p-Automata: Acceptors for Markov Chains

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Abstract

We present p-automata, which accept an entire Markov chain as input. Acceptance is determined by solving a sequence of stochastic weak and weak games. The set of languages of Markov chains obtained in this way is closed under Boolean operations. Language emptiness and containment are equi-solvable, and languages themselves are closed under bisimulation. A Markov chain (respectively, PCTL formula) determines a p-automaton whose language is the bisimulation equivalence class of that Markov chain (respectively, the set of models of that formula). We define a simulation game between p-automata, decidable in EXPTIME. Simulation under-approximates language containment, whose decidability status is presently unknown.

1 Introduction

We introduce a novel type of probabilistic automaton, motivated by the notion of alternating tree-automaton. A p-automaton reads in a Markov chain and either accepts or rejects it. This is in striking contrast to Rabin's classical probabilistic automata [14], which assign a probability of acceptance with each input (a word or tree).

Realizing such a framework faces some challenges that we managed to overcome in the work reported here. First, transitions of alternating tree-automata are Boolean formulas whose truth values are therefore 0 or 1. In p-automata, we need to be prepared to assign any value in $[0, 1]$ to such transitions. Second, deciding whether p-automaton A accepts Markov chain M is phrased in terms of a sequence of (stochastic) weak games. This requires a notion of *uniform weak* p-automaton in which two probabilistic quantifiers exist and interact in a desired manner: one quantification tallies probabilities of immediate next locations (reminiscent of the X operator in PCTL [6]); another quantification measures the probabilities of regular path sets. Third, these probabilistic quantifications treat probabilities of regular path sets as *continuous resources*. We reconsile this view of resources with an automata-theoretic treatment as follows: Transitions as positive Boolean formulas have an extended base set that combines states q with threshold obligations such as $[\![q]\!]_{>0.6}$ saying that the path set represented by q has probability greater than 0.6. We also use a $*$ operator so that, e.g., $*([\![q_1]\!]_{\geq p_1}, [\![q_2]\!]_{\geq p_2})$ specifies that the path set determined by state q_i has probability at least p_i for $i = 1, 2$; and that the sets measured by these probabilities are *disjoint*. As in the case of separation logic [13], our separation operator $*$ has no efficient translation into propositional connectives. The semantics of $*([\![q_1]\!]_{\geq p_1}, [\![q_2]\!]_{\geq p_2})$, e.g., is different from that of $[\![q_1]\!]_{\geq p_1} \wedge [\![q_2]\!]_{\geq p_2} \wedge [\![q_1 \vee q_2]\!]_{\geq p_1+p_2}$.

Motivation of work: One motivation for this work is the quest for an abstraction framework for PCTL and Markov chains that is complete in the sense of Dams & Namjoshi [3], who formulated this notion for the modal mu-calculus and Kripke structures. Completeness, in our setting, asks whether the truth of PCTL formula ϕ in infinite-state Markov chain M can be witnessed by a finite-state abstraction A of M. Dams & Namjoshi suggest to use variants of alternating tree-automata, which accept Kripke structures, as abstractions. It is therefore natural to seek a notion of alternating tree-automaton that accepts Markov chains and realizes the above completeness property.

A second motivation for this work is to create foundations for automata-based probabilistic model checking.

Contributions of paper: We define a notion of automaton that can accept an entire countable labeled Markov chain. Acceptance of such input is shown to reduce to the solution of a sequence of (stochastic) weak games, making this decidable but exponential in the size of the automaton and in the size of the input. We show that our automata are closed under Boolean operations and that languages of p-automata are closed under bisimulation. We show that Markov chains (respectively, PCTL formulas) can be translated into p-automata such that these automata have as language the bisimulation class of that Markov chain (respectively, the set of models of that PCTL formula). We then define simulation between such p-automata, decidable in EXPTIME, and show that it soundly under-approximates language containment. Finally, p-automata are a complete abstraction framework for PCTL: any infinite Markov chain satisfying a PCTL formula has a finite p-automaton that abstracts that Markov chain and whose language is contained in that of the p-automata for that PCTL formula.

Outline of paper: In Section 2 we provide background on Markov chains, PCTL, and (stochastic) weak games. Our p-automata are introduced in Section 3, their acceptance games are defined in Section 4, and our expressiveness results feature in Section 5. Simulation and its salient properties are presented in Section 6. Related and future work is discussed in Section 7. The paper concludes in Section 8. An appendix contains all proofs.

2 Background

We briefly review required background for the technical developments in this paper.

Markov Chains and PCTL. A *countable labeled Markov chain* M over a set of atomic propositions $\mathbb{AP}$ is a tuple (S, P, L, s^{in}), where S is a countable set of *locations*, $P\colon S \times S \to [0,1]$ a stochastic matrix such that $\sum_{s' \in S} P(s,s') = 1$ for all $s \in S$, location $s^{\text{in}} \in S$ is the designated *initial* one, and $L\colon S \to 2^{\mathbb{AP}}$ is a *labeling function* where $L(s)$ is the set of propositions that hold in location s. Intuitively, $P(s,s')$ is the probability that M, when in state s, transitions to state s' in one discrete time step. We write $\mathsf{succ}(s)$ for the set $\{s' \in S \mid P(s,s') > 0\}$ of *successors* of s. We say M is *finitely branching* iff for all $s \in S$, set $\mathsf{succ}(s)$ is finite. In this paper, all Markov chains are assumed to be finitely branching. We write $\mathsf{MC}_{\mathbb{AP}}$ for the set of all (finitely branching) Markov chains over $\mathbb{AP}$. A *path* π from location s in M is an infinite sequence of locations $s_0 s_1 \ldots$ with $s_0 = s$ and $P(s_i, s_{i+1}) > 0$ for all $i \geq 0$. For $Y \subseteq S$, let $P(s,Y)$ abbreviate $\sum_{s' \in Y} P(s,s')$.

For Markov chain $M = (S, P, L, s^{\text{in}})$, a *bisimulation* [12] is an equivalence relation $H \subseteq S \times S$ such that $(s,s') \in H$ implies (i) $L(s) = L(s')$ and (ii) $P(s,C) = P(s',C)$ for all equivalence classes $C \in S/H$. Bisimulations identify locations that agree on their labeling, and on the probability of transitions to equivalence classes. The union of all bisimulations for M is the greatest bisimulation $\sim$; locations s and s' are called *bisimilar* iff $s \sim s'$. This definition extends to distinct Markov chains by considering their disjoint union. In particular, two Markov chains M_1 and M_2 are bisimilar if their initial locations s_1^{in} and s_2^{in} are bisimilar.

Without loss of generality [4], one may define the probabilistic temporal logic PCTL [6] in "Greater Than Negation Normal Form": only propositions can be negated and probabilistic bounds are either $\geq$ or $>$ – see Fig. 1. The value $k = \infty$ expresses unbounded Untils, whereas $k \in \mathbb{N}$ expresses bounded Untils. We write $\phi \,\mathsf{U}\, \psi$ as shorthand for $\phi \,\mathsf{U}^{\leq\infty} \psi$, $\phi \,\mathsf{W}\, \psi$ abbreviates $\phi \,\mathsf{W}^{\leq\infty} \psi$. The usual semantics of PCTL formulas is given in Fig. 2. Path formulas α are interpreted as predicates over paths in M, and wrap PCTL formulas into "LTL" operators for Next, (strong) Until, and Weak Until. The semantics $\|\phi\|$ of PCTL formula ϕ is a subset of S, where $\mathsf{Prob}_M(s,\alpha)$ is the probability of the measurable set $\mathsf{Path}(s,\alpha)$ of paths $s_0 s_1 \ldots$ in M with $s_0 = s$ and $s_0 s_1 \ldots \models \alpha$. The construction of the measure space of path sets from cylinder path sets is standard [10]. We say that M models ϕ, denoted $M \models \phi$ if $s^{\text{in}} \in \|\phi\|$.

ϕ, ψ ::=	*PCTL formulas*	α ::=	*Path formulas*
$\mathsf{a}, \neg\mathsf{a}$	Atom	$\mathsf{X}\,\phi$	Next
$\phi \wedge \psi$	Conjunction	$\phi \,\mathsf{U}^{\leq k} \psi$	Until
$\phi \vee \psi$	Disjunction	$\phi \,\mathsf{W}^{\leq k} \psi$	Weak Until
$[\alpha]_{\bowtie p}$	Path Probability		

Figure 1. Syntax of PCTL, where $\mathsf{a} \in \mathbb{AP}$, $k \in \mathbb{N} \cup \{\infty\}$, $p \in [0,1]$, **and** $\bowtie\ \in \{>, \geq\}$

$\|\mathsf{a}\| = \{s \in S \mid \mathsf{a} \in L(s)\}$ $\quad\|\neg\mathsf{a}\| = \{s \in S \mid \mathsf{a} \notin L(s)\}$
$\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$ $\quad\|\phi \vee \psi\| = \|\phi\| \cup \|\psi\|$
$\|[\alpha]_{\bowtie p}\| = \{s \in S \mid \mathsf{Prob}_M(s,\alpha) \bowtie p\}$

- $s_0 s_1 \ldots \models \mathsf{X}\,\phi$ iff $s_1 \in \|\phi\|_M$
- $s_0 s_1 \ldots \models \phi \,\mathsf{U}^{\leq k} \psi$ iff there is $l \in \mathbb{N}$ such that $l \leq k$, $s_l \in \|\psi\|_M$ and for all $0 \leq j < l$ we have $s_j \in \|\phi\|_M$
- $s_0 s_1 \ldots \models \phi \,\mathsf{W}^{\leq k} \psi$ iff for all $l \in \mathbb{N}$ such that $0 \leq l \leq k$, either $s_l \in \|\phi\|_M$ or there is $0 \leq j \leq l$ with $s_j \in \|\psi\|_M$

Figure 2. PCTL semantics for $M = (S, P, L, s^{\text{in}})$

Weak Games. A tuple $G = ((V,E),(V_0,V_1,V_p),\kappa,\alpha)$ is a *stochastic weak game* if (V,E) is a directed graph, (V_0,V_1,V_p) a partition of V, function κ associates with every $v \in V_p$ a distribution $\kappa(v)$ of mass 1 over $E(v) = \{v' \mid (v,v') \in E\}$ such that $(v,v') \in E$ iff $\kappa(v)(v') \neq 0$; we write $\kappa(v,v')$ instead of $\kappa(v)(v')$. Set $\alpha \subseteq V$ is the winning condition. Set V_0 contains the Player 0 configurations, V_1 the Player 1 configurations, and V_p the probabilistic configurations of G. We work with *weak* games, i. e. for every maximal, strongly connected component (SCC) $V' \subseteq V$ in (V,E) either $V' \subseteq \alpha$ or $V' \cap \alpha = \{\}$. If $V_p = \{\}$, we call G simply a *weak game*. Markov chains can be thought of as stochastic weak games where $V_0 = V_1 = \{\}$ and $\alpha = V$.

A play in G is a maximal sequence $v_0 v_1 \ldots$ of nodes with $(v_i, v_{i+1}) \in E$ for all $i \in \mathbb{N}$. A play is winning for Player 0 if it is finite and ends in a Player 1 configuration, or if it is infinite and ends in a suffix of states in α. Otherwise, that play is winning for Player 1. A (pure memoryless) strategy for Player 0 is a function $\sigma\colon V_0 \to V$ with $(v,\sigma(v)) \in E$ for all $v \in V_0$. Play $v_0 v_1 \ldots$ is consistent with strategy σ if $v_{i+1} = \sigma(v_i)$ whenever $v_i \in V_0$. Strategies for Player 1 are defined analogously. Let Σ (resp. Π) be the set of all strategies for Player 0 (resp. Player 1).

Each $(\sigma,\pi) \in \Sigma \times \Pi$ from game G determines a Markov chain $M^{\sigma,\pi}$ (with sinks for dead-ends in G) whose paths are plays in G consistent with σ and π. The set of plays from $v \in V$ that Player 0 wins is measurable in $M^{\sigma,\pi}$. Let $\mathsf{val}_0^{\sigma,\pi}(v)$ be that measure, and $\mathsf{val}_1^{\sigma,\pi}(v) = 1 - \mathsf{val}_0^{\sigma,\pi}(v)$. Then $\mathsf{val}_0(v) = \sup_{\sigma\in\Sigma} \inf_{\pi\in\Pi} \mathsf{val}_0^{\sigma,\pi}(v) \in [0,1]$ and $\mathsf{val}_1(v) = \sup_{\pi\in\Pi} \inf_{\sigma\in\Sigma} \mathsf{val}_1^{\sigma,\pi}(v) \in [0,1]$ are the game values. Strategies that achieve these values are *optimal*.

$$
\begin{array}{ll}
[\![Q]\!]_{>} & = \{[\![q]\!]_{\bowtie p} \mid q \in Q, \bowtie \in \{\geq, >\}, p \in [0,1]\} \\
[\![Q]\!]^{*} & = \{*(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i\colon t_i \in [\![Q]\!]_{>}\} \\
[\![Q]\!]^{\curlyvee} & = \{\curlyvee(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i\colon t_i \in [\![Q]\!]_{>}\} \\
[\![Q]\!] & = [\![Q]\!]^{*} \cup [\![Q]\!]^{\curlyvee}
\end{array}
$$

Figure 3. Derived term sets for set Q

Theorem 1 [2, 15] *Let $G = ((V, \cdot), \ldots)$ be a stochastic weak game and $v \in V$. Then $\mathsf{val}_0(v) + \mathsf{val}_1(v) = 1$. If G is finite, $\mathsf{val}_0(v)$ is computable in NP∩co-NP, and optimal strategies exist for both players. If G is a weak game, $\mathsf{val}_0(v)$ is in $\{0, 1\}$ and linear-time computable.*

One can generalize these results to the setting in which some configurations have pre-seeded game values (in $[0, 1]$ for stochastic weak games, and in $\{0, 1\}$ for weak games).

3 Uniform Weak p-Automata

We introduce p-automata and their uniform weak variant. Traditional probabilistic automata [14] map an input to a probability of accepting it. Such an automaton A then gives rise to probabilistic languages, e. g., the set of words accepted by A with probability at least 0.8. In contrast, our p-automata either accept or reject an entire Markov chain. So a p-automaton determines a language of Markov chains.

We assume familiarity with basic notions of trees and (alternating) tree automata. For set T, let $B^+(T)$ be the set of positive Boolean *formulas* generated from elements $t \in T$, constants ff and tt, and disjunctions and conjunctions:

$$\varphi ::= t \mid \mathsf{ff} \mid \mathsf{tt} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \qquad (1)$$

Formulas in $B^+(T)$ are finite even if T is not. For set Q, we define *term* sets in Fig. 3. This uses n-ary operators $*_n$ and $\curlyvee_n$ for every $n \in \mathbb{N}$, which we write as $*$ and $\curlyvee$ throughout as n will be clear from context. Intuitively, a state $q \in Q$ of a p-automaton and its transition structure model a probabilistic path set. So $[\![q]\!]_{\bowtie p}$ holds in location s if the measure of paths that begin in s and satisfy q is $\bowtie p$. Now, $*([\![q_1]\!]_{>p_1}, [\![q_2]\!]_{\geq p_2})$, e.g., means q_1 and q_2 hold with probability greater than p_1 and greater than or equal to p_2, respectively; and that the sets supplying these probabilities are disjoint. Dually, $\curlyvee([\![q_1]\!]_{\geq p_1}, [\![q_2]\!]_{\geq p_2})$ means that either (i) there is $i \in \{0, 1\}$ such that q_i holds with probability at least p_i or (ii) the intersection of q_1 and q_2 holds with probability at least $\max(p_1 + p_2 - 1, 0)$. So $*$ and $\curlyvee$ model a "disjoint and" and "intersecting or" operator, respectively. We may write $[\![q]\!]_{\bowtie p}$ for $*([\![q]\!]_{\bowtie p})$, and similarly for $\curlyvee$.

Given $\varphi \in B^+(Q \cup [\![Q]\!])$, its closure $\mathsf{cl_p}(\varphi)$ is the set of all subformulas of φ according to (1). In particular, $*(t_1, t_2) \in \mathsf{cl_p}(\varphi)$ does not imply $t_1, t_2 \in \mathsf{cl_p}(\varphi)$. For a set Φ of formulas, let $\mathsf{cl_p}(\Phi) = \bigcup_{\varphi \in \Phi} \mathsf{cl_p}(\varphi)$.

Definition 1 *A p-automaton A is a tuple $\langle \Sigma, Q, \delta, \varphi^{\mathsf{in}}, \alpha \rangle$, where Σ is a finite input alphabet, Q is a set of states (not necessarily finite), $\delta\colon Q \times \Sigma \to B^+(Q \cup [\![Q]\!])$ the transition function, $\varphi^{\mathsf{in}} \in B^+(Q \cup [\![Q]\!])$ the initial condition, and $\alpha \subseteq Q$ an acceptance condition.*

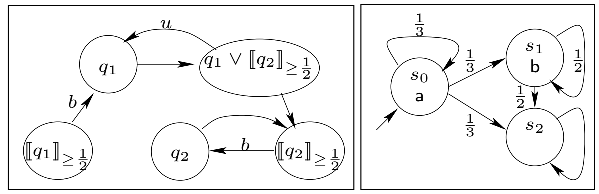

Figure 4. (a) Graph G_A of automaton A from Example 1 and (b) a Markov chain M

Throughout, automata have *states*, Markov chains have *locations*, and games have *configurations*.

Example 1 *Let $A = \langle 2^{\{\mathsf{a},\mathsf{b}\}}, \{q_1, q_2\}, \delta, [\![q_1]\!]_{\geq 0.5}, \{q_2\} \rangle$ be a p-automaton where δ is defined by*

$\delta(q_1, \{\mathsf{a}, \mathsf{b}\}) = \delta(q_1, \{\mathsf{a}\}) = q_1 \vee [\![q_2]\!]_{\geq 0.5}$
$\delta(q_2, \{\mathsf{b}\}) = \delta(q_2, \{\mathsf{a}, \mathsf{b}\}) = [\![q_2]\!]_{\geq 0.5}$
$\delta(q_1, \{\}) = \delta(q_1, \{\mathsf{b}\}) = \delta(q_2, \{\}) = \delta(q_2, \{\mathsf{a}\}) = \mathsf{ff}$

State q_2 satisfies the recursive property ϕ, that b *holds at the location presently read by q_2, and that ϕ will hold with probability at least 0.5 in the next locations. State q_1 asserts that it is possible to get to a location that satisfies q_2 along a path that satisfies* a*. The initial condition $[\![q_1]\!]_{\geq 0.5}$ means the set of paths satisfying* $\mathsf{a}\ \mathsf{U}\ \phi$ *has probability at least 0.5.*

A p-automaton $A = \langle \Sigma, Q, \delta, \ldots \rangle$ determines a labeled, directed graph $G_A = \langle Q', E, E_b, E_u \rangle$:

$$
\begin{array}{l}
Q' = Q \cup \mathsf{cl_p}(\delta(Q, \Sigma)) \\
E = \{(\varphi_1 \wedge \varphi_2, \varphi_i), (\varphi_1 \vee \varphi_2, \varphi_i) \mid \varphi_i \in Q' \setminus Q\} \\
\quad \cup \{q, \delta(q, \sigma) \mid q \in Q, \sigma \in \Sigma\} \\
E_u = \{(\varphi \wedge q, q), (q \wedge \varphi, q), (\varphi \vee q, q), (q \vee \varphi, q) \mid \varphi \in Q', q \in Q\} \\
E_b = \{(\varphi, q) \mid \varphi \in [\![Q]\!] \text{ and } q \in gs(\varphi)\}
\end{array}
$$

where $gs(\varphi)$ is the set of *guarded* states of φ: all $q' \in Q$ occurring in some term in φ. Elements $(\varphi, q) \in E_u$ are *unbounded* transitions. Elements $(\varphi, q') \in E_b$ are *bounded* transitions. And elements of E are called *simple* transitions. We mark $(\varphi, q) \in E_b$ with $*$ (and respectively, with a $\curlyvee$) if there is some $p \in [0, 1]$ for which $[\![q']\!]_{\bowtie p}$ occurs in φ within the scope of a $*$ (respectively, $\curlyvee$) operator. Note that E, E_u, and E_b are pairwise disjoint.

Let $\varphi \preceq_A \tilde{\varphi}$ iff there is a finite path from φ to $\tilde{\varphi}$ in $E \cup E_b \cup E_u$. For $\varphi \in Q \cup \mathsf{cl_p}(\delta(Q, \Sigma))$, let $(\!(\varphi)\!)$ denote the equivalence class of φ according to preorder $\preceq_A$.

Definition 2 *A p-automaton A is called* uniform *if:*

- *For each cycle in G_A, its set of transitions is either in $E \cup E_b$ or in $E \cup E_u$.*
- *For each cycle in $\langle Q, E \cup E_b \rangle$, its set of markings is either $\{\}$, $\{*\}$ or $\{\curlyvee\}$, and so cannot be $\{*, \curlyvee\}$.*

• *Preorder $\preceq_A$ induces finitely many equivalence classes. A (not necessarily uniform) p-automaton A is called* weak *if for all $q \in Q$, either $(\!(q)\!) \cap Q \subseteq \alpha$ or $(\!(q)\!) \cap \alpha = \{\}$.*

Then, A is uniform, if the full subgraph of every equivalence class in $\preceq_A$ contains only one type of non-simple transitions and at most one kind of marking $*$ or $\text{\Vtop}$. Also, all states $q' \in Q$ or formulas φ occurring in $\delta(q, \sigma)$ for some $q \in Q$ and $\sigma \in \Sigma$ can be classified as unbounded, bounded with $*$, bounded with $\text{\Vtop}$, or simple, according to SCC $(\!(q)\!)$.

Example 2 *Figure 4*(a) *depicts G_A for A of Example 1. p-Automaton A is uniform: $(\!(q_1)\!) = \{q_1, q_1 \vee [\![q_2]\!]_{\geq 0.5}\}$ and $(\!(q_2)\!) = \{q_2, [\![q_2]\!]_{\geq 0.5}\}$; in $(\!(q_1)\!)$ there are no bounded edges, in $(\!(q_2)\!)$ there are no unbounded edges; and G_A has no markings for $*$ or $\text{\Vtop}$. The SCC $(\!([\![q_1]\!]_{\geq 0.5})\!) = \{[\![q_1]\!]_{\geq 0.5}\}$ is trivial. In addition, A is weak as $\alpha = \{q_2\}$.*

Intuitively, cycles in the structure of a uniform p-automaton A take either no bounded edges or no unbounded edges, and cycles that take bounded edges don't have both markings $*$ and $\text{\Vtop}$. Subsequently, all p-automata are uniform weak unless mentioned otherwise. Uniformity makes acceptance of p-automata well defined. But, a more relaxed notion of uniformity is what really drives the proof of well-definedness: that any chain in the partial order on SCCs, defined below, has only finitely many alternations between bounded and unbounded SCCs. The requirement of weakness, though, is made merely to simplify the presentation.

4 Acceptance Games

For any $\mathbb{AP}$, p-automata $A = \langle 2^{\mathbb{AP}}, Q, \delta, \varphi^{\text{in}}, \alpha\rangle$ have $\mathsf{MC}_{\mathbb{AP}}$ as set of inputs. For $M = (S, P, L, s^{\text{in}}) \in \mathsf{MC}_{\mathbb{AP}}$, we exploit the uniform structure of A to reduce the decision of whether A accepts M to solving a sequence of weak games and stochastic weak games. The weak acceptance of A implies that these games are weak. Then the language of A is

$$\mathcal{L}(A) = \{M \in \mathsf{MC}_{\mathbb{AP}} \mid A \text{ accepts } M\}$$

For A as above, let $T = Q \cup \mathsf{cl_p}(\delta(Q, 2^{\mathbb{AP}}))$. Finite partial order $(T/\!\equiv, \leq_A)$ has set $\{(\!(t)\!) \mid t \in T\}$ ordered by $(\!(t)\!) \leq_A (\!(\tilde{t})\!)$ iff $\tilde{t} \preceq_A t$. For M as above, each non-trivial $(\!(t)\!)$ (i.e., there is a cycle in the subgraph of $(\!(t)\!)$ in G_A) determines a game

$$G_{M,(\!(t)\!)} = ((V, E), (V_0, V_1, V_p), \kappa, \tilde{\alpha})$$

Most of its configurations are in $S \times T$. The construction is such that $(s^{\text{in}}, \varphi^{\text{in}})$ occurs as configuration in exactly one of these games $G_{M,(\!(t)\!)}$, and $\mathsf{val}(s^{\text{in}}, \varphi^{\text{in}}) \in [0, 1]$. Then

$$A \text{ accepts } M \text{ iff } \mathsf{val}(s^{\text{in}}, \varphi^{\text{in}}) = 1$$

We define these games dependent on the type of $(\!(t)\!)$ and dependent on game values already computed for games of SCCs higher up (i.e. by induction): all games $G_{M,(\!(\tilde{t})\!)}$ with $(\!(t)\!) \leq_A (\!(\tilde{t})\!)$ have values $\mathsf{val}(s, t)$ already computed for all s, which we use as pre-seeded values in $G_{M,(\!(t)\!)}$. Below, we write $\mathsf{val}(s, \varphi) = \bot$ for configurations (s, φ) in $G_{M,(\!(t)\!)}$ whose game value has not yet been computed.

Case 1: For non-trivial SCC $(\!(t)\!)$ whose transitions are in $E \cup E_u$, game $G_{M,(\!(t)\!)}$ is a stochastic weak game with

$$\begin{array}{ll} V = \{(s, \tilde{t}) \mid s \in S \text{ and } \tilde{t} \preceq_A t\} & V_0 = \{(s, \varphi_1 \vee \varphi_2) \in V\} \\ V_1 = \{(s, \varphi_1 \wedge \varphi_2) \in V\} & V_p = (S \times Q) \cap V \\ \kappa(s, q)(s', \delta(q, L(s))) = P(s, s') & \tilde{\alpha} = \{\} \text{ or } V \end{array}$$

$$E = \begin{array}{l} \{((s, \varphi_1 \wedge \varphi_2), (s, \varphi_i)) \in V \times V \mid 1 \leq i \leq 2\} \cup \\ \{((s, \varphi_1 \vee \varphi_2), (s, \varphi_i)) \in V \times V \mid 1 \leq i \leq 2\} \cup \\ \{((s, q), (s', \delta(q, L(s)))) \in V \times V \mid P(s, s') > 0\} \end{array}$$

where $\tilde{\alpha}$ equals V iff some state (equally, all states) q in $(\!(t)\!)$ is in α.

By Theorem 1, for every configuration $c \in V$ we have $\mathsf{val}_0(c) \in [0, 1]$. We set $\mathsf{val}(c) = \mathsf{val}_0(c)$.

Case 2: Let $(\!(t)\!)$ be a non-trivial SCC none of whose transitions are in E_u and none have $\text{\Vtop}$ markings. For each formula $\varphi \in (\!(t)\!) \cap [\![Q]\!]^*$ of form $*([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ we define, for each $s \in S$, sets $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$. Then

$$\begin{array}{ll} V_i = \bigcup_{s,\varphi} V_i^{s,\varphi} & E = \bigcup_{s,\varphi} E^{s,\varphi} \\ V_p = \{\} & \tilde{\alpha} = \{\} \text{ or } V \end{array}$$

where $\tilde{\alpha}$ is V iff some $q \in (\!(t)\!)$ is in α, defines the weak game $G_{M,(\!(t)\!)}$. It remains to define $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$, for which we make use of values $\mathsf{val}(s, \tilde{t})$ already defined for all $s \in S$ and all $\tilde{t} \notin (\!(t)\!)$ with $(\!(t)\!) \preceq_A (\!(\tilde{t})\!)$.

As $\mathsf{succ}(s)$ and $\delta(q_i, L(s))$ are finite, so are

$$R_{s,\varphi} = \bigcup_{i=1}^{n} \{(s', \varphi') \mid s' \in \mathsf{succ}(s), \varphi' \in \mathsf{cl_p}(\delta(q_i, L(s)))\}$$

$$\mathsf{Val}_{s,\varphi} = \{0, 1, \mathsf{val}(s', \varphi') \mid (s', \varphi') \in R_{s,\varphi}, \mathsf{val}(s', \varphi') \neq \bot\}$$

For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$. Throughout, let $X \to Y$ be the set of total functions from set X to set Y. Let $\mathcal{F}_{s,\varphi}$ be $[n] \times \mathsf{succ}(s) \to \mathsf{Val}_{s,\varphi}$, the set of functions from pairs consisting of 'sub-stars' of φ and successors of s to values in $\mathsf{Val}_{s,\varphi}$. Also, any $f \in \mathcal{F}_{s,\varphi}$ is *disjoint* if there are $\{a_{i,s'} \in [0, 1] \mid i \in [n] \text{ and } s' \in \mathsf{succ}(s)\}$ such that (i) $\Sigma_{s' \in \mathsf{succ}(s)} a_{i,s'} f(i, s') P(s, s') \bowtie_i p_i$ for all $i \in [n]$ and (ii) $\Sigma_{i \in [n]} a_{i,s'} = 1$ for all $s' \in \mathsf{succ}(s)$. Intuitively, a function $f \in \mathcal{F}_{s,\varphi}$ associates with $q_1, \ldots, q_n$ and s' the value that Player 0 can achieve from configuration $(s', \delta(q_i, L(s)))$. We call f "disjoint", as all the requirements from the different q_i's can be achieved using a partition of the probability of all successors. We denote by $\mathcal{F}^*_{s,\varphi}$ the set of disjoint functions. Sets $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$ are defined in Fig. 5.

By Theorem 1, V partitions into winning regions W_0 and W_1 of configurations for Player 0 and Player 1, respectively. We set $\mathsf{val}(c) = 1$ for $c \in W_0$ and $\mathsf{val}(c) = 0$ for $c \in W_1$.

The intuition for this weak game is as follows: Configuration (s, φ) means that the transition of each q_i holds with probability $\bowtie_i p_i$ where the sets X_i measured by these probabilities are pairwise disjoint. In order to check

$V_0^{s,\varphi} =$	$\{(s,\varphi)\} \quad \cup \quad \{(s',\varphi',v) \mid s' \in \mathsf{succ}(s), \varphi' \in R_{s,\varphi}, \mathsf{val}(s',\varphi') \neq \bot, \text{ and } \mathsf{val}(s',\varphi') < v\}$	$\cup$
	$\{(s',\varphi_1 \vee \varphi_2, v) \mid s' \in \mathsf{succ}(s), \varphi_1 \vee \varphi_2 \in R_{s,\varphi}, \text{ and } \mathsf{val}(s',\varphi_1 \vee \varphi_2) = \bot\}$	
$V_1^{s,\varphi} =$	$\{(s,\varphi,f) \mid f \in \mathcal{F}^*_{s,\varphi}\} \quad \cup \quad \{(s',\varphi',v) \mid s' \in \mathsf{succ}(s), \varphi' \in R_{s,\varphi}, \mathsf{val}(s',\varphi') \neq \bot, \text{ and } \mathsf{val}(s',\varphi') \geq v\}$	$\cup$
	$\{(s',\varphi_1 \wedge \varphi_2, v) \mid s' \in \mathsf{succ}(s), \varphi_1 \wedge \varphi_2 \in R_{s,\varphi}, \text{ and } \mathsf{val}(s',\varphi_1 \wedge \varphi_2) = \bot\}$	
$E^{s,\varphi} =$	$\{((s,\varphi),(s,\varphi,f)) \mid f \in \mathcal{F}^*_{s,\varphi}\} \quad \cup \quad \{((s',\varphi',v),(s',\varphi')) \mid s' \in \mathsf{succ}(s),\ \varphi' \in [\![Q]\!] \text{ and } \mathsf{val}(s',\varphi') = \bot\}$	$\cup$
	$\{((s,\varphi,f),(s',\delta(q_i,L(s)),f(i,s'))) \mid s' \in \mathsf{succ}(s), i \in [n], \text{ and } f(i,s') > 0\}$	$\cup$
	$\{((s',\varphi_1 \vee \varphi_2,v),(s',\varphi_i,v)) \mid s' \in \mathsf{succ}(s), \varphi_1 \vee \varphi_2 \in R_{s,\varphi}, i \in \{1,2\}, \mathsf{val}(s',\varphi_1 \vee \varphi_2) = \bot\}$	$\cup$
	$\{((s',\varphi_1 \wedge \varphi_2,v),(s',\varphi_i,v)) \mid s' \in \mathsf{succ}(s), \varphi_1 \wedge \varphi_2 \in R_{s,\varphi}, i \in \{1,2\}, \mathsf{val}(s',\varphi_1 \wedge \varphi_2) = \bot\}$	

Figure 5. Components of the game $G_{M,(\!(t)\!)}$, where $(\!(t)\!)$ does not contain $\forall\!\!\!\!\ast$ transitions

that, in configuration (s,φ) Player 0 chooses some function $f \in \mathcal{F}^*_{s,\varphi}$ associating with location $s' \in \mathsf{succ}(s)$ and state q_i the value Player 0 can achieve playing from $(s',\delta(q_i,L(s)))$. The play proceeds by Player 1 choosing a successor s' of s and a state q_i and the play reaches configuration $(s',\delta(q_i,L(s)),f(i,s'))$. From such value-annotated configurations, Player 0 and Player 1 choose successors according to the usual resolution of $\vee$ and $\wedge$. In a configuration for which the value was already determined, then either Player 0 failed to achieve the value promised and loses immediately; or Player 0 succeeded to achieve the value promised and wins immediately. Otherwise, the play ends up in another configuration of the form (s',φ') for $\varphi' \in [\![Q]\!]^*$ and the play proceeds ignoring the value v (as obviously $v \leq 1$). If the play continues ad infinitum, the winner is determined according to acceptance condition $\tilde{\alpha}$.

Case 3: Let $(\!(t)\!)$ be a nontrivial SCC with no transitions in E_u and no $*$ markings. For formulas $\varphi \in (\!(t)\!) \cap [\![Q]\!]^{\forall\!\!\!\!\ast}$ of form $\forall\!\!\!\!\ast([\![q_1]\!]_{\bowtie_1 p_1},\ldots,[\![q_n]\!]_{\bowtie_n p_n})$ we reuse the definitions of $R_{s,\varphi}$, $\mathsf{Val}_{s,\varphi}$, and $\mathcal{F}_{s,\varphi}$. Weak game $G_{M,(\!(t)\!)}$ is defined as in Case 2. Sets $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$ are defined as in Fig. 5, except that functions f don't range over $\mathcal{F}^*_{s,\varphi}$ but now range over $\mathcal{F}^{\forall\!\!\!\!\ast}_{s,\varphi}$, the set of intersecting functions and the dual of $\mathcal{F}^*_{s,\varphi}$ of Case 2: function $f \in \mathcal{F}_{s,\varphi}$ is *intersecting* if for all $\{a_{i,s'} \in [0,1] \mid i \in [n] \text{ and } s' \in \mathsf{succ}(s)\}$ either (i) there is $i \in [n]$ with $\Sigma_{s' \in \mathsf{succ}(s)} a_{i,s'} f(i,s') P(s,s') \bowtie_i p_i$ or (ii) there is $s' \in \mathsf{succ}(s)$ with $\Sigma_{i \in [n]} a_{i,s'} \neq 1$.

As in Case 2, we say that wins for Player 0 have value 1, and wins for Player 1 have value 0.

The intuition for this weak game is verbatim that of the weak game in Case 2, except that Player 0 chooses a function f that is in $\mathcal{F}^{\forall\!\!\!\!\ast}_{s,\varphi}$ instead of in $\mathcal{F}^*_{s,\varphi}$.

We note that in handling $\varphi = [\![q_1]\!]_{\bowtie_1 p_1}$, i.e. when n above is 1, the definition of $*$ and $\forall\!\!\!\!\ast$ coincide. Indeed, there is exactly one option for choosing $\{a_{1,s'} \mid s' \in \mathsf{succ}(s)\}$ such that for all $s' \in \mathsf{succ}(s)$ we have $a_{1,s'} = 1$. This justifies dropping the $*$ or $\forall\!\!\!\!\ast$ when applied to one operand.

Case 4: If $(\!(t)\!)$ is a trivial SCC (i.e., contains no cycles), the games above collapse to cycle-free games with at most one alternation between Player 1 and Player 0 before arriving at configurations with pre-seeded value. Thus, $\mathsf{val}(s,t)$ can be computed directly from previously computed values, by a case analysis on $t \in B^+(Q \cup [\![Q]\!])$:

- If $t = q \in Q$, value $\mathsf{val}(s,t)$ is determined according a stochastic weak game as in Case 1: $\mathsf{val}(s,t)$ is set to $\Sigma_{s' \in S} P(s,s') \cdot \mathsf{val}(s',\delta(q,L(s)))$.
- Value $\mathsf{val}(s,*([\![q_1]\!]_{\bowtie_1 p_1},\ldots,[\![q_n]\!]_{\bowtie_n p_n}))$ is computed via a weak game as in Case 2: it is 1 if there is $\{a_{i,s'} \in [0,1] \mid i \in [n] \text{ and } s' \in \mathsf{succ}(s)\}$ with $\Sigma_{s' \in \mathsf{succ}(s)} a_{i,s'} P(s,s') \mathsf{val}(s',\delta(q_i,L(s))) \bowtie_i p_i$ for all i, and $\Sigma_{i \in [n]} a_{i,s'} = 1$ for all $s' \in \mathsf{succ}(s)$; it is 0 otherwise.
- Value $\mathsf{val}(s,\forall\!\!\!\!\ast([\![q_1]\!]_{\bowtie_1 p_1},\ldots,[\![q_n]\!]_{\bowtie_n p_n}))$ is set via a weak game as in Case 3: it is set to 1 if for all $\{a_{i,s'} \in [0,1] \mid i \in [n] \text{ and } s' \in \mathsf{succ}(s)\}$ either there is i with $\Sigma_{s \in \mathsf{succ}(s)} a_{i,s'} P(s,s') \mathsf{val}(s',\delta(q_i,L(s))) \bowtie_i p_i$ or $\Sigma_{i \in [n]} a_{i,s'} \neq 1$ for some $s' \in \mathsf{succ}(s)$; and to 0 otherwise.
- $\mathsf{val}(s,\mathsf{tt})$ and $\mathsf{val}(s,\mathsf{ff})$ are set to 1 and 0, respectively.
- Value $\mathsf{val}(s,\varphi_1 \wedge \varphi_2)$ is determined via a stochastic weak game as in Case 1: it is set to $\min_i \mathsf{val}(s',\varphi_i)$.
- Value $\mathsf{val}(s,\varphi_1 \vee \varphi_2)$ is determined via a stochastic weak game as in Case 1: it is set to $\max_i \mathsf{val}(s',\varphi_i)$.

Note that, when $n = 1$, there is no difference between $*$ and $\forall\!\!\!\!\ast$ in the second and third item above as then $\exists = \forall$.

Example 3 *We verify $M \in \mathcal{L}(A)$ for A from Example 1 and M from Fig. 4(b), where propositions that hold at location s are written within that location – e.g., $L(s_0) = \{\mathsf{a}\}$. The weak game of SCC $(\!(q_2)\!)$, shown in Fig. 6, has only accepting configurations. So Player 0 wins only $(s_1,[\![q_2]\!]_{\geq 0.5})$ and $(s_1,[\![q_2]\!]_{\geq 0.5},\{s_1 \to 1, s_2 \to 0\})$ and loses all other configurations. The stochastic weak game for SCC $(\!(q_1)\!)$, shown in Fig. 7, depicts stochastic configurations with a diamond and configurations from other SCCs are put into hexagons (with the hexagon labeled $(s_1,[\![q_2]\!]_{\geq 0.5})$ having value 1 and all others having value 0). As none of its configurations are accepting, Player 0 can only win by reaching optimal hexagons. Hexagon $(s_1,[\![q_2]\!]_{\geq 0.5})$ has value 1 and is the optimal choice for Player 0 from configuration $(s_1,q_1 \vee [\![q_2]\!]_{\geq 0.5})$. Player 0 configuration $(s_2,q_1 \vee [\![q_2]\!]_{\geq 0.5})$ has value 0. So the value for Player 0 of diamond configuration (s_0,q_1) is 0.5. Initial configuration $(s_0,[\![q_1]\!]_{\geq 0.5})$ makes up a trivial SCC (Case 4), so its value is set to 1 as $\frac{1}{3}\mathsf{val}(s_0,q_1 \vee [\![q_2]\!]_{\geq 0.5}) + \frac{1}{3}\mathsf{val}(s_1,q_1 \vee [\![q_2]\!]_{\geq 0.5}) + \frac{1}{3}\mathsf{val}(s_2,q_1 \vee [\![q_2]\!]_{\geq 0.5})$ is 0.5.*

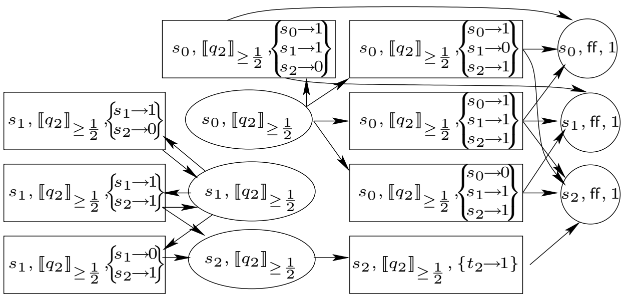

Figure 6. Case 3 of acceptance game

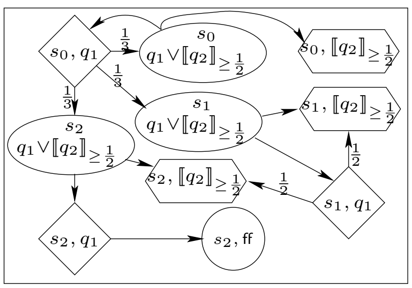

Figure 7. Case 1 of acceptance game

Theorem 2 *Given a p-automaton $A = \langle 2^{\mathbb{AP}}, \ldots \rangle$, its language $\mathcal{L}(A)$ is well defined. If A and $M \in \mathsf{MC}_{\mathbb{AP}}$ are finite, $M \in \mathcal{L}(A)$ can be decided in EXPTIME.*

For finite Markov chains M and p-automata A, checking acceptance $M \in \mathcal{L}(A)$ is exponential in the branching degree of M and $*$ operators of A, but not in the number of states or locations. Membership in EXPTIME for deciding $M \in \mathcal{L}(A)$ follows since the stochastic weak games for unbounded SCCs are in NP∩coNP, and since the weak games for bounded SCCs are solved in linear time but may have exponential size due to the number of possible *disjoint* and *intersecting* functions f.

5 Expressiveness of p-Automata

We now show (i) the languages of p-automata are closed under Boolean operations and bisimulation, and emptiness and containment of languages are equi-solvable; (ii) each Markov chain determines a p-automaton whose language is the bisimulation class of that Markov chain; and (iii) each PCTL formula determines a p-automaton whose language consists of all Markov chains satisfying that formula.

5.1 Closure of Languages

It is routine to see that p-automata are closed under union and intersection. They are also closed under complementation: Given a p-automaton $A = \langle \Sigma, Q, \delta, \varphi^{\mathsf{in}}, \alpha \rangle$, its dual is

$$\mathsf{dual}(A) = \langle \Sigma, \overline{Q}, \overline{\delta}, \mathsf{dual}(\varphi^{\mathsf{in}}), Q \setminus \alpha \rangle$$

with $\overline{Q} = \{\overline{q} \mid q \in Q\}$ and $\overline{\delta}(\overline{q}, \sigma) = \mathsf{dual}(\delta(q, \sigma))$, where $\mathsf{dual}(\varphi)$ is defined in Fig. 8. The structure of uniform weak

$\mathsf{dual}(\varphi_1 \overset{*}{\vee} \varphi_2)$	$=$	$\mathsf{dual}(\varphi_1) * \mathsf{dual}(\varphi_2)$
$\mathsf{dual}(\varphi_1 * \varphi_2)$	$=$	$\mathsf{dual}(\varphi_1) \overset{*}{\vee} \mathsf{dual}(\varphi_2)$
$\mathsf{dual}(\varphi_1 \wedge \varphi_2)$	$=$	$\mathsf{dual}(\varphi_1) \vee \mathsf{dual}(\varphi_2)$
$\mathsf{dual}(\varphi_1 \vee \varphi_2)$	$=$	$\mathsf{dual}(\varphi_1) \wedge \mathsf{dual}(\varphi_2)$
$\mathsf{dual}(q)$	$=$	$\overline{q}$
$\mathsf{dual}(\overline{q})$	$=$	q
$\mathsf{dual}(\llbracket q \rrbracket_{\bowtie p})$	$=$	$\llbracket \overline{q} \rrbracket_{\mathsf{dual}(\bowtie p)}$
$\mathsf{dual}(\geq p)$	$=$	$> 1 - p$
$\mathsf{dual}(> p)$	$=$	$\geq 1 - p$

Figure 8. Definition of $\mathsf{dual}(\varphi)$

p-automata makes sure that $\mathsf{dual}(A)$ is also uniform weak. We now show that A and $\mathsf{dual}(A)$ are complements.

Theorem 3 *Let A be a p-automata A with $\Sigma = 2^{\mathbb{AP}}$. Then $\mathcal{L}(A) = \mathsf{MC}_{\mathbb{AP}} \setminus \mathcal{L}(\mathsf{dual}(A))$.*

The key part of the proof for that theorem is to show that, for every state q of A and every location s of M, we have $\mathsf{val}(s, q) = 1 - \mathsf{val}(s, \overline{q})$ for the acceptance games.

Corollary 1 *Let $\Sigma = 2^{\mathbb{AP}}$.*

- *The set of languages accepted by p-automata with Σ is closed under Boolean operations.*
- *Language containment of p-automata with Σ reduces to language emptiness of such p-automata, and vice versa.*

Languages of p-automata are closed under bisimulation.

Lemma 1 *For p-automaton $A = \langle 2^{\mathbb{AP}}, Q, \delta, \varphi^{\mathsf{in}}, \alpha \rangle$ and $M_1, M_2 \in \mathsf{MC}_{\mathbb{AP}}$ with $M_1 \sim M_2$: $M_1 \in \mathcal{L}(A)$ iff $M_2 \in \mathcal{L}(A)$.*

To prove this, we use induction on the partial order on the SCCs in A to show that for all $t \in Q \cup \llbracket Q \rrbracket$ and for all locations s_1 in M_1 and locations s_2 in M_2 with $s_1 \sim s_2$ we have $\mathsf{val}(s_1, t) = \mathsf{val}(s_2, t)$.

5.2 Embedding of Markov Chains

A Markov chain $M = (S, P, L, s^{\mathsf{in}}) \in \mathsf{MC}_{\mathbb{AP}}$ can be converted into a p-automaton

$$A_M = \langle 2^{\mathbb{AP}}, Q, \delta, \varphi^{\mathsf{in}}, \alpha \rangle$$

whose language $\mathcal{L}(A_M)$ is the set of Markov chains bisimilar to M. For a linear order on each set $\mathsf{succ}(s')$, A_M is

$$\begin{array}{rcl}
Q & = & \{(s, s') \in S \times S \mid P(s, s') > 0\} \\
\delta((s, s'), L(s)) & = & *(\llbracket (s', s'') \rrbracket_{\geq P(s', s'')} \mid s'' \in \mathsf{succ}(s')) \\
\delta((s, s'), \sigma) & = & \mathsf{ff} \qquad\qquad \text{if } \sigma \neq L(s) \\
\varphi^{\mathsf{in}} & = & *(\llbracket (s^{\mathsf{in}}, s') \rrbracket_{\geq P(s^{\mathsf{in}}, s')} \mid P(s^{\mathsf{in}}, s) > 0) \\
\alpha & = & Q
\end{array}$$

State (s, s') represents the transition from s to s'. Labels are compared for location s. Location s' is used to require that there are successors of probability at least $P(s, s')$. This p-automaton A_M has only bounded transitions and uses only the $*$ operator. In particular, it is uniform weak.

Theorem 4 *For any Markov chain $M \in \mathsf{MC}_{\mathbb{AP}}$, $\mathcal{L}(A_M)$ is the bisimulation equivalence class of M.*

By Lemma 1, one half of Theorem 4 follows from a proof that A_M accepts M. To show this, it suffices to demonstrate that Player 0 can infinitely often reach configurations of form $(s, *([\![(s,s')]\!]_{\geq P(s,s')} \mid s' \in \mathsf{succ}(s)))$ for all locations s in M. For the other half, we use proof by contradiction: given M' with initial state t^{in} such that $M' \not\sim M$, we appeal to the partition refinement algorithm to get a coarsest partition that witnesses $s^{\text{in}} \not\sim t^{\text{in}}$. That witnessing information can then be transformed into a winning strategy for Player 1 in the acceptance game for deciding $M' \in \mathcal{L}(A_M)$, and so $M' \notin \mathcal{L}(A_M)$ follows.

The construction of A_M is the only reason why we allow p-automata with infinite state sets. Finite sets Q suffice for finite Markov chains. The conjunctive operator $*$ used in the construction of A_M effectively hides an exponential blowup. If a Markov chain is deterministic (all successors of any location disagree on their labelings), we can eliminate the use of $*$ in A_M and still secure Theorem 4. But this embedding without $*$ does break Theorem 4 for non-deterministic Markov chains (as shown in Appendix B).

5.3 Embedding of PCTL Formulas

Each PCTL formula ϕ over $\mathbb{AP}$ yields a p-automaton.

$$A_\phi = \langle 2^{\mathbb{AP}}, \mathsf{cl_t}(\phi) \cup \mathbb{AP}, \rho_x, \rho_\epsilon(\phi), F\rangle$$

that accepts exactly the Markov chains satisfying ϕ. The construction resembles the translation from CTL to alternating tree automata:

- $\mathsf{cl_t}(\phi)$ denotes the set of temporal subformulas of ϕ
- F consists of $\mathbb{AP}$ and all ψ of $\mathsf{cl_t}(\phi)$ *not* of form $\psi_1 \,\mathsf{U}\, \psi_2$
- transition function ρ_x and auxiliary function ρ_ϵ are defined in Fig. 9.

We have $\psi \in \mathsf{cl_t}(\phi)$ for subformulas $[\psi]_{\bowtie p}$ of ϕ. Also, $[\psi_1 \,\mathsf{U}\, \psi_2]_{\bowtie p}$ may be an element in $\mathsf{cl_t}(\phi)$ whereas $[\![\psi_1 \,\mathsf{U}\, \psi_2]\!]_{\bowtie p}$ can only be an element of $[\![\mathsf{cl_t}(\phi)]\!]_{>}$, it wraps $\psi_1 \,\mathsf{U}\, \psi_2$ in the probabilistic quantification $[\![\cdot]\!]_{\bowtie p}$ of A_ϕ.

Theorem 5 *For all PCTL formulas ϕ over $\mathbb{AP}$ and all $M \in \mathsf{MC}_{\mathbb{AP}}$, we have $M \models \phi$ iff $M \in \mathcal{L}(A_\phi)$*

The proof of Theorem 5 uses structural induction on PCTL formulas (i.e., state formulas) to show that for all locations s in M and all PCTL subformulas φ' of PCTL formula φ we have $s \in \|\varphi'\|$ iff $\mathsf{val}(s, \rho_\epsilon(\varphi')) = 1$ for configuration $(s, \rho_\epsilon(\varphi'))$ in the acceptance game for deciding $M \in \mathcal{L}(A_\varphi)$.

Example 4 *For $\varphi = [\mathsf{a} \,\mathsf{U}\, [\mathsf{X}\,\mathsf{b}]_{>0.5}]_{\geq 0.3}$ we have $A_\varphi = \langle 2^{\{\mathsf{a},\mathsf{b}\}}, \mathsf{cl_t}(\varphi) \cup \{\mathsf{a},\mathsf{b}\}, \rho_x, \rho_\epsilon(\varphi), F\rangle$, where $\mathsf{cl_t}(\varphi) = \{\mathsf{a} \,\mathsf{U}\, [\mathsf{X}\,\mathsf{b}]_{>0.5}, \mathsf{X}\,\mathsf{b}\}$, $\rho_\epsilon(\varphi) = (\mathsf{a} \wedge [\![\mathsf{a} \,\mathsf{U}\, [\mathsf{X}\,\mathsf{b}]_{>0.5}]\!]_{\geq 0.3}) \vee [\![\mathsf{X}\,\mathsf{b}]\!]_{>0.5}$, $F = \{\mathsf{X}\,\mathsf{b}, \mathsf{a}, \mathsf{b}\}$, and $\rho_x(\mathsf{X}\,\mathsf{b}) = \mathsf{b}$ and $\rho_x(\mathsf{a} \,\mathsf{U}\, [\mathsf{X}\,\mathsf{b}]_{>0.5}) = (\mathsf{a} \wedge \mathsf{a} \,\mathsf{U}\, [\mathsf{X}\,\mathsf{b}]_{>0.5}) \vee [\![\mathsf{X}\,\mathsf{b}]\!]_{>0.5}$.*

Corollary 1 and Theorem 5 imply that any algorithm for solving language emptiness or containment of p-automata would prove that satisfiability of PCTL is decidable [7, 1].

We now show that p-automata are more expressive than PCTL, using an adaptation of the known result by Wolper showing that LTL cannot count [16]. The p-automaton $A_W = \langle 2^{\{\mathsf{a},\mathsf{b}\}}, \{q_0, q_1\}, \delta, [\![q_0]\!]_{>0}, \{\}\rangle$ has transition function δ defined by

$$\delta(q_0, \{\mathsf{b}\}) = \delta(q_1, \{\mathsf{b}\}) = \delta(q_0, \{\mathsf{a},\mathsf{b}\}) = \delta(q_1, \{\mathsf{a},\mathsf{b}\}) = \mathsf{tt}$$
$$\delta(q_0, \{\mathsf{a}\}) = [\![q_1]\!]_{>0}$$
$$\delta(q_0, \{\}) = \mathsf{ff}$$
$$\delta(q_1, \{\}) = \delta(q_1, \{\mathsf{a}\}) = [\![q_0]\!]_{>0}$$

Lemma 2 *Every Markov chain $M \in \mathcal{L}(A_W)$ has a finite path $s_0 s_1 \ldots s_n$ with $n > 1$ such that $b \in L(s_n)$ and for all $0 \leq i < n$, either i is odd or $a \in L(s_i)$.*

We do not prove formally that $\mathcal{L}(A_W) \neq \mathcal{L}(A_\phi)$ for all PCTL formulas ϕ over $\{\mathsf{a},\mathsf{b}\}$. However, as the path from Lemma 2 is finite, its existence is equivalent to the probability of such a path being greater than 0. Thus, if it were possible to express this property in PCTL it would be possible to express it in CTL as well.

The p-automaton $A_R = \langle 2^{\{\mathsf{a}\}}, \{q_2\}, \delta, [\![q_2]\!]_{>0}, \{q_2\}\rangle$ with $\delta(q_2, \{\mathsf{a}\}) = [\![q_2]\!]_{\geq 0.5}$ and $\delta(q_2, \{\}) = \mathsf{ff}$ asserts the *recursive probabilistic* property that a location is labeled a, and that the probability of its successors with the same property is at least 0.5. We conjecture that $\mathcal{L}(A_R)$ also cannot be expressed in form $\mathcal{L}(A_\phi)$ for a PCTL formula ϕ.

6 Simulation of p-Automata

We now define simulation of p-automata as a combination of fair simulation [8], simulation for alternating word automata [5], probabilistic bisimulation [12], and the games defined in Section 3. This simulation takes into account the structure of alternating automata, their acceptance condition, and local probabilistic constraints. We show that whether B simulates A can be decided in EXPTIME and that simulation under-approximates language containment.

We define simulation through a series of games on the product of states and transitions of A and B: state u of B simulates state r of A iff Player 0 wins from configuration (r, u) in its game. More general configurations (α, β) are such that α is part of a transition of A and β is part of a transition of B. The classification of α and β as unbounded, bounded with $*$, bounded with $\mathbin{\text{\textcircled{$\vee$}}}$, or simple classifies (α, β) as one of 16 types. Here, we restrict our attention to the case that A and B do not use the $\mathbin{\text{\textcircled{$\vee$}}}$ operator. Furthermore, a state that is part of a bounded SCC in B cannot simulate a state that is part of an unbounded SCC in A. These restrictions are sufficient for handling simulation of automata that result from embedding PCTL formulas or Markov chains.

$\rho_x(\mathsf{a},\sigma) = \mathsf{tt}$ if $\mathsf{a} \in \sigma$	$\rho_x(\mathsf{a},\sigma) = \mathsf{ff}$ if $\mathsf{a} \notin \sigma$	$\rho_\epsilon(\mathsf{a}) = \mathsf{a}$ $\rho_\epsilon(\neg\mathsf{a}) = \neg\mathsf{a}$
$\rho_x(\neg\mathsf{a},\sigma) = \mathsf{tt}$ if $\mathsf{a} \notin \sigma$	$\rho_x(\neg\mathsf{a},\sigma) = \mathsf{ff}$ if $\mathsf{a} \in \sigma$	$\rho_\epsilon(\varphi_1 \vee \varphi_2) = \rho_\epsilon(\varphi_1) \vee \rho_\epsilon(\varphi_2)$
$\rho_\epsilon(\varphi_1 \wedge \varphi_2) = \rho_\epsilon(\varphi_1) \wedge \rho_\epsilon(\varphi_2)$	$\rho_\epsilon([\mathsf{X}\,\varphi_1]_{\bowtie p}) = [\![\mathsf{X}\,\varphi_1]\!]_{\bowtie p}$	$\rho_x(\mathsf{X}\,\varphi_1,\sigma) = \rho_\epsilon(\varphi_1)$

$\rho_\epsilon([\varphi_1\,\mathsf{U}\,\varphi_2]_{\bowtie p}) = (\rho_\epsilon(\varphi_1) \wedge [\![\varphi_1\,\mathsf{U}\,\varphi_2]\!]_{\bowtie p}) \vee \rho_\epsilon(\varphi_2)$	$\rho_x(\varphi_1\,\mathsf{U}\,\varphi_2,\sigma) = (\rho_\epsilon(\varphi_1) \wedge \varphi_1\,\mathsf{U}\,\varphi_2) \vee \rho_\epsilon(\varphi_2)$
$\rho_\epsilon([\varphi_1\,\mathsf{W}\,\varphi_2]_{\bowtie p}) = (\rho_\epsilon(\varphi_1) \wedge [\![\varphi_1\,\mathsf{W}\,\varphi_2]\!]_{\bowtie p}) \vee \rho_\epsilon(\varphi_2)$	$\rho_x(\varphi_1\,\mathsf{W}\,\varphi_2,\sigma) = (\rho_\epsilon(\varphi_1) \wedge \varphi_1\,\mathsf{W}\,\varphi_2) \vee \rho_\epsilon(\varphi_2)$

Figure 9. Transition function ρ_x and auxiliary function ρ_ϵ of A_φ

For sake of simplicity, p-automata $A = \langle \Sigma, Q, \delta, \varphi^{\text{in}}_a, F\rangle$ and $B = \langle \Sigma, U, \delta, \psi^{\text{in}}_b, F\rangle$ satisfy $Q \cap U = \{\}$ and we use δ for the transition function of both automata and F for both acceptance conditions. We determine whether B simulates A by a sequence of weak and stochastic weak games. The strict versions of the partial orders on equivalence classes of A and B are well-founded and so their lexicographical ordering is a well-founded ordering $\prec$ on the sets of configurations of the game. Namely, $(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis) \prec (\llparenthesis\tilde{\varphi}\rrparenthesis, \llparenthesis\tilde{\psi}\rrparenthesis)$ if either $\llparenthesis\varphi\rrparenthesis \prec_A \llparenthesis\tilde{\varphi}\rrparenthesis$ or ($\llparenthesis\varphi\rrparenthesis = \llparenthesis\tilde{\varphi}\rrparenthesis$ and $\llparenthesis\psi\rrparenthesis \prec_B \llparenthesis\tilde{\psi}\rrparenthesis$). Consider a pair of equivalence classes $(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$, where φ is in A and ψ is in B. As before, all pairs larger than $(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$ with respect to $\prec$ have already been handled: for every φ' and ψ' with $(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis) \prec (\llparenthesis\varphi'\rrparenthesis, \llparenthesis\psi'\rrparenthesis)$ value $\mathsf{val}(\varphi,\psi) \neq \bot$ is pre-seeded.

Case 1: Let $\llparenthesis\varphi\rrparenthesis$ and $\llparenthesis\psi\rrparenthesis$ be non-trivial SCCs where $\llparenthesis\varphi\rrparenthesis$ has transitions in E_u, and $\llparenthesis\psi\rrparenthesis$ has transitions in E_b without $\stackrel{*}{\triangledown}$ markings. We set $\mathsf{val}(\varphi,\psi) = 0$, bounded-with-$*$ states cannot simulates unbounded states.

Case 2: Let $\llparenthesis\varphi\rrparenthesis$ and $\llparenthesis\psi\rrparenthesis$ be non-trivial SCCs such that some transitions in $\llparenthesis\varphi\rrparenthesis$ and $\llparenthesis\psi\rrparenthesis$ are in E_u. Then $G_{\leq}(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$ is a stochastic weak game with

$$V = \{(\tilde{\varphi}, \tilde{\psi}) \mid \tilde{\varphi} \preceq_A \varphi \text{ and } \tilde{\psi} \preceq_B \psi\} \qquad V_p = \{\}$$

and V_0, V_1, and E are defined in Fig. 10. As pre-seeded values $\mathsf{val}(\tilde{\varphi}, \tilde{\psi})$ for configurations $(\tilde{\varphi}, \tilde{\psi})$ with $(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis) \prec (\llparenthesis\tilde{\varphi}\rrparenthesis, \llparenthesis\tilde{\psi}\rrparenthesis)$ may be in the open interval $(0,1)$, we treat $G_{\leq}(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$ as a stochastic weak game.

Intuitively, Player 1 resolves disjunctions on the left and conjunctions on the right and does this before Player 0 needs to move. Player 0 resolves conjunctions on the left and disjunctions on the right when Player 1 cannot move. From configurations of the form (q', u'), where q' is a state of A and u' is a state of B, Player 1 chooses a letter $\sigma \in \Sigma$ and applies the transitions of q' and u' reading σ.

Finally, an infinite play in $G_{\leq}(\llparenthesis q\rrparenthesis, \llparenthesis u\rrparenthesis)$ is winning for Player 0 if $\llparenthesis\varphi\rrparenthesis \cap Q \subseteq F$ implies $\llparenthesis\psi\rrparenthesis \cap U \subseteq F$.

By Theorem 1 every configuration c has a value. We set $\mathsf{val}(c)$ to the value of configuration c for Player 0.

Case 3: Let $\llparenthesis\varphi\rrparenthesis$ and $\llparenthesis\psi\rrparenthesis$ be non-trivial SCCs that both have transitions in E_b without $\stackrel{*}{\triangledown}$ markings. Then $G_{\leq}(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$ is a weak game. Let

$$\begin{aligned} \tilde{\varphi} &= *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n}) \\ \tilde{\psi} &= *([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_m]\!]_{\bowtie'_m p'_m}) \\ \mathcal{F}_{\tilde{\varphi},\tilde{\psi}} &= [n] \times [m] \to [0,1] \end{aligned}$$

Also, $f \in \mathcal{F}_{\tilde{\varphi},\tilde{\psi}}$ is *disjoint* if there is $\{a_{i,j} \in [0,1] \mid i \in [n] \text{ and } j \in [m]\}$ with (a) $\Sigma_{j\in[m]} a_{i,j} = 1$ for all $i \in [n]$ and (b) $\Sigma_{i\in[n]} a_{i,j} \cdot p_i \cdot f(i,j) > p'_j$ for all $j \in [m]$, or $\Sigma_{i\in[n]} a_{i,j} \cdot p_i \cdot f(i,j) = p'_j$ and either $\bowtie'_j$ is $\geq$ or there is i' with $a_{i',j} > 0$ and $\bowtie_{i'}$ is $>$. Let $\mathcal{F}^*_{\tilde{\varphi},\tilde{\psi}}$ be the set of disjoint functions. The configurations of $G_{\leq}(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$ are

$$\begin{aligned} V = \{(\tilde{\varphi}, \tilde{\psi}, f) \mid \tilde{\varphi} \in \llparenthesis\varphi\rrparenthesis, \tilde{\psi} \in \llparenthesis\psi\rrparenthesis, \text{ and } f \in \mathcal{F}^*_{\tilde{\varphi},\tilde{\psi}}\} \quad \cup \\ \{(\tilde{\varphi}, \tilde{\psi}), (\tilde{\varphi}, \tilde{\psi}, v) \mid \tilde{\varphi} \preceq_A \varphi, \tilde{\psi} \preceq_B \psi, \text{ and } v \in [0,1]\} \end{aligned}$$

and the definition of V_0, V_1, and E are given in Fig. 11. Set V above is uncountable and infinitely branching, as branching includes a choice of a function $f\colon [n] \times [m] \to [0,1]$. The techniques that were used in Section 3 can be used to make these games finite branching; and, if both A and B are finite, these games will be finite, too.

For $(\gamma, \epsilon) \in [\![Q]\!]^* \times [\![U]\!]^*$ with

$$\begin{aligned} \gamma &= *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n}) \\ \epsilon &= *([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_m]\!]_{\bowtie'_m p'_m}) \end{aligned}$$

in order to show that ϵ simulates γ, Player 0 needs to show that the probability of ϵ (and its partition) can be supported by γ. Accordingly, from (γ, ϵ) Player 0 chooses $f\colon [n] \times [m] \to [0,1]$ and moves to configuration (γ, ϵ, f). Such a configuration relates to the claim that q_i is related to u_j with proportion $f(i,j)$ and that f can be partitioned (using the $\{a_{i,j}\}$ to support the different u_j's). Then, Player 1 chooses i and j such that $f(i,j) > 0$ and an alphabet letter $\sigma \in \Sigma$, leading to a configuration of the form $(\delta(q_i,\sigma), \delta(u_j,\sigma), f(i,j))$. Conjunctions and disjunctions are resolved in the usual way until either reaching another configuration in $[\![Q]\!]^* \times [\![U]\!]^* \times [0,1]$, in which case the value $f(i,j)$ is ignored (as $f(i,j) \leq 1$), or until the play reaches a configuration with a pre-seeded value v. Then, if $f(i) \leq v$ Player 0 has fulfilled her obligation and she wins. If $f(i) > v$, Player 0 failed and she loses. An infinite play in $G_{\leq}(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$ is winning for Player 0 if $\llparenthesis\varphi\rrparenthesis \cap Q \subseteq F$ implies $\llparenthesis\psi\rrparenthesis \cap U \subseteq F$.

By Theorem 1, every $c \in V$ has a value in $\{0,1\}$ for Player 0. We set $\mathsf{val}(c)$ to that value.

Case 4: Let $\llparenthesis\varphi\rrparenthesis$ and $\llparenthesis\psi\rrparenthesis$ be non-trivial SCCs where $\llparenthesis\varphi\rrparenthesis$ has transitions in E_b without $\stackrel{*}{\triangledown}$ markings, and $\llparenthesis\psi\rrparenthesis$ has transitions in E_u. Then $G_{\leq}(\llparenthesis\varphi\rrparenthesis, \llparenthesis\psi\rrparenthesis)$ is a stochastic weak game with

V_0	$= \{c \in V \mid \exists \varphi_i, \psi_i \colon c = (\varphi_1 \wedge \varphi_2, \psi_1 \vee \psi_2)\} \quad \cup \quad \{c \in V \mid \exists q' \colon c = (q', \psi_1 \vee \psi_2), \text{or } \exists u' \colon c = (\varphi_1 \wedge \varphi_2, u')\}$
V_1	$= \{c \in V \mid \exists q', u' \colon c = (q', u')\} \quad \cup \quad \{c \in V \mid \exists \varphi_i, \psi \colon c = (\varphi_1 \vee \varphi_2, \psi), \text{or } \exists \varphi, \psi_i \colon c = (\varphi, \psi_1 \wedge \psi_2)\}$
E	$= \{((q', u'), (\delta(q', \sigma), \delta(u', \sigma)) \in V \times V \mid \sigma \in \Sigma\} \quad \cup$ $\{((\varphi_1 \vee \varphi_2, \psi), (\varphi_i, \psi)), ((\varphi, \psi_1 \wedge \psi_2), (\varphi, \psi_i)) \in V \times V \mid i \in \{1,2\}\} \quad \cup$ $\{((\varphi_1 \wedge \varphi_2, \psi_2 \vee \psi_2), (\varphi_i, \psi_j)) \in V \times V \mid i,j \in \{1,2\}\} \quad \cup$ $\{((\varphi_1 \wedge \varphi_2, u'), (\varphi_i, u')), ((q', \psi_1 \vee \psi_2), (q', \psi_i)) \in V \times V \mid i \in \{1,2\}\}$

Figure 10. Game $G_{\leq}(((\varphi)), ((\psi)))$ for $((\varphi))$ and $((\psi))$ unbounded

V_0	$= \{(\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2, v), (\alpha_1 \wedge \alpha_2, \epsilon, v), (\gamma, \beta_1 \vee \beta_2, v), (\gamma, \epsilon)\} \quad \cup \quad \{(\alpha, \beta, v) \mid \mathsf{val}(\alpha, \beta) \neq \bot \text{ and } v > \mathsf{val}(\alpha, \beta)\}$
V_1	$= \{(\gamma, \epsilon, f), (\alpha_1 \vee \alpha_2, \beta, v), (\alpha, \beta_1 \wedge \beta_2, v)\} \quad \cup \quad \{(\alpha, \beta, v) \mid \mathsf{val}(\alpha, \beta) = \bot \text{ or } v \leq \mathsf{val}(\alpha, \beta)\}$
E	$= \{((\alpha_1 \vee \alpha_2, \beta, v), (\alpha_i, \beta, v)), ((\alpha, \beta_1 \wedge \beta_2, v), (\alpha, \beta_i, v)) \mid i \in \{1,2\}\} \quad \cup \quad \{((\gamma, \epsilon), (\gamma, \epsilon, f))\} \quad \cup$ $\{((\alpha_1 \wedge \alpha_2, \epsilon, v), (\alpha_i, \epsilon, v)), ((\gamma, \beta_1 \vee \beta_2, v), (\gamma, \beta_i, v)) \mid i \in \{1,2\}\} \quad \cup$ $\{((\gamma, \epsilon, f), (\delta(q_i, \sigma), \delta(u_j, \sigma), f(i,j))) \mid f(i,j) > 0 \text{ and } \sigma \in \Sigma\} \quad \cup$ $\{((\alpha_1 \wedge \alpha_2, \beta_2 \vee \beta_2, v), (\alpha_i, \beta_j, v)) \mid i,j \in \{1,2\}\}$

Figure 11. Game $G_{\leq}(((\varphi)), ((\psi)))$ for $((\varphi))$ and $((\psi))$ bounded with $*$. Where α and β range over formulas in transitions of A and B, respectively, γ and ϵ range over formulas in $[\![Q]\!]^*$ and $[\![U]\!]^*$, respectively

$$
\begin{aligned}
V &= \{(\tilde{\varphi}, \tilde{\psi}) \mid \tilde{\varphi} \preceq_A \varphi \text{ and } \tilde{\psi} \preceq_B \psi\} \cup [\![Q]\!] \times U \times \Sigma \\
V_0 &= \{(\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2), (\alpha_1 \wedge \alpha_2, u), (\gamma, \beta_1 \vee \beta_2)\} \\
V_1 &= \{(\alpha_1 \vee \alpha_2, \beta), (\alpha, \beta_1 \wedge \beta_2), (\gamma, u)\} \\
V_p &= [\![Q]\!]^* \times U \times \Sigma \\
E &= \{((\alpha_1 \vee \alpha_2, \beta), (\alpha_i, \beta)) \mid i \in \{1,2\}\} \quad \cup \\
&\quad \{((\alpha, \beta_1 \wedge \beta_2), (\alpha, \beta_i)) \mid i \in \{1,2\}\} \quad \cup \\
&\quad \{((\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2), (\alpha_i, \beta_j)) \mid i,j \in \{1,2\}\} \quad \cup \\
&\quad \{((\gamma, u), (\gamma, u, \sigma)), ((\gamma, u, \sigma), (\delta(q_i, \sigma), \delta(u, \sigma)))\} \\
\kappa((\gamma, u, \sigma))&((\delta(q_i, \sigma), \delta(u, \sigma))) = p_i
\end{aligned}
$$

where α, α_i and β, β_i range over formulas in transitions of A and B, respectively, while γ and u range over $[\![Q]\!]^*$ and U, respectively. For probabilities p_i that do not sum up to 1, we add a sink state (losing for Player 0) that fills that gap.

An infinite play in $G_{\leq}(((\varphi)), ((\psi)))$ is winning for Player 0 if $((\varphi)) \cap Q \subseteq F$ implies $((\psi)) \cap U \subseteq F$.

By Theorem 1 every configuration c has a value. We set $\mathsf{val}(c)$ to the value of configuration c for Player 0.

Intuitively, a state u measures the probability of some regular set of paths, and a state $[\![q]\!]_{\bowtie p}$ can restrict the immediate steps taken by a Markov chain as well as enforce some regular structure on paths. Thus, this stochastic weak game establishes the conditions under which a Markov chain accepted from $[\![q]\!]_{\bowtie p}$ can be also accepted from u.

Case 5: Let $((\varphi))$ or $((\psi))$ be a trivial SCC. As in the case of acceptance games, the games defined above collapse to cycle-free games where the value of $(((\varphi)), ((\psi)))$ can be computed directly from pre-seeded values of configurations $(\tilde{\varphi}, \tilde{\psi})$ with $(((\varphi)), ((\psi))) \prec (((\tilde{\varphi})), ((\tilde{\psi})))$, covering all possible cases for $(\tilde{\varphi}, \tilde{\psi}) \in B^+(Q \cup [\![Q]\!]) \times B^+(U \cup [\![U]\!])$:

- $\mathsf{val}(\alpha_1 \vee \alpha_2, \beta) = \min_i \mathsf{val}(\alpha_i, \beta)$
- $\mathsf{val}(\alpha, \beta_1 \wedge \beta_2) = \min_i \mathsf{val}(\alpha, \beta_i)$
- $\mathsf{val}(\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2) = \max_{i,j} \mathsf{val}(\alpha_i, \beta_j)$
- $\mathsf{val}(\alpha_1 \wedge \alpha_2, \beta) = \max_i \mathsf{val}(\alpha_i, \beta)$, where $\beta \in U \cup [\![U]\!]$
- $\mathsf{val}(\alpha, \beta_1 \vee \beta_2) = \max_i \mathsf{val}(\alpha, \beta_i)$, where $\alpha \in Q \cup [\![Q]\!]$
- $\mathsf{val}(q, u) = \min_{\sigma \in \Sigma} \mathsf{val}(\delta(q, \sigma), \delta(u, \sigma))$ for $q \in Q, u \in U$
- For formulas $\gamma = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and $\epsilon = *([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_n]\!]_{\bowtie'_m p'_m})$ we set $\mathsf{val}(\gamma, \epsilon) = 1$ if there is $\{a_{i,j} \in [0,1] \mid i \in [n] \text{ and } j \in [m]\}$ with (a) $\Sigma_{j \in [m]} a_{i,j} = 1$ for all i and (b) for all j, $\Sigma_{i \in [n]} a_{i,j} \cdot p_i \cdot \min_{\sigma \in \Sigma} \mathsf{val}(\delta(q_i, \sigma), \delta(u_j, \sigma)) > p_j$ or $\Sigma_{i \in [n]} a_{i,j} \cdot p_i \cdot \min_{\sigma \in \Sigma} \mathsf{val}(\delta(q_i, \sigma), \delta(u_j, \sigma)) = p_j$, and for some i, $\bowtie_i$ is $>$ or for all j, $\bowtie'_j$ is $\geq$; and to 0 otherwise
- $\mathsf{val}(\gamma, u) = \Sigma_{i \in [n]} p_i \cdot \min_{\sigma \in \Sigma} \mathsf{val}(\delta(q_i, \sigma), \delta(u, \sigma))$ for $\gamma = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and $u \in U$
- $\mathsf{val}(q, *([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_n]\!]_{\bowtie'_m p'_m})) = 0$ for $q \in Q$

Definition 3 *We say that B simulates A, denoted $A \leq B$, if the value of configuration $(\varphi_a^{\mathsf{in}}, \varphi_b^{\mathsf{in}})$, computed in the previous sequence of games, is* 1.

Theorem 6 *Let A and B be p-automata over $2^{\mathbb{AP}}$. Then*

$$A \leq B \text{ implies } \mathcal{L}(A) \subseteq \mathcal{L}(B) \tag{2}$$

holds for all finite A and B. If A is A_M for some $M \in \mathsf{MC}_{\mathbb{AP}}$, (2) and its converse hold for all B even infinite ones.

In particular, $N \sim M$ iff $A_M \leq A_N$. We now get sound and comlete verification of $M \models \phi$ through simulations.

Corollary 2 *For infinite $M \in \mathsf{MC}_{\mathbb{AP}}$ and PCTL formula ϕ over $\mathbb{AP}$ we have $M \models \phi$ iff there is a finite p-automata A with $A_M \leq A$ and $A \leq A_\phi$.*

To see this, any such A implies $\mathcal{L}(A_M) \subseteq \mathcal{L}(A)$ and $\mathcal{L}(A) \subseteq \mathcal{L}(A_\phi)$ by both parts of Theorem 6. Thus, $M \models \phi$ holds by Theorems 4 and 5. Conversely, if there is no such A, then A_ϕ can also not be such an A. Since $A_\phi \leq A_\phi$ this implies $A_M \not\leq A_\phi$ and so $\mathcal{L}(A_M) \not\subseteq \mathcal{L}(A_\phi)$ from the converse of (2). So there is some $M' \sim M$ with $M' \not\models \phi$. Since $M' \sim M$, we get $M \not\models \phi$ as well by Lemma 1.

This method for deciding $M \models \phi$ via simulations is thus complete in the sense of [3]. To our knowledge, this is the first such completeness result for PCTL and Markov chains.

The first claim of Theorem 6 is proved as follows. Assuming $M \in \mathcal{L}(A)$ and $A \leq B$ we consider configurations (s, φ) and (φ, ψ) in the corresponding games, respectively. This determines a configuration (s, ψ) in the acceptance game for $M \in \mathcal{L}(B)$. We show an invariant, that $\mathsf{val}(s, \varphi) \cdot \mathsf{val}(\varphi, \psi) \leq \mathsf{val}(s, \psi)$ for all such "synchronized" configurations. In particular, we get $\mathsf{val}(s^{\text{in}}, \varphi^{\text{in}}) \cdot \mathsf{val}(\varphi^{\text{in}}, \psi^{\text{in}}) = 1 \cdot 1 \leq \mathsf{val}(s^{\text{in}}, \psi^{\text{in}})$ which proves $M \in \mathcal{L}(B)$. Extending this result to infinite-state automata seems to require the treatment of infinite converging products of real numbers.

The second claim of Theorem 6 follows since the simulation game collapses to an acceptance game when the automaton of the left in (2) is derived from a Markov chain.

7 Related and Future Work

The stochastic games of [11] abstract Markov decision processes as a 2-person game where two sources of nondeterminism, stemming from the MDP and the state space partition respectively, are controlled by different players. This seperation allows for more precision of abstractions but is not complete in the sense of [3], as shown in [9].

In [4], a Hintikka game was defined for satisfaction, $M \models \phi$, between Markov chains and PCTL formulas. That game resembles our acceptance game for $M \in \mathcal{L}(A_\phi)$.

We leave some research questions as future work: (i) To extend our framework so that Markov chains with infinite branching can be embedded as p-automata. (ii) To understand the difference between alternating and nondeterministic p-automata, where the latter notion still needs to be defined. (iii) To develop p-automata that embed Markov decision processes. (iv) To prove or refute equation (2) for infinite-state p-automata. (v) To remove the restriction of uniformity in p-automata and to develop a corresponding notion of games and their solution.

8 Conclusions

We presented a novel kind of automata, p-automata, that read in an entire Markov chain and either accept or reject it. We demonstrated how this acceptance can be decided through a series of stochastic weak games and weak games, at worst case exponential in the size of the automaton and in the size of the Markov chain. We proved that our automata are closed under Boolean operations, that language containment and emptiness are equi-solvable, and that the language of a p-automaton is closed under bisimulation. We showed that bisimulation equivalence classes of any Markov chain as well as the set of models of any PCTL formula are expressible as such languages. This suggests that emptiness, universality, and containment of p-automata is tightly related to the open problem of decidability of PCTL satisfiability. We then developed a fair simulation between p-automata that stem from Markov chains or PCTL formulas, decidable in EXPTIME, that under-approximates language containment. In particular, p-automata are a complete abstraction framework for PCTL: if an infinite Markov chain satisfies a PCTL formula, there is a finite p-automaton that abstracts this Markov chain and whose language is contained in that of the p-automaton for that PCTL formula.

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References

[1] T. Brázdil, V. Forejt, J. Kretínský, and A. Kucera. The Satisfiability Problem for Probabilistic CTL. In *Proc. of LICS'08*, pages 391–402. IEEE press, 2008.

[2] A. Condon. The complexity of stochastic games. *Information and Computation*, 96(2):203–224, 1992.

[3] D. Dams and K. Namjoshi. The existence of finite abstractions for branching time model checking. In *Logic in Computer Science*, 2004.

[4] H. Fecher, M. Huth, N. Piterman, and D. Wagner. Hintikka games for PCTL on labeled Markov chains. *Performance Evaluation*, 2009. To appear.

[5] C. Fritz and T. Wilke. State space reductions for alternating Büchi automata: Quotienting by simulation equivalences. In *Proc. of FSTTCS 2002*.

[6] H. Hansson and B. Jonsson. A logic for reasoning about time and reliability. *Formal Aspects of Computing*, 6:512–535, 1994.

[7] S. Hart and M. Sharir. Probabilistic propositional temporal logics. *Information and Control*, 70(2–3):97–155, 1986.

[8] T. Henzinger, O. Kupferman, and S. Rajamani. Fair simulation. *Information and Computation*, 173(1):64–81, 2002.

[9] M. Kattenbelt and M. Huth. Abstraction framework for Markov decision processes and PCTL via games. Technical report, OUCL TR RR-09-01, 2009.

[10] J. G. Kemeny, J. L. Snell, and A. W. Knapp. *Denumerable Markov Chains*. Springer Verlag, 1976. Second Edition.

[11] M. Kwiatkowska, G. Norman, and D. Parker. Game-based abstraction for markov decision processes. In *Proc. of QEST 2006*, pages 157–166. IEEE Computer Society, 2006.

[12] K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94:1–28, 1991.

[13] P. O'Hearn, J. Reynolds, and H. Yang. Local Reasoning about Programs that Alter Data Structures. In *Proc. of CSL 2001*, pages 1–19. Invited paper.

[14] M. O. Rabin. Probabilistic automata. *Information and Control*, 6:230–245, 1963.

[15] P. Ramadge and W. Wonham. The control of discrete event systems. Transactions on Control Theory, 77:81–98, 1989.

[16] P. Wolper. Temporal logic can be more expressive. *Information and Control*, 56(1–2):72–99, 1983.

A Proofs

Proof of Theorem 2: Well definedness of acceptance follows directly from Theorem 1. For finite Markov chain M and finite p-automata A we observe the following:

- The stochastic weak game arising from the combination of a Markov chain M and an unbounded SCC can be solved in NP∩co-NP.
- The weak game arising from the combination of Markov chain M and a bounded SCC may be exponential due to the large number of possible value assignment functions. Such a weak game can be solved in linear time leading to an EXPTIME upper bound.

Therefore, the sequence of weak games and stochastic weak games can be solved in EXPTIME. □

Proof of Theorem 3: We prove a stronger claim, namely that for every $s \in S$ and $\varphi \in \mathsf{cl_p}(\delta(Q), \Sigma)$ we have

$$\mathsf{val}(s, \varphi) = 1 - \mathsf{val}(s, \mathsf{dual}(\varphi))$$

The proof is by induction on the structure of the automaton. Consider an equivalence class $(\!(t)\!)$ in A. Assume by induction that the lemma holds for all the SCCs in A that are greater than $(\!(t)\!)$.

- If $(\!(t)\!)$ is a trivial SCC, the lemma follows from the dualization and the duality of $\min$ and $\max$.
- Suppose that $(\!(t)\!)$ is a nontrivial SCC and that all transitions in $(\!(t)\!)$ are unbounded. Then, the lemma follows from the dualization and the determinacy of stochastic weak games.
- Suppose that $(\!(t)\!)$ is a nontrivial SCC and that no transition in $(\!(t)\!)$ is in the scope of $\overset{*}{\triangledown}$. It follows that $(\!(\mathsf{dual}(t))\!)$ is also a nontrivial SCC and that no transition in $(\!(\mathsf{dual}(t))\!)$ is in the scope of $*$.

  Given a strategy for Player 0 in $G_{M,(\!(t)\!)}$, we show how to construct a strategy for Player 1 in $G_{M,(\!(\mathsf{dual}(t))\!)}$. The two strategies produce plays that are always in the same locations of the Markov chain M and same states of the automaton A (modulo dualization $t \mapsto \mathsf{dual}(t)$). For simplicity we denote $G_{M,(\!(t)\!)}$ by G and $G_{M,(\!(\mathsf{dual}(t))\!)}$ by $\overline{G}$.

  Consider two matching configurations (s, φ) and $(s, \mathsf{dual}(\varphi))$ in G and $\overline{G}$. Let $\varphi = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$, where $n > 1$. Consider the configuration $(s, \mathsf{dual}(\varphi))$. By playing for Player 1 in G we make Player 0 'reveal' her strategy in G and using her strategy we react to the moves of Player 0 in $\overline{G}$ by constructing a strategy for Player 1 in $\overline{G}$.

  Consider two plays ending in (s, φ) and $(s, \mathsf{dual}(\varphi))$. Let $f\colon [n] \times \mathsf{succ}(s) \to \mathsf{Val}_{s,\varphi}$ be the function chosen by Player 0 in G and let $f'\colon [n] \times \mathsf{succ}(s) \to \mathsf{Val}_{s,\mathsf{dual}(\varphi)}$ be the function chosen by Player 0 in $\overline{G}$. By definition there are $\{a_{i,s'}\}$ that witness the disjointness of f and for every i we have

  $$\Sigma_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot P(s, s') \cdot f(i, s') \bowtie_i p_i.$$

  By using the same $\{a_{i,s'}\}$ stemming from the fact that f' is intersecting, we get that there is some i such that

  $$\Sigma_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot P(s, s') \cdot f'(i, s') \mathsf{dual}(\bowtie_i) 1 - p_i.$$

  It follows that there is an $s' \in \mathsf{succ}(s)$ such that $f(i, s') + f'(i, s') > 1$. It is now Player 1's turn to move in both G and $\overline{G}$. In G we make Player 1 choose $(s', \delta(q_i, L(s)), f(i, s'))$ and the strategy for Player 1 in $\overline{G}$ is extended by $(s', \mathsf{dual}(\delta(q_i, L(s))), f'(i, s'))$. We now proceed by utilizing the duality between $\vee$ and $\wedge$ to use Player 0 choices in $\overline{G}$ to suggest moves for Player 1 in G and use Player 0 strategy in G to suggest how to extend the strategy for Player 1 in $\overline{G}$.

  Suppose that we reach configurations $(s', \varphi', f(i, s'))$ and $(s', \mathsf{dual}(\varphi'), f'(i, s'))$ such that $\mathsf{val}(s', \varphi') \neq \bot$ and $\mathsf{val}(s', \mathsf{dual}(\varphi')) \neq \bot$. Then, by assumption $\mathsf{val}(s', \varphi') = 1 - \mathsf{val}(s', \mathsf{dual}(\varphi'))$ and if $\mathsf{val}(s', \varphi') \geq f(s')$ it must be the case that $\mathsf{val}(s', \mathsf{dual}(\varphi')) < f'(s')$.

  Otherwise the game proceeds to a new configuration in $S \times [\![Q]\!]$. If the two plays are infinite, then by the duality of α and $Q \setminus \alpha$ if Player 0 wins the play in G then Player 1 wins the play in $\overline{G}$.

  Showing that a win of Player 1 in G is translated to a win of Player 0 in $\overline{G}$ is similar.
- The case that $(\!(t)\!)$ is a nontrivial SCC and that some transitions in $(\!(q)\!)$ are in scope of $\overset{*}{\triangledown}$ is similar. □

Proof of Corollary 1:

- Showing that these languages are closed under intersections and unions is trivial, and omitted. By Theorem 3, these languages are closed under complements.

- Given two p-automata A_1 and A_2, we have $\mathcal{L}(A_1) \subseteq \mathcal{L}(A_2)$ iff $\mathcal{L}(A_1) \cap \mathcal{L}(\mathsf{dual}(A_2)) = \{\}$. Therefore, checking language containment reduces to checking language emptiness, as p-automata are closed under intersection. Conversely, we can construct a p-automaton E such that $\mathcal{L}(E) = \{\}$. The language of A is empty iff $\mathcal{L}(A) \subseteq \mathcal{L}(E)$. □

Proof of Lemma 1: Let $M_i = (S_i, P_i, L_i, s_i^{\text{in}})$, for $i \in \{1, 2\}$, with the same set of labels $\mathbb{AP}$. Let $A = \langle \Sigma, Q, \delta, [\![q_0]\!]_{\bowtie p}, \alpha \rangle$, where $\Sigma = 2^{\mathbb{AP}}$. Let $\sim \subseteq S_1 \times S_2$ be the maximal bisimulation between M_1 and M_2.

We show that for every state $q \in Q$ and locations $s_1 \in S_1$, and $s_2 \in S_2$ such that $s_1 \sim s_2$, we have $\mathsf{val}(s_1, q) = \mathsf{val}(s_2, q)$. We prove this claim by induction on the partial order on the SCCs in A. Suppose that the claim holds for all SCCs greater than $(\!(q)\!)$ in the partial order. Consider the games $G_{M_1,(\!(q)\!)}$ and $G_{M_2,(\!(q)\!)}$. Consider a winning strategy σ for Player 0 in $G_{M_1,(\!(q)\!)}$. We show how this is also a winning strategy for Player 0 in $G_{M_2,(\!(q)\!)}$.

Consider a play in an unbounded SCC $(\!(q)\!)$. We build by induction a play in $G_{M_1,(\!(q)\!)}$ and a play in $G_{M_2,(\!(q)\!)}$ with the invariant that the plays end in configurations of the form (s_1, t) and (s_2, t) such that $s_1 \sim s_2$. Clearly, the initial configuration in both games satisfies this invariant. We show how to extend the play to maintain this invariant. If t is of the form $\varphi_1 \wedge \varphi_2$ and Player 1 chooses φ_i in $G_{M_2,(\!(q)\!)}$, then we emulate the same choice in $G_{M_1,(\!(q)\!)}$. If t is of the form $\varphi_1 \vee \varphi_2$, then σ instructs Player 0 to choose φ_i in $G_{M_1,(\!(q)\!)}$ and we emulate the same choice in $G_{M_2,(\!(q)\!)}$. If t is of the form q' for some state $q' \in Q$ then choices in (s_1, q') and (s_2, q') are resolved by the stochastic player. As $s_1 \sim s_2$ the successors of s_1 and s_2 can be partitioned to equivalence classes such that for each equivalence class C_1 in M_1 and C_2 in M_2 we have $P_1(s_1, C_1) = P_2(s_2, C_2)$. Consider now the measure of plays that are winning according to this composed strategy. The plays can be partitioned according to bisimulation equivalence classes and every choice has the same weight. It follows that the measure of winning plays is identical in both games.

Consider a play in a bounded SCC $(\!(q)\!)$ where no transition uses $\circledast$. Disjunctions and conjunctions are handled as above. Consider a pair of configurations (s_1, t) and (s_2, t), where $s_1 \sim s_2$ and t is of the form $*([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$. Let f_1 be the function chosen by Player 0 in $G_{M_1,(\!(q)\!)}$. As $s_1 \sim s_2$, we can find a function f_2 such that for every s_2' we have $f_2(i, s_2') = f_1(i, s_1')$ for some $s_1' \sim s_2'$ that satisfies the requirement of the game. Next, Player 1 chooses a state $s' \in \mathsf{succ}(s_2)$ and a state q_i. The same choice can be mimicked in $G_{M_1,(\!(q)\!)}$. As $s_1 \sim s_2$, it follows that $L(s_1) = L(s_2)$ and the automaton component in both configurations remains the same.

The treatment of a play in a bounded SCC $(\!(q)\!)$ where some transitions use $\circledast$ is similar. □

Proof of Theorem 4:

1. By Lemma 1, we know that $M' \sim M$ implies $M' \in \mathcal{L}(A_M)$ as soon as we have that $M \in \mathcal{L}(A_M)$. To simplify the proof of $M \in \mathcal{L}(A_M)$, we assume that all locations of M are in one SCC. Consider a location $s \in S$ and $(s, s') \in Q$. Let $\varphi_s = *([\![(s, s')]\!]_{\geq P(s,s')} \mid s' \in \mathsf{succ}(s))$. We show that from a configuration of the form (s, φ_s), Player 0 has a strategy that keeps returning to configurations of this form. As $\alpha = Q$, Player 0 can continue playing forever and wins. We start from the configuration (s, φ_s). Let $\varphi_s = *([\![(s, s_1)]\!]_{\geq P(s,s_1)}, \ldots, [\![(s, s_n)]\!]_{\geq P(s,s_n)})$. Then Player 0 chooses the function $f\colon [n] \times \mathsf{succ}(s) \to \{0, 1\}$ such that $f(i, s') = 1$ iff $s_i = s'$. The trivial assignment $a_{i,s'} = 1$ iff $s_i = s'$ shows that f is disjoint. Then, Player 1 chooses a successor $(s_i, \delta((s, s_i), L(s)), 1)$. As $\delta((s, s_i), L(s)) = \varphi_{s_i}$ the claim follows and Player 0 has a strategy to continue the play forever.

   The initial configuration in the game is $*([\![(s^{\text{in}}, s')]\!]_{\geq P(s^{\text{in}},s')} \mid s' \in \mathsf{succ}(s^{\text{in}}))$. The same intuition shows that this is winning for Player 0 as well.

2. Conversely, if $M' \not\sim M$ we show that $M' \notin \mathcal{L}(A_M)$. Let $M = (S, P, L, s^{\text{in}})$ and $M' = (T, P, L, t^{\text{in}})$. To simplify notations we assume that $S \cap T = \{\}$ and use P and L for the probability distribution and labeling of both Markov chains. We use the partition refinement algorithm that computes the bisimulation equivalence sets for a Markov chain. Let $\Xi_0 = \{S' \subseteq S \cup T \mid \forall s, s' \in S'\colon L(s) = L(s') \text{ and } S' \text{ is maximal}\}$. Clearly, Ξ_0 is a partition of $S \cup T$. Let Ξ_{i+1} be the coarsest partition of $S \cup T$ that refines Ξ_i and in addition for every $G \in \Xi_{i+1}$, for every $s, s' \in G$, and for every $G' \in \Xi_i$ we have $P(s, G') = P(s', G')$. It is well known that if $s \not\sim s'$ there is some $i_{s,s'}$ such that s and s' belong to different sets in $\Xi_{i_{s,s'}}$.[1]

[1] As our Markov chains have only finite branching, it is enough to consider $i_{s,s'} \in \mathbb{N}$. Otherwise, we may have to use transfinite induction.

By assumption, $s^{\text{in}} \not\sim t^{\text{in}}$. Let i_0 be minimal such that s^{in} and t^{in} are in different sets in Ξ_{i_0}. Denote $s_{i_0} = s^{\text{in}}$, $t_{i_0} = t^{\text{in}}$, $\varphi_{i_0} = \varphi^{\text{in}}$, and $c_{i_0} = (t_{i_0}, \varphi_{i_0})$. Consider the configuration $c_{i_j} = (t_{i_j}, \varphi_{i_j})$, where

$$\varphi_{i_j} = \mathop{*}_{s' \in \mathsf{succ}(s_{i_j})} [\![(s_{i_j}, s')]\!]_{\geq P(s_{i_j}, s')}$$

and s_{i_j} and t_{i_j} are in different sets in Ξ_{i_j}. We show that from configuration c_{i_j} Player 1 either wins immediately or finds a similar configuration for $i_{j+1} < i_j$.

If $i_j = 0$, then $L(t_{i_j}) \neq L(s_{i_j})$. Regardless of the immediate choices of $Player$ 0, we have $\delta((s_{i_j}, s'), L(t_{i_j})) = \mathsf{ff}$ and Player 1 wins.

Otherwise, $i_j > 0$. By assumption, there is some $i_{j+1} < i_j$ and $G \in \Xi_{i_{j+1}}$ such that $P(s_{i_j}, G) \neq P(t_{i_0}, G)$. Without loss of generality we assume that $P(s_{i_j}, G) > P(t_{i_j}, G)$. Indeed, if $P(s_{i_j}, G) < P(t_{i_j}, G)$, then as $P(s_{i_j}, S) = 1$ there must be a different set $G' \in \Xi_{i_{j+1}}$ such that $P(s_{i_j}, G') > P(t_{i_j}, G')$.

Let $S_{i_{j+1}} = G \cap S$. Let $(t_{i_j}, \varphi_{i_j}, f)$ be the configuration chosen by Player 0. By disjointness of f, and as $P(t_{i_j}, G) < P(s_{i_j}, G)$, there must be a location $s_{i_{j+1}} \in G$ and a location $t_{i_{j+1}} \notin G$ such that $f(t_{i_{j+1}}, s_{i_{j+1}}) > 0$. Player 1 chooses $c_{i_{j+1}} = (t_{i_{j+1}}, \varphi_{i_{j+1}}, v)$, where $\varphi_{i_{j+1}} = \delta((s_{i_j}, s_{i_{j+1}}), L(t_{i_j}))$. As $t_{i_{j+1}} \notin G$, Player 1 has forced the game to a similar configuration with $i_{j+1} < i_j$ and eventually wins by reaching Ξ_0. □

Proof of Theorem 5: We prove

For every location s of M and subformula φ' of φ we have $M, s \models \varphi'$ iff the configuration $(s, \rho_\epsilon(\varphi'))$ has value 1 for Player 0 in the acceptance game of A_φ on M.

by induction on the structure of the formula. For a proposition a, notice that the value of (s, a) depends on the values of $(s', \rho_x(\mathsf{a}, L(s)))$ for successors s' of s. By definition, $\rho_x(\mathsf{a}, L(s)) = \mathtt{tt}$ if $\mathsf{a} \in L(s)$ and ff otherwise. The claim holds similarly for negated propositions, and by induction on Boolean combinations of formulas.

Consider a subformula of the form $\varphi' = [\mathsf{X}\,\psi]_{\bowtie p}$. By induction $M, s' \models \psi$ iff the configuration $(s', \rho_\epsilon(\psi))$ is winning for Player 0. By definition $\rho_\epsilon([\mathsf{X}\,\psi]_{\bowtie p}) = [\![\mathsf{X}\,\psi]\!]_{\bowtie p}$. Consider the function $f\colon [1] \times \mathsf{succ}(s) \to [0,1]$ such that $f(1, s') = 1$ iff $\mathsf{val}(s', \rho_\epsilon(\psi)) = 1$. By assumption, $\sum_{s' \in \mathsf{succ}(s)} f(1, s')\mathsf{val}(s', \rho_\epsilon(\psi)) \bowtie p$. The claim follows.

Consider a formula of the form $\varphi' = [\psi_1 \,\mathsf{U}\, \psi_2]_{\bowtie p}$. By induction $M, s \models \psi_i$ iff the configurtaion $(s, \rho_\epsilon(\psi_i))$ is winning for Player 0, for $i \in \{1, 2\}$. Consider the stochastic weak game induced by the SCC $\psi_1 \,\mathsf{U}\, \psi_2$ in the structure of A_φ. The optimal strategy for both players is memoryless and pure. Restricting our attention to these memoryless pure strategies we can think about the game as restricted to configurations of the form $(s', \rho_\epsilon(\psi_1))$, where all configurations are probabilistic. A play that is winning for Player 0 is exactly a play that remains in states s' such that $M, s' \models \psi_1$ until reaching states s'' such that $M, s'' \models \psi_2$ (as $\psi_1 \,\mathsf{U}\, \psi_2$ is unfair). It follows that the value of $(s, \psi_1 \,\mathsf{U}\, \psi_2)$ in the stochastic game is exactly $\Pr(s, \psi_1 \,\mathsf{U}\, \psi_2)$. Finally, $\rho_\epsilon([\psi_1 \,\mathsf{U}\, \psi_2]_{\bowtie p}) = \rho_\epsilon(\psi_1) \wedge [\![\psi_1 \,\mathsf{U}\, \psi_2]\!]_{\bowtie p} \vee \rho_\epsilon(\psi_2)$. Consider a location s and the configuration $(s, \rho_\epsilon([\psi_1 \,\mathsf{U}\, \psi_2]_{\bowtie p}))$. If $(s, \rho_\epsilon(\psi_2))$ is winning for Player 0, then clearly $(s, \rho_\epsilon([\psi_1 \,\mathsf{U}\, \psi_2]_{\bowtie p}))$ is winning as well. Otherwise, by assumption $s \models [\psi_1 \,\mathsf{U}\, \psi_2]_{\bowtie p}$, so it must be the case that $s \models \psi_1$. It follows that Player 0 can choose the disjunct $\rho_\epsilon(\psi_1) \wedge [\![\psi_1 \,\mathsf{U}\, \psi_2]\!]_{\bowtie p}$. Furthermore, the function $f\colon [1] \times \mathsf{succ}(s) \to [0,1]$ that associates $\mathsf{val}(s', \psi_1 \,\mathsf{U}\, \psi_2)$ with s' is disjoint. The claim follows.

The treatment of a formula of the form $\varphi' = [\psi_1 \,\mathsf{W}\, \psi_2]_{\bowtie p}$ is similar.

The treatment of bounded Strong Until and of bounded Weak Until are variants of the above cases, and so omitted. □

Proof of Lemma 2: First we note that as both q_0 and q_1 are unfair, a winning play for Player 0 has to be finite and end with a transition that reads b. Before reaching b a winning play includes moves of the following form:

- Going from configurations of the form $(s_{2i}, [\![q_0]\!]_{>0})$ to configurations $(s_{2i+1}, [\![q_1]\!]_{>0})$ such that $P(s_{2i}, s_{2i+1}) > 0$ and $\mathsf{a} \in L(s_{2i})$.
- Going from configurations of the form $(s_{2i+1}, [\![q_1]\!]_{>0})$ to configurations $(s_{2i+2}, [\![q_0]\!]_{>0})$ such that $P(s_{2i+1}, s_{2i+2}) > 0$.

This implies the existence of a path as required. □

Proof of Theorem 6: We note that when A equals A_M for some $M \in \mathsf{MC}_{\mathbb{AP}}$, the simulation game for $A_M \leq B$ and the acceptance game for $M \in \mathcal{L}(B)$ collapse to the same game. Thus, regardless of whether A_M or B is infinite-state we have $A_M \leq B$ iff $M \in \mathcal{L}(B)$. And the latter is equivalent to $\mathcal{L}(A_M) \subseteq \mathcal{L}(B)$ by Lemma 1 and Theorem 4.

In order to prove (2) for finite-state A and B, consider a Markov chain $M = (S, P, L, s^{\text{in}})$. Consider two formulas φ and ψ such that φ appears in the transition of A and ψ appears in the transition of B.

We construct a strategy for Player 0 in G_B and plays in G_A, G_B, and $G_\leq$ such that the plays start from (s, φ), (s, ψ), and (φ, ψ), respectively and such that the values of these plays satisfy $\mathsf{val}(s, \varphi) \cdot \mathsf{val}(\varphi, \psi) \leq \mathsf{val}(s, \psi)$. Thus, we prove that $M \in \mathcal{L}(A)$ implies $M \in \mathcal{L}(B)$.

Suppose that the claim holds by induction for plays starting in configurations $(((\tilde{\varphi})), ((\tilde{\psi})))$, where $(((\tilde{\varphi})), ((\tilde{\psi}))) \prec (((\varphi)), ((\psi)))$.

- In case that $\varphi \in Q$ and $\psi \in [\![U]\!]^*$ we have $\mathsf{val}(\varphi, \psi) = 0$ and the claim holds trivially.

- Suppose that both φ and ψ are in unbounded SCCs. The game G_A is a stochastic weak game and Player 0 secures $\mathsf{val}(s, \varphi)$ in configuration (s, φ).

  Consider the configurations (s, φ), (s, ψ), and (φ, ψ) in the games G_A, G_B, and $G_\leq$, respectively.

  If φ is a disjunction, then the strategy of Player 0 in G_A instructs her to choose a disjunct φ_1 of φ. Then (φ, ψ) is a Player 1 configuration in $G_\leq$ and we instruct Player 1 to choose the successor (φ_1, ψ). If ψ is a conjunction, then Player 1 chooses a successor (s, ψ_1) of (s, ψ) in G_B. We update the game $G_\leq$ by mimicking the same choice of Player 1 from (φ, ψ). If φ is a conjunction and ψ is not a conjunction, then the strategy of Player 0 in $G_\leq$ instructs Player 0 to choose a conjunct φ_1 of φ. This choice can be mimicked in G_A in which Player 1 needs to move. If φ is not a disjunction and ψ is a disjunction, then the strategy of Player 0 in $G_\leq$ instructs Player 0 to choose a disjunct ψ_1 of φ. This choice resolves Player 0's choice in G_B.

  Consider three plays produced this way. If all plays are infinite, the claim follows from the winning condition in $G_\leq$ and the values of the plays in G_A and G_B. If one of the plays is finite then the claim follows from the induction assumption, as the play passes in $G_\leq$ to a different SCC.

- Suppose that φ and ψ are in bounded SCCs. The game G_A is a weak game and Player 0 secures $\mathsf{val}(s, \varphi)$ in configuration (s, φ). By definition $\mathsf{val}(s, \varphi) \in \{0, 1\}$. Clearly, the case $\mathsf{val}(s, \varphi) = 0$ is not interesting. Suppose that $\mathsf{val}(s, \varphi) = 1$, i.e., Player 0 wins from configuration (s, φ) in G_A. Similarly $\mathsf{val}(\varphi, \psi) \in \{0, 1\}$ in $G_\leq$. Suppose that $\mathsf{val}(\varphi, \psi) = 1$. We have to give a strategy for Player 0 in G_B such that $\mathsf{val}(s, \psi) = 1$.

  Let $\varphi = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and $\psi = *([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_m]\!]_{\bowtie'_m p'_m})$. Let $f : [n] \times \mathsf{succ}(s) \to [0, 1]$ be the function chosen by Player 0's strategy in G_A and let $f' : [n] \times [m] \to [0, 1]$ be the function chosen by Player 0's strategy in $G_\leq$. We set Player 0's strategy in G_B to choose the function $f'' : [m] \times \mathsf{succ}(s) \to [0, 1]$ where $f''(j, s')$ is the minimal value in $\mathsf{Val}_{s,\psi}$ that is at least $\max_{i \in [n]} f(i, s') \cdot f'(i, j)$. We have to show that f'' is disjoint.

  Claim 1 *f'' is disjoint.*

  Proof: Let $a_{j,s'} = \Sigma_{i \in [n]} a_{i,s'} \cdot a_{i,j}$. First, one can see that for every $s' \in \mathsf{succ}(s)$ we have

$$\sum_{j \in [m]} a_{j,s'} = \sum_{j \in [m]} \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} = \sum_{i \in [n]} a_{i,s'} \sum_{j \in [m]} a_{i,j} = \sum_{i \in [n]} a_{i,s'} = 1$$

Second, consider some $j \in [m]$. Then,

$$\begin{aligned}
& \sum_{s' \in \mathsf{succ}(s)} a_{j,s'} \cdot f''(j,s') \cdot P(s,s') \\
= & \sum_{s' \in \mathsf{succ}(s)} \left(\sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \right) \cdot f''(j,s') \cdot P(s,s') \\
\geq & \sum_{s' \in \mathsf{succ}(s)} \left(\sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \right) \cdot \max_{i \in [n]} (f(i,s') \cdot f'(i,j)) \cdot P(s,s') \\
\geq & \sum_{s' \in \mathsf{succ}(s)} \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \cdot f(i,s') \cdot f'(i,j) \cdot P(s,s') \\
= & \sum_{i \in [n]} \sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot a_{i,j} \cdot f(i,s') \cdot f'(i,j) \cdot P(s,s') \\
= & \sum_{i \in [n]} a_{i,j} \cdot f'(i,j) \cdot \sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot f(i,s') \cdot P(s,s') \\
\bowtie & \sum_{i \in [n]} a_{i,j} \cdot f'(i,j) \cdot p_i \bowtie' p_j
\end{aligned}$$

and $\bowtie$ is $>$ if for some $i \in [n]$ we have $\bowtie_i$ equals $>$ and then $\bowtie'$ is $\geq$, otherwise either $\bowtie'$ is $>$ or $\bowtie'_j$ is $\geq$ and the proof is complete. □

With f'' established as disjoint, we get back to the games. In G_B Player 1 chooses j and $s' \in \mathsf{succ}(s)$ and moves to state $(s', \delta(u_j, L(s)), f''(s', j))$. We mimic this choice in G_A by making Player 1 choose the state q_i such that $f(i,s') \cdot f'(i,j)$ is maximal and moving to $(s', \delta(q_i, L(s)), f(i,s'))$. We mimic this choice in $G_\leq$ by making Player 1 choose the states q_i, u_j, and the letter $L(s)$ leading to configuration $(\delta(q_i, L(s)), \delta(u_j, L(s)), f'(i,j))$.

If the plays continue indefinitely inside the same SCC in $G_\leq$ the claim follows from the winning condition in $G_\leq$ and the winning conditions of G_A and G_B.

If the plays exits the SCC in $G_\leq$ then the triplet of configurations is (s'', φ'', v_1), (s'', ψ'', v_2), (φ'', ψ'', v). By induction assumption $\mathsf{val}(s'', \varphi'') \cdot \mathsf{val}(\varphi'', \psi'') \leq \mathsf{val}(s'', \psi'')$ holds. Furthermore, we have to show that $\mathsf{val}(s'', \psi'') \geq v$. Let (s, φ), (s, ψ) and (φ, ψ) be the last configurations that are part of the SCC before reaching the above triplet of configurations. It follows that $\mathsf{val}(s'', \psi'') \in \mathsf{Val}_{s,\psi}$. By the choices of f, f' and f'' we know that v is the minimal value in $\mathsf{Val}_{s,\psi}$ that is at least $\max_{i \in [n]} f(i,s') \cdot f'(i,j)$. In addition, the last choice in G_A was exactly the state q_i such that i is maximal. We know that $\mathsf{val}(s'', \varphi'') \geq v_1$, that $\mathsf{val}(\varphi'', \psi'') \geq v$. It follows that $\mathsf{val}(s'', \varphi'') \cdot \mathsf{val}(\varphi'', \psi'') \geq v \cdot v_1$. But, v_2 is exactly $v \cdot v_1$ leading to the desired result.

- Suppose that φ is in a bounded SCC and ψ is in an unbounded SCC.

  The game G_A is a weak game while the games G_B and $G_\leq$ are stochastic weak games. Interesting cases are where $\mathsf{val}(s, \varphi) = 1$ and $\mathsf{val}(\varphi, \psi) > 0$. Given a strategy of Player 1 in G_B, we show how to use the winning strategies of Player 0 in G_A and $G_\leq$ to produce a winning strategy for Player 0 in G_B. We also resolve all the choices for Player 1 in G_A and $G_\leq$ leading to both $G_\leq$ and G_B being reduced to Markov decision processes. These Markov decision processes capture all the possible evolutions of the games in G_B and $G_\leq$ according to the possible choices in probabilistic configurations in G_A. We then show how to use these Markov decision processes to prove that the claim holds.

  Consider three configurations (s, φ') in G_A, (φ', ψ') in $G_\leq$, and (s, ψ') in G_B. If ψ' is a conjunction, then, in G_B, Player 1 chooses a conjunct of ψ'. The same choice is mimicked in $G_\leq$ by making Player 1 choose the same conjunct. If ψ' is a disjunction, then Player 0's strategy in $G_\leq$ instructs her to choose one disjunct. The same choice is mimicked in G_B. If φ' is a conjunction, then Player 0's strategy in $G_\leq$ chooses a conjunct of φ'. We make Player 1 in G_A choose the same conjunct. If φ' is a disjunction, then Player 0's strategy in G_A chooses a disjunct of φ'. We make Player 1 in $G_\leq$ choose the same disjunct. The remaining cases are where $\psi' = u$ is a state of B and $\varphi' = *(\llbracket q_1 \rrbracket_{\bowtie_1 p_1}, \ldots, \llbracket q_n \rrbracket_{\bowtie_n p_n})$. The configuration (s, u) in G_B is probabilistic. The configuration (φ', u) in $G_\leq$ is a Player 1 configuration. We make Player 1 choose $L(s)$ in $G_\leq$ leading to configuration $(\varphi', u, L(s))$, which is probabilistic. The configuration (s, φ') is a Player 0 configuration in G_A. The strategy of Player 0 on G_A instructs her to choose a disjoint function $f : [n] \times \mathsf{succ}(s) \to [0, 1]$.

Let $\{a_{i,s'}\}$ be witnesses to the disjointness of f. Consider a location s' that is chosen with probability $P(s, s')$ in G_B. Here, we make multiple possible choices of continuing in the games, giving rise to Markov decision processes (with a matching between the choices in them). Consider all indices i such that $a_{i,s'} > 0$. It follows that for every such index there is a way to continue unraveling the plays by making Player 1 in G_A choose the successor $(s', \delta(q_i, L(s)), f(i, s'))$ and continuing to configurations $(\delta(q_i, L(s)), \delta(u, L(s)))$ in $G_\leq$ and $(s', \delta(u, L(s)))$ in G_B. By using these strategies, this effectively creates from G_B and $G_\leq$ Markov decision processes where the choices are angelic in G_B and demonic in G_A. That is, the actual value of G_B is the best possible value in the Markov decision process arising from G_B and the value in $G_\leq$ is the worst possible value in $G_\leq$. Hence, it is enough to show one choice such that the value in the Markov decision process arising from G_B satisfies the requirement of the claim. Indeed, the actual value in G_B could only be higher while the actual value in $G_\leq$ could only be lower.

Consider now three configurations (s, φ'), (φ', ψ'), and (s, ψ'), and the resulting Markov decision processes from (φ', ψ') and (s, ψ'). By the construction of the strategy, every play starting in (s, ψ'), is associated with plays that start in (s, φ') and (φ', ψ') such that at every stage the three configurations use the same state of the Markov chain and formulas in the transitions of A and B. We consider four cases:

- Consider a triplet of configurations $(s', \tilde{\varphi})$, $(\tilde{\varphi}, \tilde{\psi}')$, and $(s', \tilde{\psi})$ such that $(\tilde{\varphi}, \tilde{\psi})$ is not in the equivalence class of $((\!(\varphi)\!), (\!(\psi)\!))$. By induction $\mathsf{val}(s', \psi'') \geq \mathsf{val}(s', \varphi'') \cdot \mathsf{val}(\varphi'', \psi'')$.

- Consider a triplet of configurations (s, φ'), (φ', ψ'), and (s, ψ') such that there is some choice in the Markov decision process that arises from G_B such that all plays starting in (φ', ψ') remain in $((\!(\varphi)\!), (\!(\psi)\!))$ and are winning for Player 0 in $G_\leq$. The matching choice of plays starting from (s, ψ') are winning for Player 0 in G_B. Indeed, if this were not the case, there were a play in G_B that is losing. It follows that the corresponding play in $G_\leq$ does not satisfy the acceptance of A and that the play in G_A is losing. However G_A is a weak game and this is impossible.

- Consider a triplet of configurations (s, φ'), (φ', ψ'), and (s, ψ') such that for all choices in the Markov decision process that arises from $G_\leq$ we have all plays starting in (φ', ψ') remain in $((\!(\varphi)\!), (\!(\psi)\!))$ and are losing for Player 0 in $G_\leq$. One can see that $\mathsf{val}(s, \psi') \geq 0$.

- Consider now a triplet (s, φ'), (φ', ψ'), and (s, ψ') such that $(\varphi', \psi') \in ((\!(\varphi)\!), (\!(\psi)\!))$ and there is no choice in the Markov decision process arising from $G_\leq$ such that (i) all paths are winning for Player 0 and (ii) for all choices the probability for Player 0 to win is positive. As the automata and the Markov chain are finite, so are the resulting Markov decision processes. It follows that the probability of winning in $G_\leq$ equals the probability of getting to one of the previous three types of configurations. Then we show that the probability to reach one of the three previous types of configurations in n steps satisfies the requirements of the Theorem, for every n. The requirement of the claim will follow.

  For every triplet (s', φ'), (φ', ψ'), and (s', ψ') let $P_0(\varphi', \psi')$ be $\mathsf{val}(\varphi', \psi')$ and $P_0(s', \psi')$ be $\mathsf{val}(s', \psi')$ if (φ', ψ') is one of the three types of configurations mentioned above. Let $P_0(\varphi', \psi')$ and $P_0(s', \psi')$ be 0, otherwise.

  Consider a triplet (s, φ'), (φ', ψ'), (s, ψ') such that $\varphi' = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_2})$ and $\psi' = u$ such that $P_0(\varphi', \psi') = P_0(s, \psi') = 0$ but for some successor s' of s there is a choice of i such that $P_0(\delta(q_i, L(s')), \delta(u, L(s))) > 0$ and

$P_0(s'', \delta(u, L(s'))) > 0$. Let I denote the set of such indices i. Then, P_1 satisfies the requirement:

$$\begin{aligned}
P_1(s,u) = & \sum_{s' \in \mathsf{succ}(s) \mid \exists i \in I . a_{i,s'} > 0} P(s,s') P_0(s', \delta(u, L(s))) & = \\
& \text{For all such s', we have } P_0(s', \delta(u, L(s))) = \mathsf{val}(s', \delta(u, L(s))). & \\
= & \sum_{s' \in \mathsf{succ}(s) \mid \exists i \in I . a_{i,s'} > 0} P(s,s') \cdot \mathsf{val}(s', \delta(u, L(s))) & \geq \\
& \text{We have already proven the requirement of the theorem for these configurations.} & \\
\geq & \sum_{s' \in \mathsf{succ}(s) \mid \exists i \in I . a_{i,s'} > 0} P(s,s') \cdot \mathsf{val}(\delta(q_i, L(s)), \delta(u, L(s))) \cdot \mathsf{val}(s', \delta(q_i, L(s))) & \geq \\
& \text{By } \mathsf{val}(s', \delta(q_i, L(s))) \geq \Sigma_{i \in I} f(i,s') \cdot a_{i,s'}. & \\
\geq & \sum_{s' \in \mathsf{succ}(s) \mid \exists i \in I . a_{i,s'} > 0} P(s,s') \cdot \sum_{i \in I} f(i,s') \cdot a_{i,s'} \cdot \mathsf{val}(\delta(q_i, L(s)), \delta(u, L(s))) & = \\
& \text{Changing the order of summation.} & \\
= & \sum_{i \in I} \mathsf{val}(\delta(q_i, L(s)), \delta(u, L(s))) \cdot \sum_{s' \in \mathsf{succ}(s) \mid \exists i \in I . a_{i,s'} > 0} f(i,s') \cdot a_{i,s'} \cdot P(s,s') & \geq \\
& \text{By } f \text{ being disjoint, } \Sigma_{s' \in \mathsf{succ}(s) \mid \exists i \in I . a_{i,s'} > 0} f(i,s') \cdot a_{i,s'} \cdot P(s,s') \geq p_i. & \\
\geq & \sum_{i \in I} \mathsf{val}(\delta(q_i, L(s)), \delta(u, L(s))) \cdot p_i = P_1(\varphi', u) &
\end{aligned}$$

Consider a triplet (s, φ'), (φ', ψ'), and (s, ψ'). The strategy defined fixes most such configurations as deterministic in their respective Markov decision processes. The only interesting case is when $\varphi' \in [\![Q]\!]$ and $\psi' \in U$. In this case (φ', ψ') and (s, ψ') are probabilistic configuration and the strategy above includes some choice in the matching between successors of (φ', ψ') and (s, ψ'). Let $\varphi' = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and $\psi' = u$. Then,

$$P_{n+1}(s,u) = \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot P_n(s', \delta(u, L(s)))$$

Recall that the way to extend the game from configuration (φ', u) (matching a move to $\delta(q_i, L(s))$ with the move to $(s', \delta(u, L(s))))$ depends on which $a_{i,s'}$ are positive in a disjoint function f.

$$P_{n+1}(\varphi', u) = \Sigma_{i \in [n]} \underset{i : a_{i,s'} > 0}{max} \mathsf{val}(s', \delta(q_i, L(s))) \cdot P_n(\delta(q_i, L(s)), \delta(u, L(s)))$$

We now assume by induction that for possible matching triplet (s, φ'), (φ', ψ'), and (s, ψ') we have:

$$P_n(s, \psi') \geq \mathsf{val}(s, \varphi') \cdot P_n(\varphi', \psi')$$

and prove the same for P_{n+1}. We concentrate on the only interesting case, where $\varphi' = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and

$\psi' = u$:

$$P_{n+1}(s,u) = \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot P_n(s', \delta(u, L(s)))$$

By induction, where $i_{s'}$ is such that $\mathsf{val}(s', \delta(q_{i'_s}, L(s))) \cdot P_n(\delta(q_{i'_s}, L(s)), \delta(u, L(s)))$ is maximal among all $i \in [n]$.

$$\geq \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$$

By $\Sigma_{i \in [n]} a_{i,s'} = 1$, for all s'.

$$= \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot (\sum_{i \in [n]} a_{i,s'}) \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot$$

By choice of $i_{s'}$ to maximize $\mathsf{val}(s', \delta(q_{i'_s}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$ as maximal.

$$\geq \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot \sum_{i \in [n]} a_{i,s'} \cdot \mathsf{val}(s', \delta(q_i, L(s))) \cdot P_n(\delta(q_i, L(s)), \delta(u, L(s)))$$

By choice of f and win in G_A, we have $\mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \geq f(i_{s'}, s')$

$$\geq \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot \sum_{i \in [n]} a_{i,s'} \cdot f(i_{s'}, s') \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$$

Change order of summation.

$$= \sum_{i \in [n]} P_n(\delta(q_i, L(s)), \delta(u, L(s))) \cdot \sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot f(i, s') \cdot P(s,s')$$

By choice of f and $a_{i,s'}$ we have $\sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot f(i,s') \cdot P(s,s') \geq p_i$.

$$\geq \sum_{i \in [n]} P_n(\delta(q_i, L(s)), \delta(u, L(s))) \cdot p_i = P_{n+1}(\varphi', u)$$

B Ancillary Material: Wrong Embedding of Markov Chains

We show that a simpler conversion of Markov chains to automata produces automata that accept Markov chains that are not necessarily bisimilar to the original. Our example uses a Markov chain where no distinct locations are bisimilar.

Consider a Markov chain $M = (S, P, L, s^{\text{in}})$. We suggest the following "very-weak" embedding of a Markov chain in an automaton. Let $\Sigma = 2^{\mathbb{AP}}$. We define the following p-automaton $A^w_M = \langle \Sigma, Q, \delta, \varphi^{\text{in}}, \alpha \rangle$, where

$$\begin{aligned}
Q &= \{(s,s') \mid P(s,s') > 0\} \\
\varphi^{\text{in}} &= \bigwedge_{s' \in \mathsf{succ}(s^{\text{in}})} [\![(s^{\text{in}}, s')]\!]_{\geq P(s^{\text{in}}, s')} \\
\alpha &= Q \\
\delta((s,s'), \sigma) &= \bigwedge_{\{s'' \mid P(s',s'') > 0\}} [\![(s', s'')]\!]_{\geq P(s',s'')} \qquad \text{if } \sigma = L(s) \\
\delta((s,s'), \sigma) &= \quad \mathsf{ff} \quad \text{if } \sigma \neq L(s)
\end{aligned}$$

A state (s, s') represents the transition from s to s'. Unlike the automaton defined in Section 5.2, this automaton uses $\wedge$ instead of $*$.

Consider the Markov chain M in Figure 12 and let M_1 be M with s_1 as initial location, and let M_2 be M with s_2 as initial location. We show that M_1 and M_2 are not bisimilar and that $A^w_{M_1}$ accepts M_2.

Lemma 3 $M_1 \not\sim M_2$.

Proof: The transitions from s_1 to locations whose label is b have probability $\frac{2}{3}$ and the transitions from s_2 to locations whose label is b have probability $\frac{1}{3}$. □

Lemma 4 *The automaton $A^w_{M_1}$ accepts M_2.*

Proof: The initial configuration is $(s_2, \varphi^{\text{in}})$. As φ^{in} is a conjunction, Player 1 can choose one of three successor configurations: $(s_2, [\![s_1, s_3]\!]_{\geq \frac{1}{3}})$, $(s_2, [\![s_1, s_4]\!]_{\geq \frac{1}{3}})$, and $(s_2, [\![s_1, s_5]\!]_{\geq \frac{1}{3}})$. One can see that Player 0 wins from the latter two.

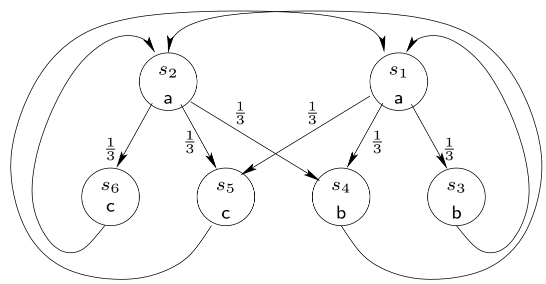

Figure 12. Markov chain whose uniform weak embedding accepts non-bisimilar Markov chains

Suppose that Player 1 chooses the configuration $(s_2, [\![s_1, s_3]\!]_{\geq \frac{1}{3}})$. Player 0 chooses the configuration $(s_2, [\![(s_1, s_3)]\!]_{\geq \frac{1}{3}}, f)$ where f is the function that sets $f(1, s_4) = 1$ and $f(1, s_5) = f(1, s_6) = 0$. The next configuration is $(s_4, [\![s_3, s_1]\!]_{\geq 1}, 1)$. We complete a cycle by going back to configuration $(s_2, \varphi^{\text{in}})$.

This completes a winning strategy for Player 0. □