

Our starting point for the issues considered here was to observe that in BI the multiplicative connectives are considered intuitionistically rather than classically, and to ask whether any computationally significant models would be admitted by a classical version of BI or, for that matter, any non-trivial models (i.e., models in which the connectives do not collapse). Our main conceptual contribution in the present paper is to make the connection between classical BI and our class of classical BI-models, which is a non-trivial class containing in particular the class of Abelian groups. We believe that our models have potentially interesting applications, and merit further investigation in their own right.

The choice of classical BI as an appropriate logical setting in which to investigate these models is justified by our main technical contribution: our display calculus $\mathrm{DL}_{\mathrm{BI}}$, which we have shown to be both cut-eliminating, and sound and complete with respect to validity in our models. Moreover, our proof of soundness and completeness, which relies upon admissibility embeddings, makes an explicit connection between proof in $\mathrm{DL}_{\mathrm{BI}}$ and proof in LBI^{+}, which is just the usual BI sequent calculus extended by three axioms. The sequent calculus LBI^{+} employs a form of deep inference in order to correctly formulate the left-introduction rules, which is unnecessary in the display calculus $\mathrm{DL}_{\mathrm{BI}}$. However, even though cut-elimination in $\mathrm{DL}_{\mathrm{BI}}$ entails a subformula property, proof search in this setting is nevertheless made daunting by the presence of the display postulates, which can obviously lead to divergence if applied blindly. It thus remains of clear interest to formulate well-behaved sequent calculus or natural deduction proof systems for classical BI, and we hope that the present paper represents a first step in this direction.

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A Appendix: Proofs

A.1 Proof of Theorem 4.6

The following definition is taken from Belnap [1]. By a *constituent* of a structure or consecution we mean an occurrence of one of its substructures.

Definition A.1 (Parameters / congruence). Let I be an instance of a rule R of $\mathrm{DL_{BI}}$. Note that I is obtained by assigning structures to the structure variables occurring in R and formulas to the formula variables occurring in R.

Any constituent of the consecutions in I occurring as part of structures assigned to structure variables in I are defined to be *parameters* of I. All other constituents are defined to be *non-parametric* in I, including those assigned to formula variables.

Constituents occupying similar positions in occurrences of structures assigned to the same structure variable are defined to be *congruent* in I.

We remark that congruence as defined above is an equivalence relation.

Belnap's analysis guarantees cut-elimination (Theorem 4.6) provided the rules of $\mathrm{DL_{BI}}$ satisfy the following 8 conditions, which are stated with reference to an instance I of a $\mathrm{DL_{BI}}$ rule R. (Here, we state a stronger, combined version of Belnap's original conditions C6 and C7, since the our rules satisfy this stronger condition.) In each case, we indicate how to verify that the condition holds for our rules.

C1. *Preservation of formulas.* Each formula which is a constituent of some premise of I is a subformula of some formula in the conclusion of I.
Verification. One observes that, in each rule, no formula variable or structure variable is lost when passing from the premises to the conclusions.

C2. *Shape-alikeness of parameters.* Congruent parameters are occurrences of the same structure.
Verification. Immediate from the definition of congruence.

C3. *Non-proliferation of parameters.* No two constituents in the conclusion of I are congruent to each other.

Verification. One just observes that, for each rule, each structure variable occurs exactly once in the conclusion.

C4. *Position-alikeness of parameters.* Congruent parameters are either all antecedent or all consequent parts of their respective consecutions.

Verification. One observes that, in each rule, no structure variable occurs both as an antecedent part and a consequent part.

C5. *Display of principal constituents.* If a formula is nonparametric in the conclusion of I, it is either the entire antecedent or the entire consequent of that conclusion. Such a formula is said to be *principal* in I.

Verification. It is easy to verify that the only non-parametric formulas in the conclusions of our rules are the two occurrences of P in (Id) and those occurring in the introduction rules for the logical connectives in Figure 4, which obviously satisfy the condition.

C6/7. *Closure under substitution for parameters.* Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas which are parameters.

Verification. This condition is satisfied because no restrictions are placed on the structural variables used in our rules.

C8. *Eliminability of matching principal formulas.* If there are inferences I_1 and I_2 with respective conclusions $X \vdash F$ and $F \vdash Y$ and with F principal in both inferences, then either $X \vdash Y$ is equal to one of $X \vdash F$ and $F \vdash Y$, or there is a derivation of $X \vdash Y$ from the premises of I_1 and I_2 in which every instance of cut has a cut-formula which is a proper subformula of F.

Verification. There are only two cases to consider. If F is atomic then $X \vdash F$ and $F \vdash Y$ are both instances of (Id). Thus we must have $X \vdash F = F \vdash Y = X \vdash Y$, and are done. Otherwise F is non-atomic and introduced in I_1 and I_2 respectively by the right and left introduction rule for the main connective of F. In this case, a derivation of the desired form can be obtained using only the display postulates of Figure 3 and cuts on subformulas of F. For example, if the considered cut is of the form:

$$\dfrac{\dfrac{X \vdash A, B}{X \vdash A \otimes B}\,(\otimes\mathrm{R}) \quad \dfrac{A \vdash Y \quad B \vdash Z}{A \otimes B \vdash Y, Z}\,(\otimes\mathrm{L})}{X \vdash Y, Z}\,(\mathrm{Cut})$$

then we can eliminate this cut in the following manner:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X \vdash A, B}{X, \flat B \vdash A}\,(\mathrm{D}\equiv) \quad A \vdash Y}{X, \flat B \vdash Y}\,(\mathrm{Cut})}{X, \flat Y \vdash B}\,(\mathrm{D}\equiv) \quad B \vdash Z}{X, \flat Y \vdash Z}\,(\mathrm{Cut})}{X \vdash Y, Z}\,(\mathrm{D}\equiv)$$

where (D≡) denotes the use of a display-equivalence.

A.2 Proof of Lemma 5.2

We fix a classical BI-model $M = \langle R, \circ, e, -, \infty \rangle$ and an environment ρ for M. First, we prove that for any $r \in R$ and CBI-formula F that $r \models F$ iff $r \models \ulcorner F \urcorner$, proceeding by structural induction on F:

Case $F \in \{P \mid P \in \mathcal{V}\} \cup \{\top, \bot, \top^*\}$. Trivial.

Case $F = F_1 \mathbin{?} F_2$, where $? \in \{\wedge, \vee, \rightarrow, *, \mathrel{-\!\!*}\}$. We are immediately done by the induction hypothesis.

Case $F = \neg G$. Immediate by the induction hypothesis.

Case $F = \bot^*$. We have $\ulcorner \bot^* \urcorner = \neg \bowtie$. By definition, $r \models \bot^*$ iff $r \notin \infty$ and $r \models \neg \bowtie$ iff $r \notin \infty$, so we are done.

Case $F = \sim G$. We require to show $r \models \sim G$ iff $r \models \neg - \ulcorner G \urcorner$, i.e., $-r \not\models G$ iff $r \not\models - \ulcorner G \urcorner$. By Lemma 3.1, $r \not\models - \ulcorner G \urcorner$ iff $-r \not\models \ulcorner G \urcorner$, so we are done by the induction hypothesis.

Case $F = F_1 \otimes F_2$. We require to show:

$$
\begin{array}{rcl}
r \models F_1 \otimes F_2 & \Leftrightarrow & r \models \ulcorner \sim(\sim F_1 * \sim F_2) \urcorner \\
& \Leftrightarrow & r \models \neg - (\neg - \ulcorner F_1 \urcorner * \neg - \ulcorner F_2 \urcorner) \\
& \Leftrightarrow & -r \not\models \neg - \ulcorner F_1 \urcorner * \neg - \ulcorner F_2 \urcorner \\
& \Leftrightarrow & \neg \exists r_1, r_2.\ -r \in r_1 \circ r_2 \text{ and } -r_1 \not\models \ulcorner F_1 \urcorner \text{ and } -r_2 \not\models \ulcorner F_2 \urcorner \\
& \Leftrightarrow & \forall r_1, r_2.\ -r \in r_1 \circ r_2 \text{ implies } -r_1 \models \ulcorner F_1 \urcorner \text{ or } -r_2 \models \ulcorner F_2 \urcorner \\
& \Leftrightarrow & r \models \ulcorner F_1 \urcorner \otimes \ulcorner F_2 \urcorner
\end{array}
$$

Note that we use Lemma 3.1 in some of the equivalences above. The required equivalence thus follows from the induction hypothesis. This completes the induction.

Now let $X \vdash Y$ be a $\mathrm{DL_{BI}}$ consecution. Noting that Ψ_X and Υ_Y are CBI-formulas, and that $\ulcorner \Psi_X \urcorner$ and $\ulcorner \Upsilon_Y \urcorner$ are BI^+-formulas, we then have:

$$
\begin{array}{rcll}
\ulcorner X \vdash Y \urcorner \text{ true in } M & \Leftrightarrow & \ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_Y \urcorner \text{ true in } M & \\
& \Leftrightarrow & \forall r \in R.\ r \models \Phi_{\ulcorner \Psi_X \urcorner} \text{ implies } r \models \ulcorner \Upsilon_Y \urcorner & \\
& \Leftrightarrow & \forall r \in R.\ r \models \ulcorner \Psi_X \urcorner \text{ implies } r \models \ulcorner \Upsilon_Y \urcorner & \\
& \Leftrightarrow & \forall r \in R.\ r \models \Psi_X \text{ implies } r \models \Upsilon_Y & \text{(by first part)} \\
& \Leftrightarrow & X \vdash Y \text{ true in } M &
\end{array}
$$

Thus $X \vdash Y$ is valid if and only if $\ulcorner X \vdash Y \urcorner$ is. □

A.3 Proof of Proposition 5.5

According to Definition 5.1, we can restate the lemma as follows: for each rule of $\mathrm{DL}_{\mathrm{BI}}$, say:

$$\frac{\{X_i \vdash Y_i \mid 1 \leq i \leq j\}}{X \vdash Y}\ j \in \{0,1,2\}$$

if $\ulcorner \Psi_{X_i} \urcorner \vdash \ulcorner \Upsilon_{Y_i} \urcorner$ is derivable for all $1 \leq i \leq j$ then so is $\ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_Y \urcorner$.

Naturally, we must distinguish a case for each proof rule of $\mathrm{DL}_{\mathrm{BI}}$. Taking into account the bidirectionality of the display postulates and some of the structural rules, there are many cases. Luckily, most of them are very straightforward. Note that all derivations shown are BI^+ derivations, except that we also use the rule symbol (=) to denote rewriting a sequent according to the definitions of Ψ_-, Υ_- and $\ulcorner - \urcorner$ (cf. Definitions 4.7 and 5.1).

Logical rules (Figure 4)

*Cases (Id), (⊥L), (⊤R), (⊥*L), (⊤*R).* The conclusions of these rules become instances of the LBI^+ axiom (Id) under the embedding $\ulcorner - \urcorner$, so we are done. E.g., in the case of ($\perp^*$L) we have:

$$\ulcorner \Psi_{\perp^*} \urcorner \vdash \ulcorner \Upsilon_\varnothing \urcorner = \ulcorner \perp^* \urcorner \vdash \ulcorner \perp^* \urcorner = \neg \bowtie \vdash \neg \bowtie$$

which is an instance of (Id).

*Cases (⊤L), (⊥R), (¬L), (¬R), (∧L), (∨R), (⊤*L), (⊥*R), (∼L), (∼R), (*L), (⊗R).* The premise and conclusion of these rules are identified under $\ulcorner - \urcorner$, so we are trivially done. E.g., in the case of (∼R) we have:

$$\ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_{\flat A} \urcorner = \ulcorner \Psi_X \urcorner \vdash \ulcorner \sim\Upsilon_A \urcorner = \ulcorner \Psi_X \urcorner \vdash \ulcorner \sim A \urcorner$$
$$\ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_{\sim A} \urcorner = \ulcorner \Psi_X \urcorner \vdash \ulcorner \sim\Psi_A \urcorner = \ulcorner \Psi_X \urcorner \vdash \ulcorner \sim A \urcorner$$

Case (Cut). This rule becomes an instance of the LBI^+ (Cut) rule under $\ulcorner - \urcorner$.

*Cases (∨L), (∧R), (→L), (→R), (⊗L), (*R), (−*L), (−*R).* These rules can be derived straightforwardly under $\ulcorner - \urcorner$ in LBI^+ using the rewrite rules given by Lemma 5.4 with the derivable equivalences in Lemma 5.3. E.g., in the case of

$(\mathrel{-\!\!*}\mathrm{L})$ we proceed as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_A \urcorner}}{\ulcorner \Psi_X \urcorner \vdash \ulcorner A \urcorner}\,(=) \qquad \dfrac{\dfrac{\vdots}{\ulcorner \Psi_B \urcorner \vdash \ulcorner \Upsilon_Y \urcorner}}{\ulcorner B \urcorner \vdash \ulcorner \Upsilon_Y \urcorner}\,(=)}{\ulcorner A \urcorner \mathrel{-\!\!*} \ulcorner B \urcorner, \ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_Y \urcorner}\,(\mathrel{-\!\!*}\mathrm{L})}{\ulcorner A \urcorner \mathrel{-\!\!*} \ulcorner B \urcorner \vdash \ulcorner \Psi_X \urcorner \mathrel{-\!\!*} \ulcorner \Upsilon_Y \urcorner}\,(\mathrel{-\!\!*}\mathrm{R})}{\ulcorner A \urcorner \mathrel{-\!\!*} \ulcorner B \urcorner \vdash \neg\!-(\ulcorner \Psi_X \urcorner * \neg\!-\ulcorner \Upsilon_Y \urcorner)}\,(\dashv\vdash\mathrm{R})}{\ulcorner A \urcorner \mathrel{-\!\!*} \ulcorner B \urcorner \vdash \neg\!-(\neg\!-\neg\!-\ulcorner \Psi_X \urcorner * \neg\!-\ulcorner \Upsilon_Y \urcorner)}\,(\dashv\vdash\mathrm{R})}{\ulcorner A \urcorner \mathrel{-\!\!*} \ulcorner B \urcorner \vdash \ulcorner \sim\!\Psi_X \otimes \Upsilon_Y \urcorner}\,(=)}{\ulcorner \Psi_{A \mathrel{-\!\!*} B} \urcorner \vdash \ulcorner \Upsilon_{\flat X, Y} \urcorner}\,(=)
$$

Structural rules (Figure 5)

Cases (WkL), (CtrL). Under the embedding $\ulcorner - \urcorner$ these are instances respectively of the LBI^+ rules (Weak) and (Contr).

Cases (WkR), (CtrR). These follow respectively from the derivability of $F \vdash F \vee G$ and $F \vee F \vdash F$ in LBI^+.

Cases (AAL), (AAR), (ACL), (ACR), (MAL), (MAR), (MCL), (MCR). The derivability of these rules under $\ulcorner - \urcorner$ follows from the associativity and commutativity of $\wedge, \vee$ and $*$, all of which are easily derivable in LBI^+.

Cases (AIL), (AIR), (MIL). These follow respectively from the derivability of $\top \wedge F \dashv\vdash F$, $F \vee \bot \dashv\vdash F$, and $\top^* * F \dashv\vdash F$ in LBI^+.

Case (MIR). This rule follows from the derivable equivalence $F \vdash \neg\!-(\neg\!-F * \neg\!-\neg \bowtie)$ given by Lemma 5.3.

Display postulates (Figure 3)
These are all similar to previous cases. The additive rules are obviously all right, and the multiplicative rules can be derived using the derivable equivalences of Lemma 5.3.

A.4 Proof of Proposition 5.8

We distinguish a case for each proof rule of LBI^+. Note that all derivations shown are $\mathrm{DL_{BI}}$ derivations; we use the rule symbol (D$\equiv$) to denote the use of a display equivalence. We freely rewrite sequents according to the definition of $\llcorner - \lrcorner$ (cf. Definition 5.6), which only affects occurrences of $\bowtie$.

Case (DNE). We require to prove that $(\llcorner F \lrcorner \to \bot) \to \bot \vdash F$ is $\mathrm{DL_{BI}}$-derivable. Using Lemma 5.7, we have $(\llcorner F \lrcorner \to \bot) \to \bot \dashv\vdash \neg\neg F$ is $\mathrm{DL_{BI}}$-derivable, so it suffices to derive $\neg\neg F \vdash F$, which is easy.

Cases (DIE), (DII). We need to show that $\llcorner F \lrcorner \dashv\vdash \llcorner - - F \lrcorner$ is is a derivable equivalence of $\mathrm{DL_{BI}}$. Expanding the definitions of $-$ and $\llcorner - \lrcorner$, we obtain:

$$\llcorner F \lrcorner \dashv\vdash (((\llcorner F \lrcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot)) \to \bot) \mathrel{-\!\!*} (\neg\bot^* \to \bot)) \to \bot$$

Using Lemma 5.7 it is straightforward (though tedious) to show the following is $\mathrm{DL_{BI}}$-derivable:

$$\neg\!\sim\!\neg\!\sim\! F \dashv\vdash (((\llcorner F \lrcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot)) \to \bot) \mathrel{-\!\!*} (\neg\bot^* \to \bot)) \to \bot$$

It thus suffices to show that $\neg \sim\neg \sim F \dashv\vdash F$ is $\mathrm{DL_{BI}}$-derivable, which is again straightforward using Lemma 5.7.

Case (Id). We need to show $\llcorner F \lrcorner \vdash \llcorner F \lrcorner$ is $\mathrm{DL_{BI}}$-provable, which is the case by Proposition 4.5.

Cases ($\top R$), ($\vee R_1$), ($\vee R_2$), ($\wedge R$), ($\to R$), ($* R$), ($-\!\!* R$). These rules all have easy derivations using the corresponding $\mathrm{DL_{BI}}$ rule and, in some cases, the additive structural rules. E.g., in the case of ($\wedge$R) we proceed as follows:

$$\dfrac{\dfrac{\llcorner \Gamma \lrcorner \vdash \llcorner F_1 \lrcorner \quad \llcorner \Gamma \lrcorner \vdash \llcorner F_2 \lrcorner}{\llcorner \Gamma \lrcorner ; \llcorner \Gamma \lrcorner \vdash \llcorner F_1 \lrcorner \wedge \llcorner F_2 \lrcorner}\,(\wedge\mathrm{R})}{\llcorner \Gamma \lrcorner \vdash \llcorner F_1 \lrcorner \wedge \llcorner F_2 \lrcorner}\,(\mathrm{CtrL})$$

Cases (Weak), (Contr), (Cut), ($\bot L$), ($\vee L$), ($\wedge L$), ($\to L$), ($* L$), ($-\!\!* L$). These rules all operate inside bunches on the left of sequents. We observe that $\llcorner \Gamma \lrcorner$ is a structure for any bunch Γ and that $\llcorner \Delta \lrcorner$ is always an antecedent part of $\llcorner \Gamma \lrcorner(\llcorner \Delta \lrcorner)$. By the display theorem (Theorem 4.3) we can display the sub-bunch on which the rule operates as the entire antecedent of a display-equivalent consecution. We can then apply the corresponding rule of $\mathrm{DL_{BI}}$ to this antecedent and then simply invert the display postulate steps used previously to restore the original context. For example, in the case of ($\to$L) we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\llcorner \Delta \lrcorner \vdash \llcorner F_1 \lrcorner \qquad \dfrac{\dfrac{\dfrac{\llcorner \Gamma \lrcorner(\llcorner \Delta \lrcorner ; \llcorner F_2 \lrcorner) \vdash \llcorner F \lrcorner}{\llcorner \Delta \lrcorner ; \llcorner F_2 \lrcorner \vdash X}\,(\mathrm{D}\equiv)}{\llcorner F_2 \lrcorner ; \llcorner \Delta \lrcorner \vdash X}\,(\mathrm{ACL})}{\llcorner F_2 \lrcorner \vdash \sharp \llcorner \Delta \lrcorner ; X}\,(\mathrm{D}\equiv)}{\llcorner F_1 \lrcorner \to \llcorner F_2 \lrcorner \vdash \sharp \llcorner \Delta \lrcorner ; \sharp \llcorner \Delta \lrcorner ; X}\,(\to\mathrm{L})}{\llcorner \Delta \lrcorner ; \llcorner \Delta \lrcorner \vdash \sharp(\llcorner F_1 \lrcorner \to \llcorner F_2 \lrcorner) ; X}\,(\mathrm{D}\equiv)}{\llcorner \Delta \lrcorner \vdash \sharp(\llcorner F_1 \lrcorner \to \llcorner F_2 \lrcorner) ; X}\,(\mathrm{CtrL})}{\llcorner \Delta \lrcorner ; \llcorner F_1 \lrcorner \to \llcorner F_2 \lrcorner \vdash X}\,(\mathrm{D}\equiv)}{\llcorner \Gamma \lrcorner(\llcorner \Delta \lrcorner ; \llcorner F_1 \lrcorner \to \llcorner F_2 \lrcorner) \vdash \llcorner F \lrcorner}\,(\mathrm{D}\equiv)$$

where X is a placeholder for the structure that results as the consequent from displaying Y in the consecution $\llcorner\Gamma\lrcorner(Y) \vdash \llcorner F \lrcorner$.

Case (Equiv). We require to show that if $\llcorner\Gamma'\lrcorner \vdash \llcorner F\lrcorner$ is derivable and $\Gamma \equiv \Gamma'$ then $\llcorner\Gamma\lrcorner \vdash \llcorner F\lrcorner$ is derivable. We also note that $\equiv$ can be straightforwardly expressed as an inductive relation, and proceed by rule induction on $\Gamma \equiv \Gamma'$. The associativity, commutativity, and unitary cases follow from the induction hypothesis and the corresponding structural rules of $\mathrm{DL_{BI}}$. For the congruence case, we need to show that if $\Delta \equiv \Delta'$ and $\llcorner\Gamma\lrcorner(\llcorner\Delta'\lrcorner) \vdash \llcorner F\lrcorner$ is derivable then so is $\llcorner\Gamma\lrcorner(\llcorner\Delta\lrcorner) \vdash \llcorner F\lrcorner$. We use the display theorem as before and the induction hypothesis as follows:

$$\dfrac{\dfrac{\llcorner\Gamma\lrcorner(\llcorner\Delta'\lrcorner) \vdash \llcorner F\lrcorner}{\llcorner\Delta'\lrcorner \vdash X}\,(\mathrm{D}\equiv) \quad \vdots\,(\text{I.H.}) \quad \llcorner\Delta\lrcorner \vdash X}{\llcorner\Gamma\lrcorner(\llcorner\Delta\lrcorner) \vdash \llcorner F\lrcorner}\,(\mathrm{D}\equiv)$$

where X is used as a placeholder as in the previous cases. This completes the proof. $\square$

A.5 Proof of Lemma 5.9

The proof proceeds in three stages.

Stage 1. We prove by structural induction on CBI-formulas F that $F \dashv\vdash \ulcorner F \urcorner$ is derivable in $\mathrm{DL_{BI}}$. In the following, we make use of the fact that derivable equivalence $\dashv\vdash$ in $\mathrm{DL_{BI}}$ is an equivalence relation. In particular, it is reflexive by Proposition 4.5 and transitive by the rule (Cut).

Case $F \in \{P \mid P \in \mathcal{V}\} \cup \{\top, \bot, \top^*\}$. We have $\ulcorner F \urcorner = F$ and are thus immediately done since $F \dashv\vdash F$ holds in $\mathrm{DL_{BI}}$.

Case $F = F_1 \mathbin{?} F_2$ where $? \in \{\wedge, \vee, \rightarrow, *, -\!\!*\}$. We have $\ulcorner F_1 \mathbin{?} F_2 \urcorner = \ulcorner F_1 \urcorner \mathbin{?} \ulcorner F_2 \urcorner$ and the case is then straightforward by the induction hypothesis. E.g., in the case $F = F_1 -\!\!* F_2$ we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\overset{(\text{I.H.})}{\vdots}\; \ulcorner F_1\urcorner \vdash F_1 \qquad \overset{(\text{I.H.})}{\vdots}\; F_2 \vdash \ulcorner F_2 \urcorner}{F_1 -\!\!* F_2 \vdash \flat\ulcorner F_1\urcorner, \ulcorner F_2\urcorner}\,(-\!\!*\mathrm{L})}{F_1 -\!\!* F_2, \ulcorner F_1\urcorner \vdash \ulcorner F_2\urcorner}\,(\mathrm{D}\equiv)}{F_1 -\!\!* F_2 \vdash \ulcorner F_1\urcorner -\!\!* \ulcorner F_2\urcorner}\,(-\!\!*\mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{\overset{(\text{I.H.})}{\vdots}\; F_1 \vdash \ulcorner F_1\urcorner \qquad \overset{(\text{I.H.})}{\vdots}\; \ulcorner F_2\urcorner \vdash F_2}{\ulcorner F_1\urcorner -\!\!* \ulcorner F_2\urcorner \vdash \flat F_1, F_2}\,(-\!\!*\mathrm{L})}{\ulcorner F_1\urcorner -\!\!* \ulcorner F_2\urcorner, F_1 \vdash F_2}\,(\mathrm{D}\equiv)}{\ulcorner F_1\urcorner -\!\!* \ulcorner F_2\urcorner \vdash F_1 -\!\!* F_2}\,(-\!\!*\mathrm{R})$$

Case $F = \neg G$. We have $\ulcorner \neg G \urcorner = \ulcorner G \urcorner \rightarrow \bot$. By Lemma 5.7, $\ulcorner G \urcorner \rightarrow \bot \dashv\vdash \neg\ulcorner G \urcorner$ is provable in $\mathrm{DL_{BI}}$. It thus suffices to prove $\neg G \dashv\vdash \neg\ulcorner G\urcorner$, which follows straightforwardly from the induction hypothesis.

Case $F = \bot^*$. We have $\ulcorner \bot^* \urcorner = \llcorner \bowtie \to \bot \lrcorner = \neg\bot^* \to \bot$. We have $\neg\bot^* \to \bot \dashv\vdash \neg\neg\bot^* \dashv\vdash \bot^*$ by Lemma 5.7 and so are done.

Case $F = \sim G$. We have $\ulcorner \sim G \urcorner = ((\ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot)) \to \bot) \to \bot$. Using Lemma 5.7 and transitivity of $\dashv\vdash$ we have:

$$((\ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot)) \to \bot) \to \bot \dashv\vdash \ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot)$$

Also, again by Lemma 5.7, we have $\sim G \dashv\vdash G \mathrel{-\!\!*} \bot^*$. To complete the case it suffices to show $G \mathrel{-\!\!*} \bot^* \dashv\vdash \ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot)$ is $\mathrm{DL_{BI}}$-provable, which we do as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{(I.H.)}\\ \vdots \\ \ulcorner G \urcorner \vdash G\end{array} \quad \begin{array}{c}\text{(Prop. 4.5)}\\ \vdots \\ \bot^* \vdash \bot^*\end{array}}{G \mathrel{-\!\!*} \bot^* \vdash \flat\ulcorner G \urcorner, \bot^*}\,(\mathrel{-\!\!*}\text{L})}{G \mathrel{-\!\!*} \bot^*, \ulcorner G \urcorner \vdash \bot^*}\,(\text{D}\equiv) \quad \begin{array}{c}\text{(Lemma 5.7)}\\ \vdots \\ \bot^* \vdash \neg\bot^* \to \bot\end{array}}{G \mathrel{-\!\!*} \bot^*, \ulcorner G \urcorner \vdash \neg\bot^* \to \bot}\,(\text{Cut})}{G \mathrel{-\!\!*} \bot^* \vdash \ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot)}\,(\mathrel{-\!\!*}\text{R})
\qquad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{(I.H.)}\\ \vdots \\ G \vdash \ulcorner G \urcorner\end{array} \quad \begin{array}{c}\text{(Lemma 5.7)}\\ \vdots \\ \neg\bot^* \to \bot \vdash \bot^*\end{array}}{\ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot) \vdash \flat G, \bot^*}\,(\mathrel{-\!\!*}\text{L})}{\ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot), G \vdash \bot^*}\,(\text{D}\equiv)}{\ulcorner G \urcorner \mathrel{-\!\!*} (\neg\bot^* \to \bot) \vdash G \mathrel{-\!\!*} \bot^*}\,(\mathrel{-\!\!*}\text{R})$$

Case $F = F_1 \otimes F_2$. We have $\ulcorner F_1 \otimes F_2 \urcorner = \ulcorner \sim(\sim F_1 * \sim F_2) \urcorner$. Since $F_1 \otimes F_2 \dashv\vdash \sim(\sim F_1 * \sim F_2)$ is $(\text{D}\equiv)_B I$-provable by Lemma 5.7, it suffices to show that $\sim(\sim F_1 * \sim F_2) \dashv\vdash \ulcorner \sim(\sim F_1 * \sim F_2) \urcorner$ is $\mathrm{DL_{BI}}$-provable, for which the techniques used in the cases for $\sim$ and $*$ above are sufficient. This completes the stage.

Stage 2. We prove by structural induction on $\mathrm{DL_{BI}}$-structures X that $X \vdash \Psi_X$ and $\Upsilon_X \vdash X$ are derivable in $\mathrm{DL_{BI}}$. The case where X is a formula F follows by the fact that $F \vdash F$ is $\mathrm{DL_{BI}}$-provable for all formulas F (Proposition 4.5). The other cases all follow straightforwardly from the induction hypothesis. E.g., when $X = \flat Y$ we proceed as follows (writing $(=)$ to denote rewriting a consecution according to the definitions of Ψ_{-} and Υ_{-} given by Definition 4.7):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{(I.H.)}\\ \vdots \\ \Upsilon_Y \vdash Y\end{array}}{\flat Y \vdash \flat\Upsilon_Y}\,(\text{D}\equiv)}{\flat Y \vdash \sim\Upsilon_Y}\,(\sim\text{R})}{\flat Y \vdash \Psi_{\flat Y}}\,(=)
\qquad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{(I.H.)}\\ \vdots \\ Y \vdash \Psi_Y\end{array}}{\flat\Psi_Y \vdash \flat Y}\,(\text{D}\equiv)}{\sim\Psi_Y \vdash \flat Y}\,(\sim\text{L})}{\Upsilon_{\flat Y} \vdash \flat Y}\,(=)$$

Stage 3. By the lemma assumption and the definitions of $\ulcorner - \urcorner$ and $\llcorner - \lrcorner$ we have that $\ulcorner \Psi_X \urcorner \vdash \ulcorner \Upsilon_Y \urcorner$ is $\mathrm{DL_{BI}}$-provable. Noting that $\ulcorner \Psi_X \urcorner$ and $\ulcorner \Upsilon_Y \urcorner$ are CBI-formulas, we can then construct a $\mathrm{DL_{BI}}$-proof of $X \vdash Y$ as follows:

$$
\dfrac{
  \dfrac{
    \begin{array}{c}(\text{Stage 2})\\ \vdots \\ X \vdash \Psi_X\end{array}
    \qquad
    \begin{array}{c}(\text{Stage 1})\\ \vdots \\ \Psi_X \vdash \llcorner\!\ulcorner \Psi_X \urcorner\!\lrcorner\end{array}
  }{X \vdash \llcorner\!\ulcorner \Psi_X \urcorner\!\lrcorner}\;(\text{Cut})
  \qquad
  \dfrac{
    \begin{array}{c}(\text{assumption})\\ \vdots \\ \llcorner\!\ulcorner \Psi_X \urcorner\!\lrcorner \vdash \llcorner\!\ulcorner \Upsilon_Y \urcorner\!\lrcorner\end{array}
    \qquad
    \dfrac{
      \begin{array}{c}(\text{Stage 1})\\ \vdots \\ \llcorner\!\ulcorner \Upsilon_Y \urcorner\!\lrcorner \vdash \Upsilon_Y\end{array}
      \qquad
      \begin{array}{c}(\text{Stage 2})\\ \vdots \\ \Upsilon_Y \vdash Y\end{array}
    }{\llcorner\!\ulcorner \Upsilon_Y \urcorner\!\lrcorner \vdash Y}\;(\text{Cut})
  }{\llcorner\!\ulcorner \Psi_X \urcorner\!\lrcorner \vdash Y}\;(\text{Cut})
}{X \vdash Y}\;(\text{Cut})
$$

This completes the proof. □