TOPICAL CATEGORIES OF DOMAINS

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Steven Vickers,
Department of Computing,
Imperial College,
180 Queen's Gate,
London,
SW7 2BZ.

email: sjv@doc.ic.ac.uk

Abstract

It is shown how many techniques of categorical domain theory can be expressed in the general context of topical categories (where "topical" means internal in the category **Top** of Grothendieck toposes with geometric morphisms). The underlying topos machinery is hidden by using a geometric form of constructive mathematics, which enables toposes as "generalized topological spaces" to be treated in a transparently spatial way, and also shows the constructivity of the arguments. The theory of strongly algebraic (SFP) domains is given as a case study in which the topical category is Cartesian closed.

Properties of local toposes and of lifting of toposes (sconing) are summarized, and it is shown that the category of toposes has a fixpoint object in the sense of Crole and Pitts. This is used to show that for a local topos, all endomaps have initial algebras, and this provides a general context in which to describe fixpoint constructions including the solution of domain equations involving constructors of mixed variance. Covariance with respect to embedding-projection pairs or adjunctions arises in a natural way.

The paper also provides a summary of constructive results concerning Kuratowski finite sets, including a novel strong induction principle; and shows that the topical categories of sets, finite sets and decidable sets are not Cartesian closed (unlike the cases of finite decidable sets and strongly algebraic domains).

1. Introduction

1.1 "Stone-Grothendieck mathematics"

"Always topologize!" (Stone 1938)
"A topos is a generalized topological space." (Grothendieck 1972)

Taken together, these two dicta imply a general mathematical programme of topologization in which classes are replaced by toposes: instead of the class of widgets we consider the topos *classifying* widgets (i.e. the topos whose points are widgets). The topos not only determines the class of widgets, but simultaneously (and inseparably) embodies the generalized topological structure on the generalized space of widgets.

This resort to toposes may seem at first sight an unacceptably burdensome technical overhead, but in fact the practical mathematical consequences can be surprisingly unobtrusive. Toposes classify geometric theories, and it is quite in order to treat a topos as a "space" whose points are the models of the theory and to treat a geometric morphism as a transformation of points of one such space into points of another. The required "continuity" of these transformations is guaranteed by a constructive discipline which is in fact the mathematical core, expressed in a slogan "continuity = uniformity + geometricity". Our Stone-Grothendieck generalized topologization is really a matter of doing mathematics geometrically. Introductory accounts of the ideas are in Vickers (1996, 1997a); more technical justification is in Vickers (1997b).

However, it has to be admitted that it is not immediately evident how to do this painlessly. To a great extent this is because of a dual nature of toposes: as Mac Lane and Moerdijk (1992) say right at the outset, a topos can be considered both as a "generalized topological space" and as a "generalized universe of sets". But the formal definitions say that the topos *is* the generalized universe of sets, and in these terms it is extremely difficult to sustain the generalized space view. Hence although this view is a fundamental one of which experienced topos theorists are fully aware, it tends to get obscured in the exposition.

We shall explicitly separate the two viewpoints by reserving the word *topos* for the generalized spaces. (This runs counter to the general usage, but has etymological support in that it treats toposes as the objects of which *topo*logy is the study.) The generalized universes of sets – specifically, those categories (otherwise known as Grothendieck toposes) that satisfy the conditions of Giraud's theorem (see Johnstone (1977)) – will be called *geometric universes* or *GU*s (in Vickers (1993a, 1995a, 1995b) they are called Giraud frames or G-frames). The distinction is analogous to that between locales and frames (in Johnstone (1982); or, in Joyal and Tierney (1984), the distinction between spaces and locales), and indeed we hope that the techniques of spatial reasoning for locales that are investigated in Vickers (1995) can be developed for toposes too. Similarly, a GU *homomorphism* will be a functor that preserves finite limits and arbitrary colimits – hence, the inverse image part of a geometric morphism. For a topos D, the corresponding geometric universe will be written $\mathcal{S}$D.

The present paper is in large measure a case study for this topologization programme in which it is applied to domain theory, and one deliberate aim is to give a topos-theoretic account that looks as much as possible just like constructive domain theory. A preliminary account in the form of lecture slides has already appeared (Vickers 1992a).

Let us now lay down the ground rules for this geometric mathematics (technical justification is in Vickers (1997b)).

(1) "Geometric" mathematics comprises those constructions and properties that can be interpreted in any geometric universe and are preserved by GU homomorphisms.

(2) If a certain structures are described as being the models of a geometric theory T, that is to say, they are specified by structure and properties within geometric mathematics, then there is a corresponding "classifying" topos [T] of which those structures are the points.

Recall the usual notion of a *geometric theory presentation* – a many-sorted, infinitary, first-order theory presentation, in which the axioms take the form $\phi \vdash_{\mathbf{x}} \psi$. Here $\mathbf{x}$ is a finite list of sorted variables and ϕ and ψ are *geometric* formulae (the only connectives allowed are arbitrary disjunction, finitary conjunction, sorted equality and existential quantification) whose free variables are all taken from $\mathbf{x}$. (Details can be found in Makkai and Reyes (1977); also in Johnstone (1977) and Mac Lane and Moerdijk (1992), though for simplicity they treat the *coherent* theories, restricted to the finitary logic.) We shall be more liberal and admit presentations that use geometric constructions as type constructors, to create new types out of the given sorts (the base types). Function and predicate symbols will be allowed to use the derived types in their arities.

(3) If such a theory in (2) is "essentially propositional", that is to say, it has no sorts (other than what can be constructed geometrically out of thin air), then the corresponding topos is actually a locale. (We have no need to distinguish between locales and localic toposes, since our notation explicitly distinguishes between frames ΩD and generalised universes $\mathcal{S}D$.)

(4) Suppose D and E are two toposes. Then construction of points of E out of points of D, if it is geometric, describes a geometric morphism (or *map*) from D to E.

(5) Geometric morphisms between locales are the same as continuous maps.

Consequently, we describe a locale or a topos by giving a geometric description of its points; and we describe a continuous map or a geometric morphism by giving a geometric description of how it transforms points to points. No discussion of topology is needed – the geometricity already covers that –, and so locales appear as "topology-free spaces".

We shall examine what is allowed in this geometric mathematics, but first let us mention some things that are *not* allowed.

- *The logic is non-classical.* Intuitionistic logic is valid in geometric universes, but in general excluded middle and choice are not valid. More subtly, intuitionistic negation is not preserved by GU homomorphisms, and nor are implication and universal quantification – so we can't use them in general, though we shall on occasion use the intuitionistic formulae in proving geometric results. The geometric logic is therefore more restricted than intuitionistic logic. However, if we can prove or postulate that two propositions P and Q are logical complements ($P \wedge Q \vdash$ **false**, **true** $\vdash P \vee Q$), then that fact is preserved by GU homomorphisms and so gives an instance of a geometric negation.
- We can't use exponentials X^Y, powersets $\mathcal{P}X$, or the subobject classifier Ω – none of these is preserved by GU homomorphisms.

I shall not attempt to formalize the geometric constructions, but they include finite limits, set-indexed colimits, image factorization, monicness, epiness, inclusion between subobjects, finite intersections and arbitrary set-indexed unions of subobjects, existential quantification, free algebra constructions, ℕ (natural numbers), ℚ (rationals), Kuratowski finiteness, finite powersets (free semilattices), universal quantification bounded over finite objects.

A couple of specific issues worth mentioning are decidability and finiteness. Equality is part of the geometric logic, but inequality is not (because there is no negation). Nonetheless, certain "decidable" sets come equipped with inequality, a relation complementary to equality – two good examples are ℕ and ℚ. Finiteness is – as remarked above – *Kuratowski finiteness* (Johnstone 1977): X is Kuratowski finite iff the free semilattice $\mathcal{F}X$ has an element T such that $\{x\} \subseteq T$ for every x. This notion can sometimes behave surprisingly. For instance, subsets of finite sets, or intersections of finite subsets, need not themselves be finite. Section 2.1 provides a technical discussion.

Notes –

(i) If T is a geometric theory, then the corresponding geometric universe $\mathcal{S}$[T] is exactly the category that is usually referred to as the classifying topos of T. The notation can be read either as $\mathcal{S}$heaves over the topos [T], or as $\mathcal{S}$ets with an adjoined generic model of T.

(ii) When we refer to the points of [T], the models of T, these models might be in an arbitrary geometric universe $\mathcal{S}$D. D is known as the *stage of definition* of the point, and the theory of classifying toposes shows that points of [T] at stage D are the same as maps from D to [T]. Models in the initial GU $\mathcal{S} = \mathcal{S}1$ of sets, i.e. maps from 1 to [T], are known as *global* points.

(iii) If f and g are two maps from [T] to [T'], then a natural transformation from f to g is a geometric construction, given a model M of T, of a homomorphism from f(M) to g(M).

(iv) Toposes, maps and natural transformations are the 0-, 1- and 2-cells of a 2-category **Top**. We shall look at it more closely later, but let us note immediately that the hom-categories **Top**([T], [T']) (which is equivalent to the category of models of T' in $\mathcal{S}$[T]) are not arbitrary categories – they have all filtered colimits (Johnstone 1977).

1.2 Topologizing domain theory

It has long been recognized that domains are topological spaces under their Scott topology. Normally, they are also sober (by Johnstone (1981) this is not true of arbitrary dcpos, but it holds for all continuous posets) and hence can be equivalently treated as locales. By the remarks above, therefore, domains are normally also toposes. It turns out that domain theoretic constructions such as products, coproducts and exponentials are special cases of the more general topos constructions, and we shall prove this. In particular, the existence of least fixpoints for continuous endomaps of domains with bottom turns out to be a special case of the existence of initial fixpoints for arbitrary endomaps of local toposes (toposes with initial points) – in effect, local toposes are algebraically complete in the appropriately transferred sense of Freyd (1991).

However, the methods go considerably beyond this. Nice enough domains can be presented by information systems of various flavours (e.g. Larsen and Winskel 1984; Vickers 1993; or indeed the slightly different methods of Abramsky 1991) which are the models of geometric theories, and moreover the continuous maps between the domains are equivalent to "approximable mappings" between information systems, which are also the models of geometric theories. Fixing a flavour of information system, we therefore get two toposes [IS] and [AM]. (AM is the theory of two information systems and an approximable mapping between them.) We also have maps src and tar: [AM] → [IS] giving the source and target, a map id: [IS] → [AM] giving the identity approximable mappings, and more that in short make an internal category in **Top** – a *topical category.* (The fact that **Top** is a 2-category greatly complicates the idea of internal category in it, and a definitive account of such things (Hyland and Moerdijk unpublished) hasn't appeared yet. However, the topical categories we study will all in a certain sense represent full subcategories of **Top**, in that the approximable mappings correspond to arbitrary maps (as geometric morphisms) between the corresponding domains (as toposes), and this gives us a somewhat more solid base on which to rest the internal category structure.) We find that the topical category has, internally, much of the structure of the corresponding category of domains, and in particular for strongly algebraic domains the topical category is internally Cartesian closed. This is a stronger result than appears at first sight, for with some other well-known CCCs such as **Set**, the Cartesian closedness is *not* internal in the corresponding topical category: essentially this is because exponentiation of sets is not geometric.

The topos setting now begins to pay off more decisively. In particular, we can use the result mentioned above on algebraic completeness of local toposes to find not only fixpoints within domains, but also fixpoints among domains, i.e. solutions of domain equations. This is most easily seen for domains with bottom, when [IS] is local (the singleton information system $\{\bot\}$ is initial) and any map F: [IS] $\rightarrow$ [IS], i.e. any uniform, geometric construction of information systems from information systems, has an initial algebra: this will solve the domain equation D = F(D). The key point is that toposes automatically have all the filtered colimits that abstract categorical domain theory has to postulate, and the uniform, geometric definitions of geometric morphisms suffice to give us the required continuity, preservation of these filtered colimits.

Note that F is necessarily functorial, but that is with respect to the homomorphisms between information systems – 2-cells in **Top** – and not the approximable mappings. In the strongly algebraic case, which is internally Cartesian closed,we have a map F(X) = [X$\Rightarrow$X] that is not functorial with respect to continuous maps. However, the homomorphisms turn out to correspond to adjunctions between the domains so that we painlessly discover the well-known technical trick from domain theory that regains functoriality. (Actually, domain theory normally uses embedding-projection pairs, not general adjunctions. The difference corresponds to the constructivist issue of whether the information system order is decidable or not.)

1.3 Overview of the paper

Following this introduction, we move in Section 2 to the technical background. Much of this is already known, and I think none of it will cause any surprise to topos theorists though perhaps some of the detailed proofs have not been set out before. However, I do not know of convenient references and certainly not in the "generalized space" language that I am trying to use.

In section 3 we look at some examples of topical categories, and in particular at two ways of constructing them. An "intrinsic" topical category captures the idea, given any topos D, of a category whose objects are points of D and whose morphisms are homomorphisms. These are simple, but inadequate for our domain theory. We need the slightly more complicated notion of "display" topical category. This starts from an exponentiable map p: E $\rightarrow$ D, and captures the idea of a category whose objects are points of D, but whose morphisms are maps between pullbacks of p.

Section 4 treats the particular case of strongly algebraic domains in some detail. Its domain-theoretic substance is, virtually entirely, taken from Abramsky (1991). Its purpose is not so much to present the results in a new way, different from Abramsky's – the apparent differences are actually ones of expositional taste rather than anything else – but to show how unobtrusive the new, topos-theoretic foundations are.

Section 5 addresses domain equations and their solution.

2. Technical background

This section gathers together diverse technical results under four headings:

2.1 Finite power sets
2.2 The 2-category **Top** of toposes
2.3 Lifting in **Top**
2.4 Algebraic dcpos

2.1 Finite power sets

The geometric account of finiteness (by which we mean *Kuratowski* finiteness (Kock et al. 1975; Johnstone 1977) has some unexpected behaviour, a notorious example being that subsets of finite sets need not themselves be finite (Kock et al. 1975). Nonetheless, it fits well with observational intuitions that a set is finite iff you can give a finite list of all its elements. (But note that if equality is not decidable then you can't necessarily eliminate duplicates from the list.) Two finite sets are equal iff every element of each is also an element of the other. To understand the paradox of subsets, suppose S is finite and $T = \{s \in S: \phi(s)\}$. To list *all* the elements of T, we also need negative information $\neg\phi(s)$ in order to know which elements of S can be omitted from the list.

We recap here some basic properties and constructions relating to finite sets, and in particular the fact that *bounded universal quantification over a finite set is geometric* (Johnstone and Linton 1978). Much of this seems to be well-known folklore, but I don't know of any convenient reference for the ideas and shall summarize them here.

The first step is to construct the finite power set $\mathcal{F}X$ over X, and this is done as the free (join) semilattice. As it happens, by a theorem of Mikkelson this can be constructed in any elementary topos as the $\cup$-subsemilattice of $\wp X$ generated by the singletons (see Theorem 9.16 in Johnstone 1977). However, in the context of geometric universes it is perhaps more convenient to see the construction as a special case of the existence of free algebras for *any* single-sorted algebraic theory that is finitary enough (Theorem 6.43 in Johnstone 1977). Moreover, by Lemma 6.44 there, the free algebra construction is preserved by GU homomorphisms: in other words, free algebra constructions are "geometric".

$\mathcal{F}X$ is the set of *Kuratowski finite* subsets of X. From now on we shall omit "Kuratowski": when we say *finite*, we mean Kuratowski finite.

We have already noted that subsets of finite sets need not be finite; here are some other unexpected behaviours.

- Finite unions of finite sets are undoubtedly finite (just concatenate the lists of elements), but finite intersections are not. For a start, the empty intersection of finite subsets of X is the whole of X, which certainly need not be finite. More subtly, if S and T are finite then $S \cap T$ need not be because to discover what are all the elements of $S \cap T$ you must be able to determine the negative information of when $x \notin S$ (or T).

- The cardinality of a finite set is not defined in general. To know that you have counted exactly how many elements there are in {w,x,y,z}, you need to know all the equalities *and inequalities* amongst the elements, and the negative information is not always available geometrically.

(Often the problem is one of decidability, i.e. lack of negative information. For instance, if ϕ is decidable and S is finite then $\{u \in S: \phi(u)\}$ *is* finite; and if X has decidable equality, a binary predicate $\neq$ that's a complement of =, then $\mathcal{F}X$ has binary intersections and there is a cardinality function from $\mathcal{F}X$ to $\mathbb{N}$.)

Definition 2.1.1 *(Finitely bounded universal quantification)* Let $\phi(x, y)$ be a predicate on X×Y. Then the predicate $\forall x \in S.\ \phi(x, y)$ on $\mathcal{F}X \times Y$ is defined as

$$\bigvee_{n \in \mathbb{N}} \exists x_1,\ldots,x_n.\ (S = \{x_1,\ldots,x_n\} \wedge \bigwedge_{i=1}^{n} \phi(x_i, y))$$

(An alternative and perhaps more correct definition would make direct use of the free semilattice property of $\mathcal{F}X$.) The definition makes explicit that this is a geometric construction on ϕ. To show that it really is bounded universal quantification, one shows the characterizing proof theoretic adjunction:

Proposition 2.1.2 Let $\phi(x, y)$ and $\psi(y)$ be predicates on X×Y and on Y. Then

$$\psi(y) \vdash_{S,y} \forall x \in S.\ \phi(x,y) \qquad \text{iff} \qquad \psi(y) \wedge x \in S \vdash_{S,y,x} \phi(x,y)$$]

Note that if $f: X \to \wp Y$ corresponds to $\phi(x,y)$ (i.e. $\phi(x,y)$ iff $y \in f(x)$), then this extends to a unique semilattice homomorphism from $\mathcal{F}X$ to $\wp Y$ under $\cap$, and this corresponds to $\forall x \in S.\ \phi(x,y)$.

Next, we give some basic inductive and recursive tools for dealing with finite sets.

Theorem 2.1.3 *(Simple $\mathcal{F}$-induction)*
Let $\phi(S)$ be a predicate on $\mathcal{F}X$ such that $\phi(\emptyset)$ (base case), and if $\phi(S)$ then $\phi(\{x\} \cup S)$ for all x:X (induction step). Then $\phi(S)$ holds for all S.
Proof Let M be the subset of $\mathcal{F}X$ comprising those elements S for which $\forall T{:}\mathcal{F}X.\ (\phi(T) \to \phi(S \cup T))$. M is a subsemilattice, and by the induction step it contains the singletons, so it is the whole of $\mathcal{F}X$. From $S \in M$, and the base case $\phi(\emptyset)$, we deduce $\phi(S)$.

Note that although the statement of this Theorem is geometric, the proof is not – it uses intuitionistic formulae. We conjecture that there is a geometric proof.]

(In 2.1.11 we shall prove a stronger induction principle.)

Lemma 2.1.4 *($\mathcal{F}$-recursion)*
Let $f: X \times Y \to Y$ satisfy

(i) $\forall x,x',y.\ f(x, f(x', y)) = f(x', f(x, y))$
(ii) $\forall x,y.\ f(x, f(x, y)) = f(x, y)$

Then there is a unique $g: \mathcal{F}X \times Y \to Y$ such that

$$\forall y.\ g(\emptyset, y) = y$$
$$\forall x,y.\ g(\{x\}, y) = f(x, y)$$
$$\forall S,T,y.\ g(S \cup T, y) = g(S, g(T, y))$$

Proof Let f': $X \rightarrow Y^Y$ be the curried form of f. Let M_0 be the image of f' in Y^Y, which is a monoid under composition, and let M be the submonoid generated by M_0. Conditions (i) and (ii) say that the elements of M_0 are commuting idempotents. Because they commute, M is commutative, for consider the centralizer of M_0 in Y^Y – the set of elements that commute with everything in M_0. This is a submonoid containing M_0, and hence containing all of M, and so everything in M commutes with everything in M_0. Therefore the centralizer of M contains M_0 and hence all of M, so M is commutative. Now we can show that the set of idempotent elements of M is a submonoid containing all of M_0, and hence is the whole of M, so M is a semilattice. It follows that f' factors uniquely via a semilattice homomorphism g': $\mathcal{F}X \rightarrow M$, which uncurries to the required g.]

Theorem 2.1.5 Let f: $X \times Y \rightarrow Y$ satisfy conditions (i) and (ii) of 2.1.4, and let y_0: Y. Then there is a unique h: $\mathcal{F}X \rightarrow Y$ satisfying

- $h(\emptyset) = y_0$
- $\forall x,S.\ h(\{x\} \cup S) = f(x, h(S))$

Proof Let g be the function obtained in Lemma 2.1.4, and define $h(S) = g(S, y_0)$. Then $h(\emptyset) = g(\emptyset, y_0) = y_0$, and $h(\{x\} \cup S) = g(\{x\} \cup S, y_0) = g(\{x\}, g(S, y_0)) = f(x, h(S))$. Uniqueness follows by $\mathcal{F}$-induction.]

Using $\mathcal{F}$-induction, we can easily prove a number of results, all of which I should think are already known.

Theorem 2.1.6

(i) $\forall x \in S.\ (\phi(x) \vee \psi(x)) \vdash_{S:\mathcal{F}X} \exists S_\phi, S_\psi.\ (S = S_\phi \cup S_\psi \wedge \forall x \in S_\phi.\ \phi(x) \wedge \forall x \in S_\psi.\ \psi(x))$

(ii) (Decidable subsets of finite sets are finite: Kock et al. (1975).)
If S is finite and $\phi(x)$ is decidable, then $\{x \in S: \phi(x)\}$ is finite.
(Use (i) with ψ the complement of ϕ.)

(iii) (Johnstone 1984) $\forall x \in S.\ (\phi(x) \vee \psi(x)) \vdash_{S:\mathcal{F}X} \forall x \in S.\ \phi(x) \vee \exists x \in S.\ \psi(x)$
Note that the analogous deduction with S infinite is intuitionistically unsound, so this result is saying something about finite boundedness. It is directly analogous to the relation "$\Box(\phi \vee \psi) \vdash \Box\phi \vee \Diamond\psi$" seen in the Vietoris powerlocale.

(iv) If ϕ is decidable, with complement ψ, then $\forall x \in S.\ \phi(x)$ and $\exists x \in S.\ \psi(x)$ are complements.

(v) If X has decidable equality, then on $\mathcal{F}X$ we have that $\in$ is decidable ($x_0 \notin S$ is equivalent to $\forall x \in S.\ x \neq x_0$), that the intersection of two finite sets is still finite (use (ii) with $S \cap T = \{x \in S.\ x \in T\}$; see Acuña-Ortega and Linton (1979)) and that each finite set has a cardinality.

(vi) $\forall x \in S.\ \exists y{:}Y.\ \phi(x,y) \vdash_{S:\mathcal{F}X} \exists U{:}\mathcal{F}(X \times Y).\ (\mathrm{fst}(U) = S \wedge U \subseteq \phi)$]

We shall now use $\mathcal{F}$-induction and recursion to prove a sequence of finiteness results: that if S and T are finite, then so are $S \times T$, $\mathcal{F}S$ and the set FT(S, T) of finite total relations from S to T. The framework of the proof is the same in each case, and can be illustrated with $S \times T$. For arbitrary types X and Y, $\times$ can be treated as a function from $\mathcal{F}X \times \mathcal{F}Y$ to $\mathcal{F}(X \times Y)$. Defining the function is not too difficult (using $\mathcal{F}$-recursion, the free semilattice property and so on), but more important to us is its *specification*, that $S \times T = \{(x,y): x \in S \wedge y \in T\}$ – in other words,

$$(x,y) \in S \times T \dashv\vdash_{x:X, y:Y, S:\mathcal{F}X, T:\mathcal{F}Y} x \in S \wedge y \in T$$

To show that the recursive definition works correctly, i.e. that it satisfies its specification, one can use $\mathcal{F}$-induction in a routine sort of way, but in practice this amounts to an assumption that the recursive calls work correctly and we shall make this assumption without comment. (Compare this with the method of *recursion variants* as set out in Morgan (1990) or Broda et al. (1994).)

Proposition 2.1.7 (Kock et al. 1975) If S and T are finite then so is $S \times T$.

Proof

Let X, Y be any types. We define $\times$: $\mathcal{F}X \times \mathcal{F}Y \to \mathcal{F}(X \times Y)$ such that

$$(x,y) \in S \times T \dashv\vdash_{x:X, y:Y, S:\mathcal{F}X, T:\mathcal{F}Y} x \in S \wedge y \in T$$

If b: Y, then there is a unique semilattice homomorphism from $\mathcal{F}X$ to $\mathcal{F}(X \times Y)$, written $S \mapsto S \times \{b\}$, such that $\{a\} \times \{b\} = \{(a, b)\}$. We have $(x,y) \in S \times \{b\} \dashv\vdash_{x:X, y:Y, S:\mathcal{F}X} x \in S \wedge y = b$, for $\{S: \mathcal{F}X \mid \forall x:X, y:Y.\ (x,y) \in S \times \{b\} \leftrightarrow x \in S \wedge y = b\}$ is a subsemilattice of $\mathcal{F}X$ containing the generators $\{a\}$. (Note how we have temporarily dipped into intuitionistic logic in order to derive a geometric result.) Now fixing S: $\mathcal{F}X$, we can define $T \mapsto S \times T$ to be the unique semilattice homomorphism from $\mathcal{F}Y$ to $\mathcal{F}(X \times Y)$ such that $S \times \{b\}$ is as already defined. Then $\{T: \mathcal{F}Y \mid \forall x:X, y:Y.\ (x,y) \in S \times T \leftrightarrow x \in S \wedge y \in T\}$ is a subsemilattice of $\mathcal{F}X$ containing the generators $\{b\}$ and so contains every T. This shows that the definition satisfies the specification.]

Corollary 2.1.8

(i) $\vdash_{S:\mathcal{F}X,\ T_1, T_2:\mathcal{F}Y} S \times (T_1 \cup T_2) = S \times T_1 \cup S \times T_2$

(ii) $\vdash_{S_1, S_2:\mathcal{F}X,\ T:\mathcal{F}Y} (S_1 \cup S_2) \times T = S_1 \times T \cup S_2 \times T$

Proof (i) Of course, this is immediate from the *construction* of $\times$ in Proposition 2.1.7, but we can also prove it from the *specification:* $(x,y) \in S \times (T_1 \cup T_2) \Leftrightarrow x \in S \wedge y \in (T_1 \cup T_2) \Leftrightarrow x \in S \wedge (y \in T_1 \vee y \in T_2) \Leftrightarrow (x \in S \wedge y \in T_1) \vee (x \in S \wedge y \in T_2) \Leftrightarrow (x,y) \in S \times T_1 \cup S \times T_2$.

(ii) is less immediate from the construction but follows just as easily from the specification.]

Proposition 2.1.9 (Kock et al. 1975) If S is finite, then so is $\mathcal{F}S$.

Proof

If X is any type, we desire a function $\mathcal{F}$: $\mathcal{F}X \to \mathcal{F}\mathcal{F}X$ such that $T \in \mathcal{F}(S) \dashv\vdash_{S,T:\mathcal{F}X} T \subseteq S$. Let $\mathcal{F}$ be the unique function such that

$$\mathcal{F}(\emptyset) = \{\emptyset\}$$
$$\mathcal{F}(\{a\} \cup S) = \mathcal{F}(S) \cup \{\{a\} \cup T \mid T \in \mathcal{F}(S)\}$$

($\{\{a\} \cup T \mid T \in \mathcal{F}(S)\}$ is the direct image of $\mathcal{F}(S)$ under the function from $\mathcal{F}X$ to itself that maps T to $\{a\} \cup T$.) Of course, we must check the conditions for $\mathcal{F}$-recursion. In other words, if $\mathcal{U}$: $\mathcal{F}\mathcal{F}X$, then we want

$$(\mathcal{U} \cup \{\{a\} \cup T \mid T \in \mathcal{U}\}) \cup \{\{b\} \cup U \mid U \in \mathcal{U} \cup \{\{a\} \cup T \mid T \in \mathcal{U}\}\}$$
$$= \ldots \text{ same thing with a and b interchanged}$$

which is clear because the expression reduces to

$$\mathcal{U} \cup \{\{a\} \cup T \mid T \in \mathcal{U}\} \cup \{\{b\} \cup U \mid U \in \mathcal{U}\} \cup \{\{a\} \cup \{b\} \cup T \mid T \in \mathcal{U}\}$$

Also, we want that when a = b the expression reduces to $\mathcal{U} \cup \{\{a\} \cup T \mid T \in \mathcal{U}\}$, which again is clear.

Now we must show that the definition satisfies the specification. $T \in \mathscr{F}(\emptyset) \dashv\vdash T = \emptyset \dashv\vdash T \subseteq \emptyset$, and it remains to show the case for $\{a\} \cup S$.

$$T \in \mathscr{F}(\{a\} \cup S) \dashv\vdash T \in \mathscr{F}(S) \vee \exists U \in \mathscr{F}(S).\ T = \{a\} \cup U$$
$$\dashv\vdash T \subseteq S \vee \exists U{:}\mathcal{F}\mathcal{F}X.\ (U \subseteq S \wedge T = \{a\} \cup U)$$

(Note the assumption, justified as an $\mathcal{F}$-induction hypothesis, that $T \in \mathscr{F}(S) \dashv\vdash T \subseteq S$.) Certainly this implies that $T \subseteq \{a\} \cup S$. For the converse, if $T \subseteq \{a\} \cup S$ then we can find T_1 and T_2 in $\mathcal{F}X$ such that $T = T_1 \cup T_2$, $T_1 \subseteq \{a\}$ and $T_2 \subseteq S$. If $T_1 = \emptyset$ then $T = T_2 \subseteq S$, while if T_1 is inhabited then it is $\{a\}$ and so $T = \{a\} \cup T_2$ with $T_2 \subseteq S$.]

We also write $\mathscr{F}_1(S)$ for the set of inhabited elements of $\mathscr{F}(S)$, i.e. those $T \subseteq S$ satisfying the decidable predicate $\exists y.\ y \in T$ (its complement is $T = \emptyset$).

The following proposition is included not for its general importance, but because it is used later on (in section 4.5) at a point where one might more naturally expect to use the set of functions from S to T. However, for finiteness of the set of functions we should require decidability of S so that single valuedness of a relation R could be expressed as $\forall (x,y),(x',y') \in R.\ (x \neq x' \vee y = y')$.

Lemma 2.1.10 If S and T are finite, then so is the set FT(S, T) of finite total relations from S to T.

Proof

If X and Y are types then we desire $FT: \mathcal{F}X \times \mathcal{F}Y \to \mathcal{F}\mathscr{F}(X \times Y)$ such that

$$R \in FT(S, T) \dashv\vdash_{R:\mathscr{F}(X \times Y)} R \subseteq S \times T \wedge \forall x \in S.\ \exists y \in T.\ (x, y) \in R$$

We define FT to be the unique function such that

$$FT(\emptyset, T) = \{\emptyset\}$$
$$FT(\{a\} \cup S, T) = \{R \cup \{a\} \times T' \mid R \in FT(S, T) \wedge T' \in \mathscr{F}_1(T)\}$$

Again, it is not hard to show that this definition satisfies the conditions for $\mathcal{F}$-recursion.

When $S = \emptyset$ we have $R \in FT(\emptyset, T) \dashv\vdash R = \emptyset \dashv\vdash R \subseteq \emptyset \times T \wedge \forall x \in \emptyset.\ \exists y \in T.\ (x, y) \in R$ as required. For the other case,

$$R \in FT(\{a\} \cup S, T) \dashv\vdash \exists R',T'.\ (R' \in FT(S, T) \wedge T' \in \mathscr{F}_1(T) \wedge R = R' \cup \{a\} \times T')$$
$$\dashv\vdash \exists R',T'.\ (R' \subseteq S \times T \wedge \forall x \in S.\ \exists y \in T.\ (x, y) \in R' \wedge T' \in \mathscr{F}_1(T) \wedge R = R' \cup \{a\} \times T')$$

This certainly implies that $R \subseteq (\{a\} \cup S) \times T$ and $\forall x \in \{a\} \cup S.\ \exists y \in T.\ (x, y) \in R$. For the converse, from $R \subseteq (\{a\} \cup S) \times T = \{a\} \times T \cup S \times T$ we deduce that there are finite R_1 and R_2 such that $R = R_1 \cup R_2$, $R_1 \subseteq \{a\} \times T$ and $R_2 \subseteq S \times T$. We can find $b \in T$ such that $(a,b) \in R$; let $R_1' = R_1 \cup \{(a,b)\} \subseteq \{a\} \times T$. If T' is the direct image of R_1' under the projection to T then $T' \in \mathscr{F}_1(T)$ (inhabited because it contains b), and $R_1' = \{a\} \times T'$. Next, $\forall x \in S.\ \exists y \in T.\ (x, y) \in R$ and so if $S = \{x_1, \ldots, x_n\}$ we can find $\{y_1, \ldots, y_n\} \subseteq T$ such that $(x_i, y_i) \in R$ $(1 \leq i \leq n)$. Let $R' = R_2 \cup \{(x_1,y_1), \ldots, (x_n, y_n)\}$. $R' \subseteq S \times T$, $\forall x \in S.\ \exists y \in T.\ (x, y) \in R'$, and $R = R' \cup \{a\} \times T'$, so we have found R' and T' as required.]

We finish this section by strengthening the principle of $\mathcal{F}$-induction considerably, strengthening the induction hypothesis. (The only place where we need the stronger principle is in our account of Abramsky's normalization result for function spaces, Section 4.5.)

Theorem 2.1.11 *The principle of strong $\mathcal{F}$-induction*
Let $P \subseteq \mathcal{F}X$ be a predicate satisfying –

$$\forall x \in S.\ \exists U{:}\mathcal{F}X.\ (S = \{x\} \cup U \wedge P(U)) \vdash_{S:\mathcal{F}X} P(S) \qquad (*)$$

Then P satisfies

$$\vdash_{S:\mathcal{F}X} P(S)$$

Proof Note that the induction hypothesis $\forall x \in S.\ \exists U \in \mathcal{F}(S).\ (S = \{x\} \cup U \wedge P(U))$ implies $S = \emptyset \vee \exists x{:}X.\ \exists U{:}\mathcal{F}X.\ (S = \{x\} \cup U \wedge P(U))$, which is a collected form of induction hypothesis for simple $\mathcal{F}$-induction, so this is a formally stronger induction principle: any proof that uses the simple induction principle can easily be turned into a proof using strong induction.

The proof of validity of the strong principle is by induction on the size n of a representation $S = \{x_1, \ldots, x_n\}$ (remember that in the absence of decidable equality we don't have a well-defined cardinality of S), and one role of the Theorem is to package up such induction and give a reasoning principle that does not have to refer to the representation. I am grateful to Paul Taylor for a discussion that led to a rigorous proof along these lines to replace a more complicated one that I originally had.

Let us write $\mathcal{B}X$ for the free commutative monoid over X. One should think of its elements as the finite bags, or multisets, over X. We write + (bag sum) and 0 (empty bag) for the monoid operation and its unit, and $\{|-|\}$ for the injection of generators (so $\{|x|\}$ is the singleton bag containing x). We also write $\#: \mathcal{B}X \to \mathbb{N}$ for the monoid homomorphism with $\#\{|x|\} = 1$ (so #B is the total size of B), and $\sigma: \mathcal{B}X \to \mathcal{F}X$ for the monoid homomorphism with $\sigma\{|x|\} = \{x\}$ (so σB is the set of elements of B).

It is straightforward to prove the following induction principle on $\mathcal{B}X$: if $P \subseteq \mathcal{B}X$ is such that P(0), and whenever P(B) then $P(\{|x|\}+B)$, then P(B) holds for all B. We can now show –

(1) If $\#B = 0$ then $B = 0$ (easy by bag induction)

(2) σ is onto: for the image of $\mathcal{B}X$ is a submonoid of $\mathcal{F}X$ that contains all the generators $\{x\}$.

(3) If $x \in \sigma B$ then there is some C such that $B = \{|x|\}+C$. The base case, $B = 0$, is obvious – $x \in \sigma 0$ is impossible. If $x \in \sigma(\{|y|\}+B') = \{y\} \cup \sigma B'$, then either $x = y$ or $x \in \sigma B'$. If $x = y$, then $\{|y|\}+B' = \{|x|\}+B'$. If $x \in \sigma B'$, then by induction $B' = \{|x|\}+C'$ for some C', so $\{|y|\}+B' = \{|x|\}+(\{|y|\}+C')$.

Now let P be as stated in the overall theorem. It suffices to show $\forall n{:}\mathbb{N}.\ Q(n)$, where

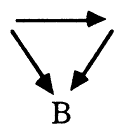
$$Q(n) \equiv_{def} \forall B{:}\mathcal{B}X.\ (\#B = n \to P(\sigma B))$$

When $n = 0$, we have $P(\emptyset)$ because the induction hypothesis (left-hand side of (*)) holds vacuously. Otherwise, suppose $\#B = n+1$. If $x \in \sigma B$ then $B = \{|x|\}+B'$ for some B', and $\#B' = n$. By induction on n we have $P(\sigma B')$, and $\sigma B = \{x\} \cup \sigma B'$. It follows from (*) that $P(\sigma B)$ as required.]

2.2 *The 2-category* **Top** *of toposes*

We shall describe here some aspects of categorical structure of the category **Top** of toposes (Grothendieck toposes with geometric morphisms between them) and of its slices **Top/B**. Though the constructions are well known, we shall need to describe them in terms of the theories classified – in effect, in terms of the points of the toposes.

It is worth bearing in mind that **Top** is in fact a *2-category:* each hom-class **Top**(D, E) is a category, and a large one at that (though locally small). As a consequence, it is generally too much to expect diagrams in **Top** to commute "on the nose", i.e up to equality – commuting is usually only up to isomorphism. In broad terms, this is because in a category equality between objects is less important that isomorphism. Moreover, universal properties should properly be described in a 2-categorical form. For instance a product D×E is a representing object for a functor from **Top** to **Cat**, taking a topos F to the category **Top**(F, D)×**Top**(F, E): "representing object" means that for every F the functor **Top**(F, D×E) to **Top**(F, D)×**Top**(F, E), mapping f to (f;fst, f;snd), is an equivalence of categories.

We shall need to work not only in **Top** itself, but also in the slice toposes **Top/B**. The 2-categorical laxness that we shall allow is that the morphisms, triangles of the form , are to commute up to a given isomorphism. An important issue will be whether the constructions we describe are preserved by the pullback functors between slice categories: in fact, they all are.

Terminal object

The terminal topos 1 classifies the empty theory (no vocabulary, no axioms). $\mathcal{S}1$ = **Set**.

Pullbacks

Let D and E be two toposes over a base B: in other words we are given geometric morphisms f: D $\rightarrow$ B and g: E $\rightarrow$ B. To avoid having to name too many things, weshall use restriction map notation so that (for instance) if x is a point of D then $x|_B = f(x)$.

The pullback $D \times_B E$ classifies triples (x, y, θ) where x and y are points of D and E, and $\theta: x|_B \cong y|_B$.

This construction covers pullbacks and binary products in slices **Top/C**.

Comma squares

Again let D and E be toposes over B. The comma object $D >_B E$ classifies triples (x, y, θ) where x and y are points of D and E, and $\theta: x|_B \rightarrow y|_B$ is a homomorphism. Again this covers comma squares in slices. A particular case of this is the *inserter,* when a single topos D lies over B in two different ways f, g: D $\rightarrow$ B. the comma object $(D,f) >_B (D,g)$ is the inserter for f to g.

Initial object

The initial topos Ø classifies the contradictory theory (no vocabulary, axiom **true** ⊢ **false**). To see why, suppose we have a point of Ø, in other words we have contradiction **false**. Then any interpretation of vocabulary will give a model for any theory, and any two interpretations will be isomorphic. The same topos Ø is initial in every slice, and is preserved by pullback.

Bagtoposes

If D is a topos, then there is also a topos B_LD, its lower bagtopos, whose points are pairs (S, $(x_\lambda)_{\lambda\in S}$) where S is a set and $(x_\lambda)_{\lambda\in S}$ is an S-indexed family of points of D. In terms of geometric theory presentations, the slightly intricate construction involves adding a new sort (for S) and functions from the old sorts to S in such a way that the fibres over elements of S are models of the old theory. This can be universally characterized as a partial product (Johnstone 1992); it is a notable example of a case where care is needed in giving a proper 2-categorical account of the universal property (Johnstone 1993). We shall not go into the details here but use these construction mainly to make geometric sense of phrases such as "set-indexed family of points".

Coproducts

Let D and E be toposes. Then their coproduct D+E classifies tuples (I, $(x_\lambda)_{\lambda\in I}$, J, $(y_\mu)_{\mu\in J}$, θ) where $(x_\lambda)_{\lambda\in I}$ is an I-indexed family of points of D, $(y_\mu)_{\mu\in J}$ is a J-indexed family of points of E, and θ: I+J ≅ 1. (Classically, of course, this is either a point of D or a point of E.)

We write it this way to make it clear using bagtoposes that we have a geometric theory, but in practice we can use a more perspicuous notation. The subsingletons I and J with I+J ≅ 1 are equivalent to a Boolean value (complemented proposition) $p \equiv \exists\lambda.\ \lambda \in I$, and the I- and J-indexed families are equivalent to a point x of D defined if p, and a point y of E if ¬p. Let us therefore write this point of D+E as a conditional **if** p **then** x **else** y. The injection inj_1: D → D+E maps x to **if true then** x **else** –, and inj_2 maps y to **if false then** – **else** y.

An important feature of the **if** … **then** … **else** … notation is that it embodies a filtered diagram. Consider **if** p **then** u **else** v, where u and v are points of a single topos. This gives a diagram whose shape is the discrete category {*∈ 1: p} + {*∈ 1: ¬p} ≅ 1, and whose nodes are u for each * in {*∈ 1: p} and v for each * in {*∈ 1: ¬p}. The diagram is filtered and hence has a colimit (another point of the same topos). We shall write "**if** p **then** u **else** v" to denote this filtered colimit.

To see that D+E is a coproduct, consider maps f: D → F and g: E → F. A point u = **if** p **then** x **else** y of D+E is isomorphic to the filtered colimit

if p **then** (**if true then** x **else** –) **else** (**if false then** – **else** y)
  = **if** p **then** inj_1 x **else** inj_2 y

Because geometric morphisms preserve filtered colimits of points, the copairing [f,g] has to map u to the colimit of the image diagram in F, namely **if** p **then** f(x) **else** g(y).

(The alert reader may well be worried by this recourse to preservation of filtered colimits when the diagrams are not external ones, but in fact sense can be made of the argument.)

If D and E are toposes over B, then D+E is still the coproduct in **Top/B**. Moreover, if f: C → B, then f*(D+E) is equivalent to f*D + f*E. A point of f*D + f*E is of the form **if** p **then** (x, z', φ) **else** (y, z", ψ) where x and y are points of D and E, z' and z" are points of C, φ: f(z') ≅ $x|_B$ and ψ: f(z") ≅ $y|_B$. If we let z = **if** p **then** z' **else** z", then ((**if** p **then** x **else** y), z, θ) is a point of f*(D+E), θ being the isomorphism

f(z) = f(**if** p **then** z' **else** z") ≅ **if** p **then** f(z') **else** f(z")

$\cong$ **if** p **then** $x|_B$ **else** $y|_B$ = (**if** p **then** x **else** $y)|_B$

This gives an equivalence between $f^*D + f^*E$ and $f^*(D+E)$.

Exponentials

Top is not Cartesian closed. However, many of the toposes we shall be dealing with are exponentiable, so exponentials often exist. Let us note that pullback functors between slices preserve existing exponentials. For suppose $q: E \to B$ is a geometric morphism and that D_1 and D_2 are toposes over B such that the exponential $D_1 \Rightarrow_B D_2$ in **Top/B** exists. If F is a topos over E, then

maps $F \to q^*(D_1 \Rightarrow_B D_2)$	over E
~ maps $F \to D_1 \Rightarrow_B D_2$	over B
~ maps $F \times_B D_1 \to D_2$	over B
~ maps $F \times_E q^*D_1 \to D_2$	over B ($F \times_E q^*D_1 \cong F \times_B D_1$)
~ maps $F \times_E q^*D_1 \to q^*D_2$	over E

It follows that $q^*(D_1 \Rightarrow_B D_2)$ is an exponential $q^*D_1 \Rightarrow_E q^*D_2$.

2.3 Lifting in **Top**

It is convenient to summarize here general results about lifting of toposes, commonly known as *scone* or *Freyd cover* (Johnstone and Moerdijk 1989; Johnstone 1992). Some of the coherence questions that arise are quite intricate, and we shall defer detailed discussion of them (in more general 2-categories than **Top**) to a later study. Here we shall be content with sketching the concrete constructions.

Definition 2.3.1 (Johnstone and Moerdijk 1989) A topos D is *local* iff the unique (up to isomorphism) map $!: D \to 1$ has a left adjoint $\bot: 1 \to X$ ($\bot$ pronounced "bottom"). Being a left adjoint means exactly that the global point $\bot$ is initial amongst all points: if $f: Y \to X$, then there is a unique 2-cell from $!;\bot$ to f.

(Clearly this definition can be extrapolated to general 2-categories. In particular, in poset-enriched categories, $\bot$ is indeed a bottom point of D and so we use this term rather than Johnstone and Moerdijk's "centre" which seems topographically wrong.)

A map between two local toposes is *strict* iff it preserves $\bot$ (up to isomorphism).

We now have a sub-2-category **LTop** of **Top**, full on 2-cells, whose objects and morphisms are the local toposes and strict maps.

We shall feel free to extend these definitions to slice categories **Top/B** by change of base, getting the notions of toposes or maps being local or strict *over B*.

The essence of *lifting* is that it provides a left adjoint to the inclusion **LTop** $\to$ **Top** – this is exactly what lifting of domains does in a rather simpler context. It is less straightforward in our 2-categorical context, but Johnstone (1992) shows that the *scone* or *Freyd cover* construction has the right properties. In Johnstone and Moerdijk (1989), $\mathcal{L}X$ is written $\hat{X}$.

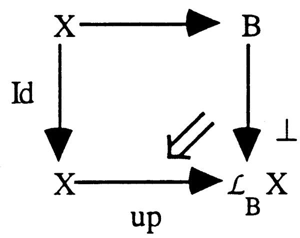
Definition 2.3.2 Let X be a topos over base B. (We shall use restriction notation for the map from X to B.) The *scone* or *lifting* of X over B, $\mathcal{L}_B X$, classifies triples $(x, I, (y_\lambda, \theta_\lambda)_{\lambda \in I})$ where x is a point of B, I is a subsingleton, and $(y_\lambda, \theta_\lambda)_{\lambda \in I}$ is an I-indexed family of pairs, y_λ a point of X and θ_λ: $y_\lambda|_B \equiv x$. This is again a topos over B, by the map that forgets everything except x.

Over B, it has an initial point given by $x \mapsto (x, \emptyset, (-,-))$, and a map up: $X \to \mathcal{L}_B X$ given by $y \mapsto (y|_B, 1, (y, Id))$.

Proposition 2.3.3 $\mathcal{L}_B$ provides lifting in **Top/B**.

Proof

This and further properties of lifting (e.g. that it has coKZ properties) follow from the fact that $\mathcal{L}_B X$ is a cocomma object in **Top**:



(The morphism up is the unit of the monad $\mathcal{L}_B$.)]

We now turn to discuss the axiomatization by Crole and Pitts (1992) of lift. They require a *fixpoint-object* Ω: Ω is an "initial lift algebra" (with structure morphism σ: $\mathcal{L}\Omega \to \Omega$), equipped with a global point ω: $1 \to \Omega$ that is an equalizer for Id_Ω and up;σ: $\Omega \to \Omega$. However, "algebra" here is used in a sense that is weaker than that of Eilenberg-Moore algebra for lift *qua* monad, so let us avoid confusion by using the word *structure* for the weaker sense.

Definition 2.3.4 Let F: $\mathcal{C} \to \mathcal{C}$ be an endofunctor of a category $\mathcal{C}$. Then an F-*structure* is an object X of $\mathcal{C}$ equipped with a morphism α: FX $\to$ X.

Theorem 2.3.5 Each slice **Top/B** has a fixpoint-object.

(Note that in the 2-category **Top** even the statement of this theorem raises coherence questions that we are neglecting for the time being.)

Proof

Let us first prove the case when B = 1. Define the locale Ω to classify inhabited initial segments of $\mathbb{N}$ (so Ω is the ideal completion of $(\mathbb{N}, \leq)$). Ω is local (its initial point is $\{0\}$), and we also have a map suc: $\Omega \to \Omega$, suc(S) = $\{0\} \cup \{n+1 : n \in S\}$ which, by the universal property of $\mathcal{L}$, extends to σ: $\mathcal{L}\Omega \to \Omega$.

Let us note straightaway that if we define the global point ω: $1 \to \Omega$ to be the whole of $\mathbb{N}$, then this is the equalizer for Id and up;σ = suc. For if S = suc(S) then $0 \in S$ and $n+1 \in S$ for every $n \in S$, so by induction S = $\mathbb{N}$. Notice that ω is a *final* point of Ω – Ω is colocal as well as local.

Now suppose F: $\mathcal{L}D \to D$ is a structure for $\mathcal{L}$. We require an essentially unique map it(F): $\Omega \to$ D that is an $\mathcal{L}$-structure homomorphism, in other words σ;it(F) $\cong$ $\mathcal{L}$(it(F));F. Now an ideal S of $\mathbb{N}$ is a filtered colimit of the principal ideals $\downarrow$n such that $n \in S$, so the action of it(F) is determined by its action on principal ideals and the inclusions between them. We have –

$$\text{it}(F)(\downarrow 0) = \text{it}(F)\circ\sigma(\bot) \cong F\circ\mathcal{L}(\text{it}(F))(\bot) = F(\bot_{\mathcal{L}D})$$

$$\text{it}(F)(\downarrow(n+1)) = \text{it}(F)\circ\text{suc}(\downarrow n) = \text{it}(F)\circ\sigma\circ\text{up}(\downarrow n) \cong F\circ\mathcal{L}(\text{it}(F))\circ\text{up}(\downarrow n) \cong F\circ\text{up}\circ\text{it}(F)(\downarrow n)$$

By induction, this proves uniqueness of it(F) on principal ideals. Let us write x_n for it(F)($\downarrow$n).

$$\mathrm{it}(F)(\downarrow 0 \sqsubseteq \downarrow 1) = \mathrm{it}(F) \circ \sigma(\bot \to \mathrm{up}(\downarrow 0)) \cong F \circ \mathcal{L}(\mathrm{it}(F))(\bot \to \mathrm{up}(\downarrow 0)) = F(!: \bot \to \mathrm{up}(x_0))$$
$$\mathrm{it}(F)(\downarrow(n+1) \sqsubseteq \downarrow(n+2)) = \mathrm{it}(F) \circ \sigma \circ \mathrm{up}(\downarrow n \sqsubseteq \downarrow(n+1)) \cong F \circ \mathrm{up} \circ \mathrm{it}(F)(\downarrow n \sqsubseteq \downarrow(n+1))$$

This proves uniqueness of it(F) on inclusions between principal ideals, and hence (taking filtered colimits) on arbitrary points of Ω. It also proves existence by allowing us to define it(F)(S) as the filtered colimit of the corresponding diagram (over $n \in S$) of points of D.

The argument for **Top**/B is similar, but parametrized by points of B. The fixpoint object in **Top**/B is $B \times \Omega$, and this is an $\mathcal{L}_B$-structure by $Id_B \times \sigma$: $B \times \mathcal{L}\Omega \to B \times \Omega$ (using the fact that $\mathcal{L}_B(B \times \Omega) \cong B \times \mathcal{L}\Omega$).]

We use the fixpoint object to prove a remarkable property of local toposes, namely that they are the topical analogues of Freyd's (1991) *algebraically complete* categories, i.e. those for which every endofunctor has an initial structure. By considering the identity endofunctor one can prove that every algebraically complete category has an initial object, but the converse is far from true. However, we show that if a *topos* D has an initial point (that is to say, it is local), then every endomap F has an initial structure constructed using filtered colimits of points. To make this precise, we consider the topos [F-Str] that classifies F-structures. (An F-structure is a point x of D equipped with a homomorpism α: Fx $\to$ x.) This is the inserter for F to Id_D.

We first set out some easy facts about toposes [F-Str] that are familiar from the category context (Freyd 1991).

Proposition 2.3.6 Let F be an endomap of a topos D.

(i) Let α: Fx $\to$ x and β: Fy $\to$ y be two F-structures. Then homomorphisms from α to β are homomorphisms f: x $\to$ y such that α;f = Ff;β.

(ii) F extends to an endomap of [F-Str], mapping α: Fx $\to$ x to Fα: $F^2x \to$ Fx. Moreover, there is a natural transformation from this F to $Id_{[F\text{-}Str]}$.

(iii) If [F-Str] is local, with initial point α: Fa $\to$ a, then α is an isomorphism.

Proof

(i) This is quite obvious. (The issue is that the general notion of homomorphism between models of a geometric theory has already been defined.)

(ii) Also obvious. α itself provides the homomorphism from Fx to x.

(iii) We briefly recall the usual argument. By initiality there is a unique F-structure homomorphism α': a $\to$ Fa. α';α is the unique F-structure endomorphism on a, and so is equal to the identity. Then because α' is an F-algebra homomorphism we have $\alpha;\alpha' = F\alpha';F\alpha = F(\alpha';\alpha) = F(Id_a) = Id_{Fa}$.]

Proposition 2.3.7 Let D and E be toposes, and F: D $\to$ E and G: E $\to$ D maps. Then [GF-Str] is local iff [FG-Str] is.

Proof

Suppose [GF-Str] is local, with initial point α: GFa $\to$ a. By an obvious generalization of 2.3.6 (ii), F and G extend to maps F: [GF-Str] $\to$ [FG-Str] and G: [FG-Str] $\to$ [GF-Str]; we show that Fα is an initial point of [FG-Str]. Suppose that β: FGy $\to$ y is an FG-structure, and let f: a $\to$ Gy be the unique GF-structure homomorphism. Then Ff;β: Fa $\to$ y is an FG-structure homomorphism. For uniqueness, let g: Fa → y be another. Then α^{-1};Gg: a → Gy is the unique GF-structure homomorphism and so equals f. Then Ff;β = Fα^{-1};FGg;β = Fα^{-1};Fα;g = g.]

Theorem 2.3.8 *(In* **Top**, *local ⇔ algebraically complete .)*
Let D be a topos. Then D is local iff for every map F: D → D, [F-Str] is local.
Proof
(Again, in the proof here we are neglecting coherence issues.)
⇐: Take F = Id_D, and let α: a → a be the initial point in [Id-Str]. There is a unique Id-structure homomorphism from (a, α) to itself, i.e. a unique homomorphism f: a → a such that α;f = f;α. But both Id_a and α satisfy this, so α = Id_a. Now by considering the unique Id-structure homomorphism from (a, Id_a) to (b, Id_b), we see that there is a unique homomorphism from a to any point b of D. Hence D is local with initial point a.
⇒: Let us write S for the topos [F-Str]. By Proposition 2.3.3, we have a unique strict map F': $\mathcal{L}$D → D, and hence it(F'): Ω → D the unique $\mathcal{L}$-structure homomorphism. Let a = it(F')(ω). Then

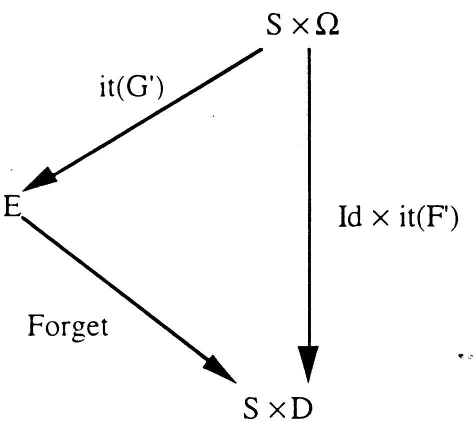
$$a = \mathrm{it}(F')\circ\sigma\circ\mathrm{up}(\omega) \cong F'\circ\mathcal{L}(\mathrm{it}(F'))\circ\mathrm{up}(\omega) \cong F'\circ\mathrm{up}\circ\mathrm{it}(F')(\omega) = F(a)$$

and the isomorphism makes a an F-structure A = (α: Fa → a). This will be our initial point ⊥ of S.

Now let E classify diagrams of the form $Fb \xrightarrow{\beta} b \xleftarrow{f} y$ where b and y are points of D and β and f are homomorphisms. E is a topos over S (by the map that picks out the structure β: Fb → b), and moreover it is local over S: the initial point over the algebra b has y = ⊥. We can define a map G: E → E over S mapping the above diagram to

$$Fb \xrightarrow{\beta} b \xleftarrow{\beta} Fb \xleftarrow{Ff} Fy$$

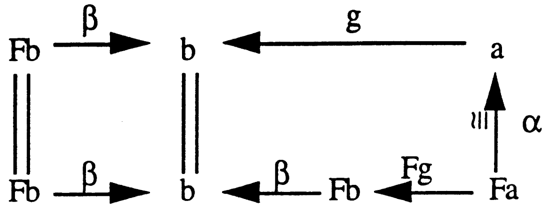
From G we get a map G': $\mathcal{L}_S$ E → E, strict over S. The forgetful map from E to S × D (forgets f) is an $\mathcal{L}_S$-structure homomorphism, and so we have a diagram of $\mathcal{L}_S$-structures



By initiality of S × Ω, it follows that for an F-structure B = (β: Fb → b), it(G')(B, ω) has the form

$$Fb \xrightarrow{\beta} b \xleftarrow{g} a$$

By the equalizing property of ω, we have an isomorphism

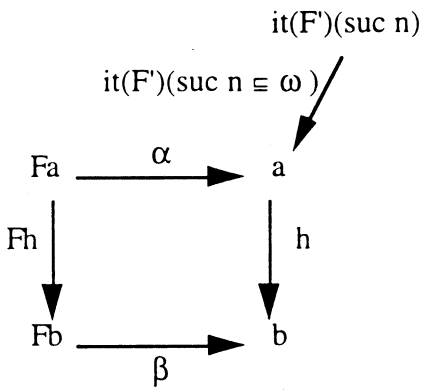


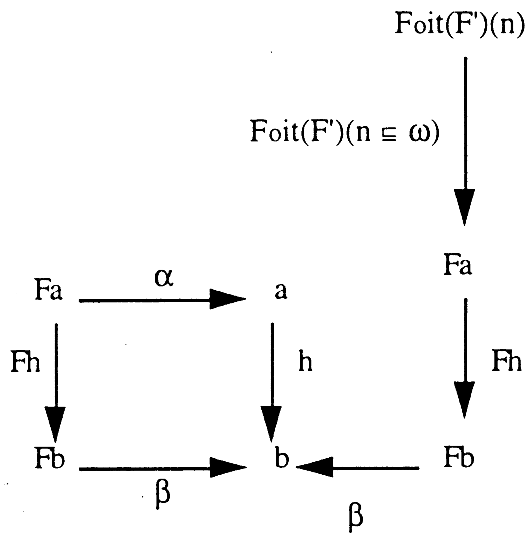
so that g is an F-structure homomorphism.

To prove uniqueness, let S_1 classify F-structures β: Fb $\to$ b equipped with F-structure homomorphisms h: a $\to$ b, and let E_1 classify points of S_1 equipped with homomorphisms f: y $\to$ b. E_1 is local over S_1 (take y = $\perp$). We have a map G_1: $E_1 \to E_1$ over S_1, defined just like G, and hence G_1': $\mathcal{L}_{S_1}E_1 \to E_1$. Define a map H: $S_1 \times \Omega \to E_1$ by

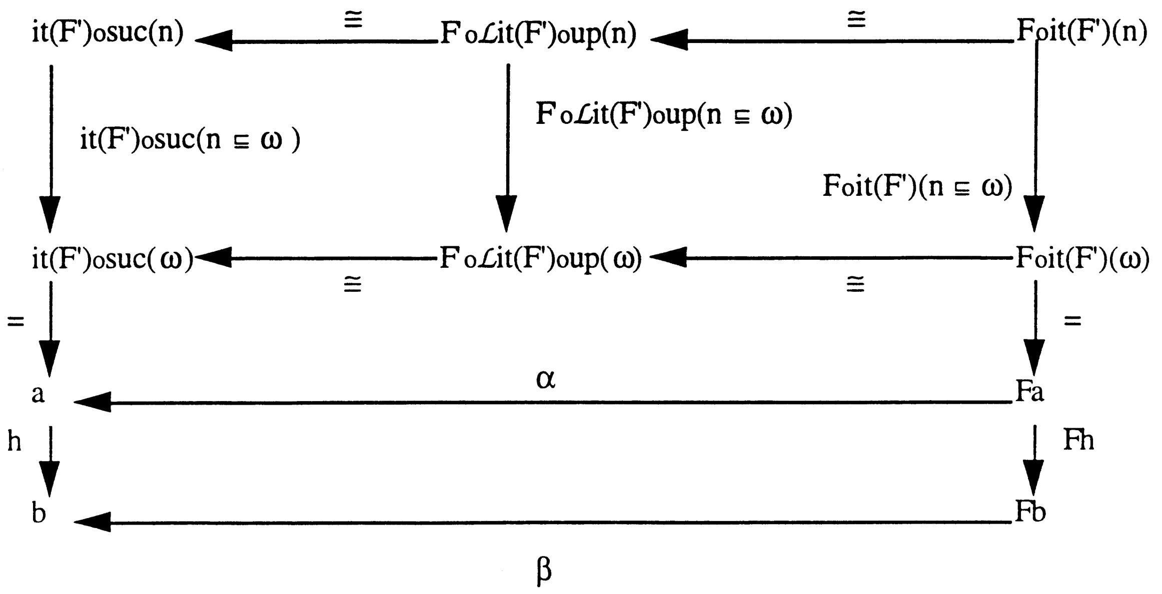
$$H((B,h),n) = (B,h) \text{ with } it(F')(n \sqsubseteq \omega);h: it(F')(n) \to it(F')(\omega) = a \to b$$

We show that H is an $\mathcal{L}_{S_1}$-structure homomorphism, so we check that two maps agree on $\mathcal{L}_{S_1}(S_1 \times \Omega) \cong S_1 \times \mathcal{L}\Omega$ – they are $(S_1 \times \sigma)$;H and $\mathcal{L}_{S_1}H;G_1'$. Since every point of $\mathcal{L}\Omega$ is a filtered colimit of points $\perp$ and points n of Ω, it suffices to check on these. For ((B,h),$\perp$), we find that both images in E_1 have homomorphisms $\perp \to$ b which must be equal by initiality of $\perp$ in D. For ((B,h),n), we find that the two images are





To show these are isomorphic, consider the diagram:



Here, the bottom part commutes because h is an F-structure homomorphism, the middle part by definition of α, the top right because up is a natural transformation from Id to $\mathcal{L}$, and the top left because it(F') is an $\mathcal{L}$-structure homomorphism.

This shows that H is a (hence the unique) $\mathcal{L}_{S_1}$-structure homomorphism from $S_1 \times \Omega$ to E_1. But there is another H', defined by H'((B,h),n) = it(G')(B,n) with h tacked on, and so H = H'. Applying them both to ((B,h),ω), we see that h = g and hence there is a unique F-structure homomorphism from A to B.]

Corollary 2.3.9 Let D be an arbitrary topos, F: $\mathcal{L}$D → D a map. Then [(up;F)-Str] is local.

Proof

$\mathcal{L}$D is local, so by 2.3.8 [(F;up)-Str] is local. Now apply 2.3.7.]

2.4 Algebraic dcpos

The localic theory of algebraic dcpos is well-known, but we shall recall some of it here for three reasons.

(1) The strongly algebraic domains that are the main concern of this paper *are* algebraic dcpos, and many of the points discussed here will be needed later in the special case.

(2) They provide a simple example to illustrate the "Display categories" in Section 3.2.

(3) We wish to illustrate the idea that a locale *is* a special kind of topos.

Let us first recall the localic theory of algebraic dcpos. It is constructive, and hence holds in a general geometric universe.

Proposition 2.4.1 Let X be a poset. Then the following frames are isomorphic:

(i) Fr ⟨↑s (s ∈ X) | $\uparrow t \le \uparrow s \quad (s \sqsubseteq t)$

$\mathbf{true} \le \bigvee_{s \in X} \uparrow s$

$\uparrow s \wedge \uparrow t \le \bigvee \{\uparrow u : u \in X, s \sqsubseteq u, t \sqsubseteq u\}$ ⟩

(ii) The Alexandroff topology on X (that is to say, the frame of upper-closed subsets of X)

(iii) The Scott topology on Idl X.]

The points of the corresponding locale are in order-isomorphism with the ideals of X. We therefore write Idl X for the locale, which is constructively spatial. There is an equivalent predicate presentation, using a unary predicate $I \subseteq X$ satisfying –

$$I(t) \wedge s \sqsubseteq t \vdash_{st} I(s)$$
$$\vdash \exists s.\ I(s)$$
$$I(s) \wedge I(t) \vdash_{st} \exists u.\ (I(u) \wedge s \sqsubseteq u \wedge t \sqsubseteq u)$$

Definition 2.4.2 A locale is an *algebraic dcpo* iff it is homeomorphic to Idl X for some poset X.

We still have the more usual order-theoretic characterizations:

Proposition 2.4.3 A locale is an algebraic dcpo iff its frame is the Scott topology of a dcpo D satisfying one of the following equivalent conditions:

(i) D is order-isomorphic to the ideal completion of the poset KD of its compact elements.
(ii) Every element of D is a directed join of compact elements below it.

Then its global points are in order-isomorphism with D.]

If we do this for the generic poset $(X, \sqsubseteq)$ in $\mathcal{S}$[poset] then its ideal completion Idl(X) is a locale [poset][ideal] over [poset], and every algebraic dcpo (over any topos) is a pullback of it. We may therefore consider [poset][ideal] $\rightarrow$ [poset] as the *algebraic dcpo classifier.* "Classifier" here has the same sense as in "subobject classifier", not as in "classifying topos". In an elementary topos, the subobjects are the pullbacks of the subobject classifier true: $1 \rightarrow \Omega$, and in the category of toposes the algebraic posets are the pullbacks of the algebraic dcpo classifier.

To summarize: The ideal completion of a poset can be constructed "generically", as a geometric morphism [poset][ideal] $\rightarrow$ [poset]. All other instances of the construction, over any topos, can be obtained from this one as pullbacks.

An important corollary from these results is that algebraic dcpos are exponentiable (Lemma 4.1 of Johnstone and Joyal (1982)) in the category of toposes. Of course, it is better known that they are exponentiable in the category of locales, i.e. locally compact. The corresponding property for toposes is slightly stronger – such locales are known as "metastably locally compact" –, but the results of Johnstone and Joyal are enough to show that it holds for algebraic dcpos.

If X and Y are posets, then by analysing the frame homomorphisms from Fr $\langle \uparrow t\ (t \in Y) \mid \dots \rangle$ to Alex X one easily sees that continuous maps from Idl(X) to Idl(Y) can be described equivalently as certain relations f from X to Y – explicitly, they are those relations satisfying –

- $s' \sqsupseteq s\ f\ t \sqsupseteq t' \vdash s'\ f\ t'$
- $\vdash \exists t \in Y.\ s\ f\ t$
- $s\ f\ t_1 \wedge s\ f\ t_2 \vdash \exists t \in Y.\ (s\ f\ t \wedge t \sqsupseteq t_1 \wedge t \sqsupseteq t_2)$

Such relations are known as *approximable mappings* from X to Y. The identity approximable mapping is $\sqsupseteq$, and composition is by relational composition.

Note that the last two axioms are the nullary and binary case of a more general form that can be proved from the special cases by induction on n:

$$\bigwedge_{i=1}^{n} s\ f\ t_i \vdash \exists t.\ (s\ f\ t \wedge \bigwedge_{i=1}^{n} t \sqsupseteq t_i)$$

We can state this more succinctly using finite sets: if T is a finite subset of X_2, then

$$\forall t' \in T.\ s\ f\ t' \vdash \exists t.\ (s\ f\ t \wedge \forall t' \in T.\ t \sqsupseteq t')$$

Note also that approximable mappings are geometric in the sense that there is a geometric theory whose models are pairs (X, Y) of posets, together with an approximable mapping (appearing as a binary predicate) between them. Let us write AM for this theory. There are then two posets X_s and X_t in $\mathcal{S}$[AM], i.e. two geometric morphisms from [AM] to [poset], so [AM] is a topos over $[\text{poset}]^2$. Thinking spatially, the fibre over a given pair of posets is to be the space of continuous maps from $\text{Idl}(X_s)$ to $\text{Idl}(X_t)$, so perhaps [AM] should be the exponential $(\text{Idl}(X_s) \Rightarrow_{[\text{poset}]^2} \text{Idl}(X_t))$ over $[\text{poset}]^2$. Indeed it is.

Theorem 2.4.4 The exponential $(\text{Idl}(X_s) \Rightarrow_{[\text{poset}]^2} \text{Idl}(X_t))$ exists and classifies the theory AM.
Proof This is an application of Lemma 4.1 in Johnstone and Joyal (1982). Let us sketch the proof in this simple case. If D is a topos over $[\text{poset}]^2$, then let X and Y be the two corresponding posets in $\mathcal{S}$D. It can be calculated that Idl(X) classifies the theory of pairs (x, F) where x is a point of D and F is a flat presheaf on X(x): hence by Diaconescu's theorem, $\mathcal{S}$Idl(X) is equivalent to the geometric universe of internal X-diagrams in $\mathcal{S}$D. A geometric morphism from Idl(X) to Idl(Y) is an ideal of the constant internal X-diagram corresponding to Y, but this can be calculated to be just an approximable mapping from X to Y.]

This result gives us a universal characterization of the topos [AM] that does not depend on the presentation we gave for the theory AM.

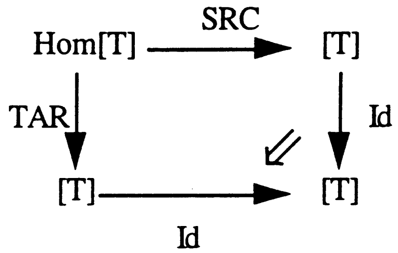
3 Examples of topical categories

3.1 Intrinsic categories

An important aspect of the 2-categorical structure of **Top** is that it allows us to imagine each topos D = [T] as a category – not as its geometric universe $\mathcal{S}$D, but as an idealization of pt D (i.e. **Top**(1, D)) that transcends the possible insufficiency of models of T in **Set**.

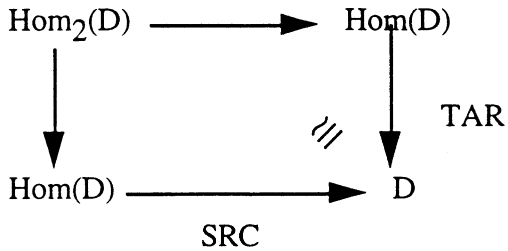
The way this works as a practical technique is that if you have an aspect of categories that can be expressed using the 2-categorical structure of **Cat**, then that expression can be translated to **Top**. For instance, a category C has an initial object iff the unique functor $!: C \to 1$ has a left adjoint. Transferring this property to **Top** gives a natural notion of "topos with initial point" (and these are the local toposes of Section 2.3). C has finite coproducts iff every diagonal functor $\Delta_n: C \to C^n$ has a left adjoint, and in **Top** we get the notion of "topos with all finite coproducts". A result of Johnstone's (1992) can then be naturally read as saying that a certain bagdomain construction freely adjoins finite coproducts to a topos.

This somewhat mystical category of generalized points is manifested as a topical category, because if T is a geometric theory then the theory of two models with a homomorphism between them is also geometric. Clearly we seek a comma square

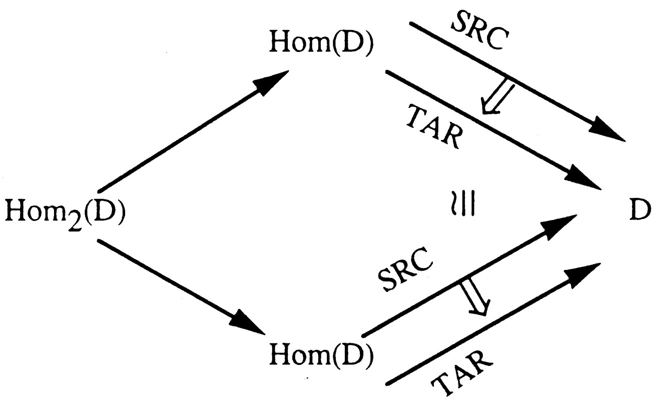


Alternatively, Hom[T] can be expressed as the exponential ($ ⇒ [T]) where $ is the Sierpinski locale.

We can now make Hom(D) $\rightrightarrows$ D into a topical category. ID: D → Hom(D) corresponds to $D \begin{smallmatrix}\xrightarrow{Id}\\ \Downarrow Id\\ \xrightarrow[Id]{}\end{smallmatrix} D$. For composition COMP, let $Hom_2(D)$ be the pullback



Then we have



and hence COMP: $Hom_2(D)$ → Hom(D). By definition, ID and COMP interact correctly with SRC and TAR, while the unit laws and associativity follow from the corresponding properties of 2-cells. We call this topical category the *intrinsic* category on D.

The topical categories that we shall use to "topologize" categorical domain theory are actually not intrinsic categories – they are examples of the *display* categories that we shall introduce in the next section. However, wherever you have a topos you have an intrinsic category, and it turns out that some of those associated with the display categories for domains have particular domain-theoretic significance: one, for instance, corresponds to a category of domains with embedding-projection pairs for morphisms.

3.1.1 "The topos of sets is not Cartesian closed"

We prove this to suggest that topical CCC's are less common than you might expect. Of course, the statement must be understood rather carefully. "The topos of sets" means the topos *classifying* sets, i.e. [Set]. This has an intrinsic categorical structure *topically:* its morphism topos is Hom[Set] = [Fn], classifying two sets and a function between them. It is this topical category that is not Cartesian closed, i.e. it cannot be extended with the (essentially algebraic, and hence topically meaningful) structure of a Cartesian closed category.

The basic idea is that if [Set] were a topical CCC, then exponentiation would have to be covariant in both arguments, and this is impossible.

Suppose we are given a topical CCC C, with toposes C_0 and C_1 classifying objects and morphisms, and various maps including an exponential EXP: $C_0^2 \rightarrow C_0$. If we take global points, then we get classes pt C_0 and pt C_1 of objects and morphisms, with various functions making an ordinary (though large) CCC pt C. This includes pt EXP: (pt $C_0)^2 \rightarrow$ pt C_0, which is determined uniquely up to isomorphism by the category structure of pt C. Of course, with respect to the morphisms in pt C_1, pt EXP is contravariant in the first argument and covariant in the second. On the other hand, pt C_0 is not just a class – it is a category in its own right, and with respect to the morphisms there, pt EXP is covariant in both arguments.

Now consider the case of the intrinsic topical category on [Set], and suppose that it is Cartesian closed. The global points give the category **Set**, and EXP(X, Y) is ordinary exponentiation Y^X. But pt C_0, i.e. pt [Set], is also **Set**, so we have a covariant functor EXP: $\mathbf{Set}^2 \rightarrow \mathbf{Set}$ such that EXP(X, Y) $\cong Y^X$. Now consider (!, Id): $(\emptyset, \emptyset) \rightarrow (1, \emptyset)$ in $\mathbf{Set}^2$. EXP(!, Id) is a function from $\emptyset^\emptyset \cong 1$ to $\emptyset^1 \cong \emptyset$, which is impossible.

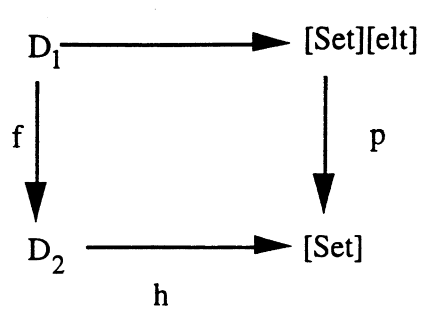
It is also instructive, under the assumption that [Set] is intrinsically a topical CCC, to consider the GU homomorphism $\mathcal{S}$EXP: $\mathcal{S}$[Set] $\rightarrow$ $\mathcal{S}$[Set]^2. This is defined by a single object of $\mathcal{S}$[Set]^2, and one can show that it would have to be Y^X where X and Y are the two generic sets in $\mathcal{S}$[Set]^2. But one can also calculate that Y^X is isomorphic to Y – essentially because the only functions that can be defined from one generic set to another are the constant functions. The assumption that [Set] is intrinsically a topical CCC implies that this exponentiation is generic, and hence that in any geometric universe we have $Y^X \cong Y$ for all objects X and Y – an obvious nonsense.

One might well ask whether *any* topos other than 1 is intrinsically Cartesian closed in this sense.

3.2. "Display" categories

We follow with another family of examples of topical categories, which we shall call "display categories". Hyland and Pitts (1989) use pullback-stable classes of morphisms to model dependent types, and we shall use this idea in **Top** in the case where the class is generated by a single "classifying" morphism p of which every other morphism in the class is a pullback. The paradigm example is the way a subobject classifier t: $1 \rightarrow \Omega$ classifies monics in an elementary topos – monics are pullbacks of it. Similarly, we treat p as classifying the pullbacks of it.

The prime example in toposes is the étale classifier, the forgetful map from [Set][elt] $\rightarrow$ [Set]: a geometric morphism f: $D_1 \rightarrow D_2$ is étale iff it is a pullback



By definition of the object classifier [Set], h corresponds to an object X of $\mathcal{S}D_2$, and then $\mathcal{S}D_1$ is equivalent to the slice geometric universe $\mathcal{S}D_2/X$.

Now for the subobject classifier, the morphism along the bottom is *uniquely* determined by the monic m. For the étale maps it's defined uniquely only up to isomorphism, and in fact there are examples where even that doesn't hold. Hence what p classifies is really pullbacks of it equipped with specified pullback squares and so the notion of "classifying" is somewhat weaker that with the subobject classifier; but the comparison is still useful.

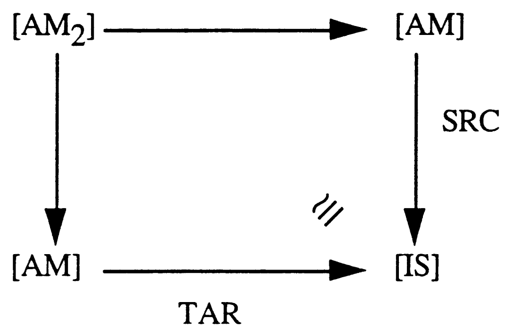
Let us call such pullbacks *p-displays.* If f: $D_1 \rightarrow D_2$ (equipped with a pullback square) is a p-display, let us call D_1 a *p-topos* over D_2. Given a topos D, consider p-**Top**/D, the full subcategory of **Top**/D whose objects are p-toposes. Our interest lies in devising p to capture various notions of topos – mostly locales, actually – over (arbitrary) D as p-**Top**/D. We have already seen how to capture discrete spaces (étale maps) using p = forget: [Set][elt] → [Set], and then p-**Top**/D ≈ $\mathcal{S}D$. A second main example is that of algebraic dcpos: if we take p to be the forgetful map (forgets the ideal) from [poset][ideal] to [poset], then it's the ideal completion of the generic poset, and we have already argued that it classifies algebraic dcpos. (Note that there is a discordance here with the way Hyland and Pitts (1989) use the word "algebraic". For them, an algebraic topos D is one that classifies an essentially algebraic theory: its geometric universe $\mathcal{S}D$ is a presheaf category $\mathbf{Set}^{C^{op}}$ for some *lex* category C, and a localic algebraic topos is one for which C is a poset – hence a meet semilattice. Our algebraic dcpos are locales D (localic toposes) for which $\mathcal{S}D$ is $\mathbf{Set}^{C^{op}}$ for an *arbitrary* poset C, and it would indeed be natural for us to define a topos to be algebraic if $\mathcal{S}D$ is $\mathbf{Set}^{C^{op}}$ for an arbitrary category.) Other kinds of locales that can be captured (sometimes in more than one constructively inequivalent way) include continuous dcpos, Scott domains, strongly algebraic (SFP) domains, Stone locales, spectral locales, etc., etc. – we shall discuss some these more fully in a later section. The main body of the paper will be concerned with strongly algebraic domains because of their computer science interest.

Let us fix notation for a general scheme so far: we have a theory IS of *information systems* (e.g. the theory of posets), a theory IS+pt of *points* of information systems (e.g. ideals of posets), and a map p: [IS][pt] → [IS]. Pullbacks of this will be called *domains* (though *predomains* would often be a more conventional term, because we don't usually assume bottom points). We shall assume that p is exponentiable, and therefore have $[AM] = (P_s{}^*p \Rightarrow_{[IS]^2} P_t{}^*p)$, where P_s and P_t are the two projections from $[IS]^2$ to [IS].

[AM] is a topos over $[IS]^2$, so we certainly have two geometric morphisms SRC and TAR from [AM] to [IS] (corresponding to P_s and P_t). Let us show that these form the source and target maps of a topical category. Remember that the characterization of [AM] as an exponential over $[IS]^2$ enables us to define maps from any D into [AM] as pairs (P_s, P_t) of maps from D to [IS] together with a map from $P_s{}^*p$ to $P_t{}^*p$ over D.

The identity map ID: [IS] → [AM] is given by the pair (Id, Id) of maps from [IS] to itself, together with the identity map from Id*p to Id*p.

Let $[AM_2]$ be the theory of two composable approximable mappings, in other words the pullback



We have three maps from $[AM_2]$ to [IS] – or, more carefully, four maps with an isomorphism between the middle two. Accordingly, we get four domains over $[AM_2]$. The two maps to [AM] give maps over $[AM_2]$ between the first two domains and between the last two, and the isomorphism gives an equivalence between the middle two. Composing gives a map between the first and last, corresponding to a map from $[AM_2]$ to [AM]. This is COMP, for composition.

By definition, ID and COMP interact correctly with SRC and TAR. The unit laws and associativity follow essentially from the corresponding properties of maps, though we have somewhat glossed over the 2-categorical aspects here. We shall call the resulting topical category the *display category* obtained from p.

Example When p is the étale classifier, then its display category is equivalent to the intrinsic category on [Set]. This is because maps between discrete locales are equivalent to functions (homomorphisms) between the corresponding sets.

Our main aim in this paper is to show how a specific class of domains, namely the strongly algebraic (or SFP) domains, can be put into a topical setting to exemplify an account of categorical domain theory that in many respects works much more smoothly than the usual one. The strongly algebraic domains are chosen for exactly the usual reason, namely that the (topical) category of them is Cartesian closed and supports domain theoretic constructions including the Plotkin power domain.

3.2.1 Capturing extra structure on the display category

Categories of domains are usually Cartesian (finite products), and if you're lucky they're Cartesian closed. If you don't require bottoms, then they're also coCartesian. All these kinds of extra structure can be expressed using essentially algebraic (finite limit) theories, and so are meaningful for internal categories in any category with finite limits. Unfortunately, the category of toposes is actually a 2-category, and pullback squares commute only up to isomorphism. Because of these complications we shall not here attempt to work with a proper 2-categorical definition of "internal category" (Hyland and Moerdijk unpublished).

Instead, we shall show how properties of p lead to p-**Top**/B inheriting structure from **Top**/B. For instance, p-**Top**/B can inherit terminal objects from **Top**/B as follows. Suppose there is a global point TERM of [IS] (i.e. TERM: 1 → [IS]) that looks as though it ought to be the terminal object in an internal category sense. For any topos B we get a corresponding p-topos over B, namely (!;TERM)*[IS][pt], and what we do is to show that this is terminal in **Top**/B. Similarly, p-**Top**/B can inherit binary products from **Top/B**. We give a map PROD: $[IS]^2 \to [IS]$, and show that for any two p-toposes D_1 and D_2 over B, corresponding to f: $B \to [IS]^2$, the pullback (f;PROD)*[IS][pt] is equivalent to the product $D_1 \times_B D_2$ in **Top/B**.

3.2.2 Examples

We can now extend the negative result of 3.1.1 to cover more particular sets – specifically, finite sets and decidable sets. These give two toposes over [Set]:

[FinSet] is presented with a constant T: $\mathcal{F}X$ and an axiom $\vdash_{x:X} x \in T$

[DecSet] is presented with a binary relation $\neq$: $\wp(X \times X)$ and axioms to make it the complement of equality:

$$\mathbf{true} \vdash_{x,y:X} x = y \vee x \neq y$$

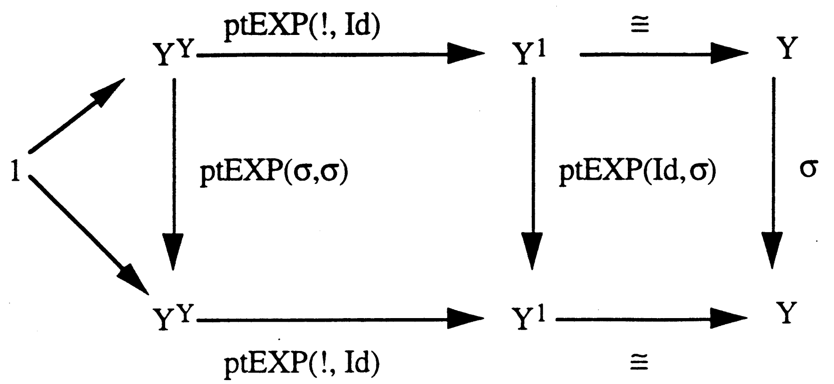
$$x \neq x \vdash_{x:X} \mathbf{false}$$

These are localic over [Set], because they are presented without any new sorts. They are not subtoposes of [Set], even though non-constructively you might think of finiteness or decidability as particular properties of a set X (i.e. just extra axioms). Actually, they represent extra *structure* on X, and this shows up in the homomorphisms. Because T or $\neq$ must be preserved, homomorphisms of finite or decidable sets must be, respectively, onto or 1-1.

It is normal to presume that the category **Set** of sets is Cartesian closed, but we have shown that [Set] is not (topically). We might therefore ask whether perhaps [DecSet] is – maybe in [Set] we omitted too much of the constructive structure. The answer is No, but let us first take care to phrase the question properly. We are not interested in the intrinsic topical category on [DecSet], because that corresponds to the category of sets *with 1-1 functions* and that is certainly not Cartesian closed. (It does not even have a terminal object, nor binary products.) To get a topical category whose morphisms correspond to *all* functions between decidable sets, we take the display category arising from the étale classifier when pulled back to [DecSet].

Let us now show that this display category is not Cartesian closed. Supposing that it is, then taking global points just as in 3.1.1 we find that pt EXP is a covariant functor from $\mathbf{Set}_{mon}{}^2$ to $\mathbf{Set}_{mon}$, where $\mathbf{Set}_{mon}$ is the category of sets with monos, and that, on objects, it takes (X, Y) to Y^X. The argument is now the same as before, because the functions ! and Id used there are both 1-1.

Now let us turn to the finite sets. Classically, the category $\mathbf{Set}_{fin}$ of finite sets is Cartesian closed. Again, we pull back the étale classifier to [FinSet] and consider the display category that arises. We find that pt EXP is a covariant functor from $\mathbf{Set}_{fo}{}^2$ to $\mathbf{Set}_{fo}$, where $\mathbf{Set}_{fo}$ is the category of finite sets with onto functions, and that, on objects, it takes (X, Y) to Y^X. Now let Y be any non-empty finite set. !: $Y \to 1$ is onto, and so is Id_Y, so there is a function pt EXP(!, Id): $Y^Y \to Y^1 \cong Y$. Let $h_Y \in Y$ be the image of Id_Y under this function. We show that h_Y is invariant under all permutations of Y, for let σ: $Y \to Y$ be one. We have a diagram –

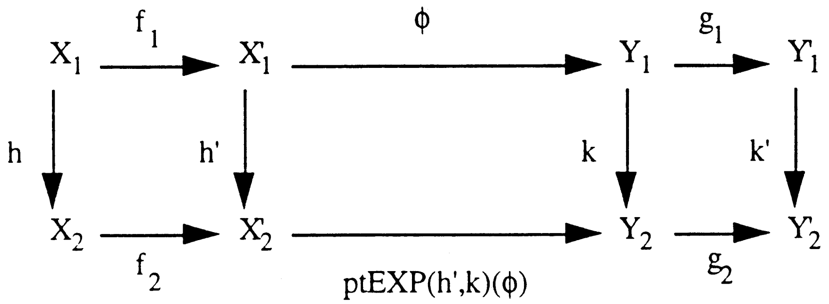


(*)

The two unlabelled morphisms from 1 to Y^Y both select the identity function.

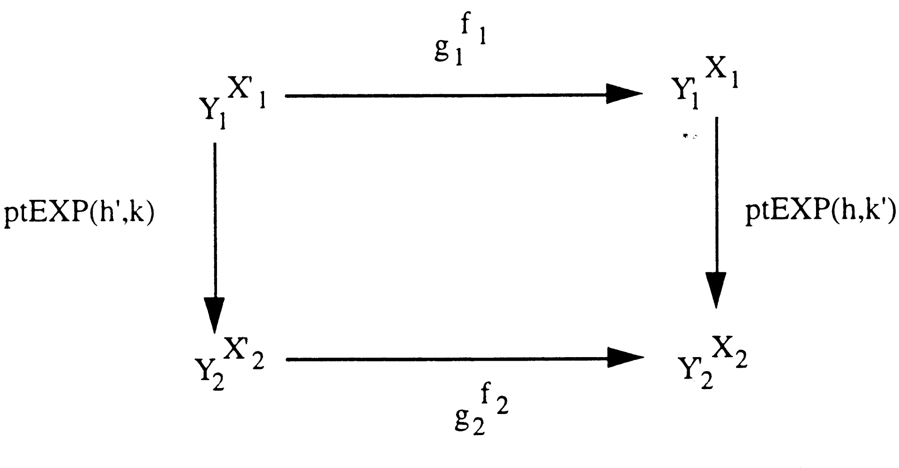
We wish to show that this diagram commutes, for this will show that $\sigma(h_Y) = h_Y$ are desired. The left hand square is obvious, because both arms come to ptEXP(!,σ). For the others we need to investigate the vertical maps. We prove a Lemma that could easily be modified to cover other topical display categories.

Lemma 3.2.2.1 Let there be a diagram as follows, in which the objects are finite sets, h, h', k and k' are all surjective, and the two outer squares commute.



Then the middle square commutes, and the bottom composite edge of the overall rectangle is equal to ptEXP(h, k')(f_1;φ;g_1).

Proof We prove the second part first. Consider the exponential on functions, contravariant in the first argument and covariant in the second. This exists simply by virtue of the internal CCC structure. If C_1 is the topos classifying two finite sets and a function between them, then this exponential is a map MEXP: $C_1{}^2 \rightarrow C_1$. This is functorial with respect to homomorphisms between points, and a homomorphism between points of $C_1{}^2$ is just a pair of squares like the outer two in the given diagram, and so we obtain a commutative square (a homomorphism between points of C_1) like this:



Applying the two arms of this to an element ϕ of $Y_1{}^{X_1'}$ gives our result.

Commutativity of the middle square follows by taking $X_1 = X_2 = X_1'$, $f_1 = h = Id_{X_1'}$ and $f_2 = h'$, and $Y_1' = Y_2' = Y_2$, $k' = g_2 = Id_{Y_2}$ and $g_1 = k$. Then at the bottom we have h';ptEXP(h',k)(ϕ);Id = ptEXP(Id,Id)(ϕ;k) = ϕ;k.]

Let us apply the lemma in the case when all the objects are Y and h' = k = σ. We see that ptEXP(σ, σ)(Id) = σ^{-1}; Id; σ = Id, in other words the left-hand triangle in (*) commutes. Now if we take $X_1' = X_2' = 1$ then we see that ptEXP(Id, σ)(ϕ) = ϕ;σ for all ϕ: 1 $\rightarrow$ Y, and this translates into commutativity of the right-hand square.

Finally, the contradiction follows by taking Y = 2 and σ to be the swap permutation, which has no fixpoints.

For a more positive example, we can now describe a display category that *is* Cartesian closed – that of finite decidable sets. (The argument is already present to some extent in Acuña-Ortega and Linton (1979).) Its display map p: [FinDecSet][elt] $\rightarrow$ [FinDecSet] is the pullback along [FinDecSet] $\rightarrow$ [Set] of the étale classifier. (Note that the intrinsic category on [FinDecSet] is not Cartesian closed at all. Indeed, it is a topical groupoid, for a homomorphism between finite decidable sets must be 1-1 and onto, and hence an isomorphism.) The main point of difficulty lies in defining the exponentials. If X and Y are finite decidable, then [X $\Rightarrow$ Y] can be defined geometrically as

$$\{f \in \mathcal{F}(X \times Y) \mid \forall x \in X.\ \exists y \in Y.\ (x,y) \in f \wedge \forall (x_1,y_1), (x_2, y_2) \in f.\ (x_1 \neq x_2 \vee y_1 = y_2)\}$$

Using 2.1.6 and 2.1.9, this is finite decidable.

4. Strongly algebraic domains

The notion of strongly algebraic (or SFP) domains is due to Plotkin (1976), who gave a variety of mathematical formulations. There are various flavours, and for our present purposes we shall assume neither a bottom point, nor second countability (i.e. the set KD of compact points need not be countable).

Recall the classical definition: an algebraic dcpo D is a *strongly algebraic domain* iff

(i) Every finite subset S of KD has a *finite, complete* set MUB(S) of minimal upper bounds in KD. Here "complete" means that every upper bound of S is greater than one of those in MUB(S).

(ii) Given $S \subseteq_{fin} KD$, define $MUB_0(S) = S$, and

$$MUB_{i+1}(S) = \bigcup \{MUB(U): U \subseteq MUB_i(S)\}$$
$$MUB_\omega(S) = \bigcup_i MUB_i(S), \quad \text{the } \textit{MUB-closure} \text{ of } S$$

We require that $MUB_\omega(S)$ should be be finite for every S.

We shall describe a geometric theory whose models are "strongly algebraic information systems" – those posets satisfying the conditions for KD given above. However, there are certain issues raised by the constructive constraints.

First, is the order decidable? We shall discuss the issues here later (Section 4.7) in more detail, but let us say straight away that we shall not assume decidability. In fact, taking the order decidable or not gives two distinct constructive theories of strongly algebraic information systems. The undecidable version that we present here – which is the harder one when it comes to describing domain constructors – is essentially that given in Abramsky (1991).

Second, the requirement of *minimality* for the bounds in MUB(S) is problematic if the order is undecidable. Classically, if S is a finite subset of a poset then we can discard the non-minimal elements to obtain a subset Min(S) comprising the minimal elements of S, but constructively this is impossible without decidability of $\sqsubseteq$. If it were possible, then homomorphisms between posets – i.e. monotone functions – would have to preserve Min. This is not so, as can be seen by considering the monotone function from 2 (discrete poset on two elements 0 and 1) to $I = \{\bot, \top\}$, with $\bot \sqsubseteq \top$. $\mathrm{Min}\{0, 1\} = \{0, 1\}$, but $\mathrm{Min}\{\bot, \top\} = \{\bot\}$, which is not the image of $\{0, 1\}$. When $\sqsubseteq$ is decidable, then homomorphisms must also preserve $\not\sqsubseteq$ and hence are order embeddings. These do preserve Min, and indeed Min(S) can be expressed geometrically as $\{t \in S: \forall s \in S.\ (s \not\sqsubseteq t \vee s = t)\}$. We shall drop the insistence on minimality and simply require, for each finite set S, the existence of a finite set T that is a complete set of upper bounds of S (and we write CUB(S, T) to express this).

Finite MUB closures are similar: instead of describing $\mathrm{MUB}_\omega(S)$ explicitly and requiring it to be finite, we shall postulate the existence of *some* finite set $T \supseteq S$ such that every finite subset of T has a complete set of upper bounds contained in T.

We express this as a geometric theory as follows:

Definition 4.1.1 The theory *IS* of *(strongly algebraic) information systems* is presented as follows:

(1) a single sort, X (whose elements are commonly called *tokens*)

(2) binary predicate $\sqsubseteq \subseteq X \times X$

(3) axioms to make $\sqsubseteq$ a partial order:

3.1 $\vdash_{t:X} t \sqsubseteq t$

3.2 $s \sqsubseteq t \wedge t \sqsubseteq u \vdash_{s,t,u:X} s \sqsubseteq u$

3.3 $s \sqsubseteq t \wedge t \sqsubseteq s \vdash_{s,t:X} s = t$

(4) a binary predicate $\mathrm{CUB} \subseteq \mathcal{F}X \times \mathcal{F}X$

(5) axioms to say that if CUB(S, T) then T is a complete set of upper bounds for S:

5.1 $\mathrm{CUB}(S, T) \vdash_{S,T:\mathcal{F}X} \forall s \in S.\ \forall t \in T.\ s \sqsubseteq t$

5.2 $\mathrm{CUB}(S, T) \wedge \forall s \in S.\ s \sqsubseteq u \vdash_{S,T:\mathcal{F}X,\ u:X} \exists t \in T.\ t \sqsubseteq u$

(6) an axiom to say that every finite set of tokens has a finite complete set of upper bounds:

$$\vdash_{S:\mathcal{F}X} \exists T:\mathcal{F}X.\ \mathrm{CUB}(S, T)$$

(7) an axiom to ensure that if T is a finite complete set of upper bounds for S, then CUB(S,T):

$$\forall s \in S.\ \forall t \in T.\ s \sqsubseteq t \wedge \mathrm{CUB}(S, T') \wedge \forall t' \in T'.\ \exists t \in T.\ t \sqsubseteq t'$$
$$\vdash_{S,T,T':\mathcal{F}X} \mathrm{CUB}(S, T)$$

(8) an axiom to say that every finite set of tokens has a finite MUB-closure

$$\vdash_{S:\mathcal{F}X} \exists T:\mathcal{F}X.\ (S \subseteq T \wedge \mathrm{CUBcl}(T))$$

where $\mathrm{CUBcl}(T) \equiv_{def} \forall U \subseteq_{fin} T.\ \exists V \subseteq_{fin} T.\ \mathrm{CUB}(U, V)$

Notes –

(i) (6) is a consequence of (8) and hence superfluous. However, we make it explicit in order to point out that (1)-(7) axiomatize the *spectral algebraic* or *2/3 SFP* information systems.

(ii) $\mathrm{CUB}(S, T) \Leftrightarrow \forall s \in S.\ \forall t \in T.\ s \sqsubseteq t \wedge \forall u.\ ((\forall s \in S.\ s \sqsubseteq u) \rightarrow \exists t \in T.\ t \sqsubseteq u)$
The $\Rightarrow$ direction is just a rewriting of axioms (5). For $\Leftarrow$, choose T' such that CUB(S, T'). If $t' \in T'$, then $\forall s \in S.\ s \sqsubseteq t'$ and so $\exists t \in T.\ t \sqsubseteq t'$; we can now use (7).

(iii) It follows from (ii) that the axioms for CUB make it uniquely determined by $\sqsubseteq$. Hence the map [IS] $\rightarrow$ [poset] is a monomorphism of toposes, though it is not an inclusion. (If it were, i.e. if [IS] were a subtopos of [poset], then its structure would have to be inherited from [poset] and in particular the homomorphisms of information systems would just be the monotone functions between posets. But we shall see later that preservation of CUB makes them more restricted.)

(iv) *Classically,* this new definition is equivalent to the old one: a poset $(X, \sqsubseteq)$ is equivalent to the set of compact points of a strongly algebraic domain iff it can be equipped with a predicate CUB making it a model for IS. For the $\Rightarrow$ direction we can define CUB(S, T) iff T is a complete set of upper bounds of S, and for $\Leftarrow$, suppose $S \subseteq_{fin} X$ and CUB(S, T). By taking the minimal elements of T, we get a finite complete set of minimal upper bounds of S. Let $U \supseteq S$ be finite and CUB-closed. The chain $(\mathrm{MUB}_i(S))$ can be constructed in U, and so $\mathrm{MUB}_\omega(S) \subseteq U$ is finite.

(v) In practice, we don't need to describe CUB fully. For suppose X, $\sqsubseteq$ and CUB_0 satisfy axioms (1) - (6) in Definition 4.1.1 for X, $\sqsubseteq$ and CUB. Then we can make a unique spectral algebraic information system using X and $\sqsubseteq$ by defining

$$\mathrm{CUB}(S, T) \equiv_{def} \forall s \in S.\ \forall t \in T.\ s \sqsubseteq t \wedge \exists T' : \mathcal{F}X.\ (\mathrm{CUB}_0(S, T') \wedge \forall t' \in T'.\ \exists t \in T.\ t \sqsubseteq t')$$

(vi) A discrete poset (i.e. a set X) can be equipped with the structure of a strongly algebraic information system iff it is finite and decidable. If X is equipped with CUB, then for some finite S we have $\mathrm{CUB}(\emptyset, S)$, from which we see that $X = S$ is finite; and $s \neq t$ iff $\mathrm{CUB}(\{s,t\}, \emptyset)$. Conversely, if X is finite decidable then CUB(S,T) iff $S = \emptyset$ and $T = X$, or $S = \{s\} = T$ for some s, or there are $s \neq t$ in S and $T = \emptyset$.

Definition 4.1.2 A *strongly algebraic domain* is the ideal completion of a strongly algebraic information system. More precisely, there is an obvious map from [IS] to [poset], and the pullback along this of the algebraic dcpo classifier is the *strongly algebraic classifier.* We shall usually write it as p: [IS][pt] $\rightarrow$ [IS]. A strongly algebraic domain (over a given topos) is a pullback of the classifier.

We have already mentioned that our domains without bottom might more usually be called predomains. However, a more subtle connotation of "predomain" is "something whose lift is a domain" so that one can move between domains and predomains by adding or removing bottom. This is not possible for us. For instance, all flat domains (lifted sets) are strongly algebraic, but by note (vi) above the unlifted sets are not, except when finite decidable.

From the general theory of algebraic dcpos, we know that maps between strongly algebraic domains are given by approximable mappings. Hence we get a theory *AM* of *strongly algebraic* approximable mappings, i.e. those for which the source and target posets are both strongly algebraic information systems.

Besides the continuous maps between domains, it is interesting also to consider the *homomorphisms* between information systems, defined in the standard way for models of a geometric theory.

Definition 4.1.3 Let X and Y be two strongly algebraic (or, indeed, spectral algebraic) information systems. A *homomorphism* from X to Y is a monotone function f: X → Y that preserves CUB:

$$\mathrm{CUB}(S, T) \Rightarrow \mathrm{CUB}(f(S), f(T))$$

Proposition 4.1.4 Let X and Y be strongly algebraic (or spectral algebraic) information systems. Then there is a bijection between –

- homomorphisms from X to Y
- adjunctions between Idl(X) and Idl(Y)

Proof Let f: X → Y be a homomorphism; in preserving $\sqsubseteq$ it is monotone. We define approximable mappings ϕ: X → Y and ψ: Y → X by

$$s\ \phi\ t \quad \text{iff } f(s) \sqsupseteq t$$
$$t\ \psi\ s \quad \text{iff } t \sqsupseteq f(s)$$

To see that ψ is indeed an approximable mapping, let S: $\mathcal{F}$X with $t \sqsupseteq f(s)$ for all $s \in S$, and suppose CUB(S, T). Because f preserves CUB, we also have CUB(f(S), f(T)). t is an upper bound for f(S), so we can find $t' \in T$ such that $t \sqsupseteq f(t')$, i.e. $t\ \psi\ t'$ and $t' \sqsupseteq s$ for all $s \in S$.

If $s \sqsupseteq s'$ then $s\ \phi\ f(s)\ \psi\ s'$, so $Id_X \sqsubseteq \phi;\psi$. If $t\ \psi\ s\ \phi\ t'$, then $t \sqsupseteq f(s) \sqsupseteq t'$, so $\psi;\phi \sqsubseteq Id_Y$. Hence (ϕ, ψ) is an adjoint pair.

Now let (ϕ, ψ) be an arbitrary adjoint pair of approximable mappings. If s: X then $s \sqsupseteq s$, so we can find t with $s\ \phi\ t\ \psi\ s$. Moreover, this t is unique, for if t' is another then $t\ \psi\ s\ \phi\ t'$ so that $t \sqsupseteq t'$, and by symmetry $t' \sqsupseteq t$. Let us write f(s) for this t. f is monotone, for if $s \sqsupseteq s'$ then $f(s)\ \psi\ s \sqsupseteq s'\ \phi\ f(s')$, so $f(s)\ (\psi;\phi)\ f(s')$ and so $f(s) \sqsupseteq f(s')$. To show that f preserves CUB, suppose CUB(S, T): we must show that CUB(f(S), f(T)). By monotonicity, f(T) is a set of upper bounds of f(S). To show completeness, suppose u is an upper bound of f(S): $\forall s \in S.\ u \sqsupseteq f(s)$. Then $u\ \psi\ s$, so S has an upper bound s' such that $u\ \psi\ s'$. We can find $t \in T$ such that $s' \sqsupseteq t$, so $u\ \psi\ t\ \phi\ f(t)$ and $u \sqsupseteq f(t)$.

If we start with f: X → Y and construct (ϕ, ψ) as above, then certainly $s\ \phi\ f(s)\ \psi\ s$, so f is recovered from (ϕ, ψ) in the way described. The other way round, suppose we start with (ϕ, ψ) and construct f. Then first, $s\ \phi\ t \Leftrightarrow f(s) \sqsupseteq t$. If $s\ \phi\ t$ then $f(s)\ \psi\ s\ \phi\ t$, so $f(s) \sqsupseteq t$, while if $f(s) \sqsupseteq t$ then $s\ \phi\ f(s) \sqsupseteq t$, so $s\ \phi\ t$. Next, $t\ \psi\ s \Leftrightarrow t \sqsupseteq f(s)$. If $t\ \psi\ s$ then $t\ \psi\ s\ \phi\ f(s)$, so $t \sqsupseteq f(s)$, while if $t \sqsupseteq f(s)$ then $t \sqsupseteq f(s)\ \psi\ s$, so $t\ \psi\ s$. Hence ϕ and ψ can be recovered from f as described above.]

Leading on from this, one can show that, using the definitions of Johnstone [93], the strongly algebraic classifier is both a fibration (homomorphism f gives map ψ) and an opfibration (f gives ϕ).

We shall now look at constructions on strongly algebraic domains – products, coproducts, function spaces and so on. (It is worth noticing that the general techniques seen in Hyland and Pitts (1989) indicate how to go beyond these domain constructions to the construction of terms for maps between domains (as also appear in Abramsky (1991).) For each of these constructions we show how to construct a new information system out of old ones, and there are usually two issues. First, does the new one have the right points? (Is the corresponding pullback of p the topos that we asked for in section 3.2.1?) Second, is the constructed information system still strongly algebraic?

In fact, this work is largely indebted to that of Abramsky (1991). He gave a localic account of SFP domains (with bottom) by describing a formal language for the compact opens that appear in various constructions – specifically, products, coalesced sums, lifting, functions spaces, the Plotkin power domain, and solutions of recursive domain equations. (Our treatment differs slightly, in a way that has been suggested by Abramsky himself, in that our syntax uses the information systems – the posets of compact points – instead of the distributive lattices of compact opens. This generally simplifies the presentation – though perhaps not for the function spaces – but the difference is not a deep one.)

Part of Abramsky's method relies on certain predicates on the terms that represent compact opens: binary predicates ≤ and =, and unary predicates **C** and **T** (**C**(a) means that a is a coprime compact open, **T**(a) means that a ≠ **true**). Because of the presence of the recursive solutions to domain equations, the definitions of these predicates are also recursive and so it is essential that the predicates occur positively in the definitions. For instance, one cannot ensure merely by definitional fiat that if ¬(a = **true**) then **T**(a), because the recursive nature means that one only gradually discovers which a's are equal to **true**. **T** must be defined by positive means, after which it is possible to prove that **T**(a) ⇔ a ≠ **true**.

Because of this, the requirements of positivity and constructivity called for by the use of geometric logic were also called for on quite immediate computational grounds in Abramsky's work, and so essentially the work of constructivizing has already been done by him. But one can also look at this in reverse: the use of geometric logic implied by the topologization programme automatically imposes strong constructive constraints that turn out to be necessary in syntactic computation. (Compare this with the lack of constraints imposed by classical logic in Vickers (1989): the apparently simpler treatment there sometimes uses arguments that are constructively useless in Abramsky's formal system. A good example is the account of strongly algebraic function spaces.)

A more significant difference is Abramsky's restriction to *local* domains (i.e. with bottom). This makes it necessary to have a different treatment of sums (because our coproducts are not local), and to construct amalgamated sums one requires a predicate to describe the negative information of when a token is not bottom – this appears as Abramsky's "termination" predicate **T** mentioned above. This issue is discussed further in Section 5.

Pre-information systems

In Definition 4.1.1 we defined our information systems to be partial orders. It is actually often more convenient to work with preorders. For instance, for the Plotkin powerdomain PD, the tokens can then be considered to be finite sets of tokens of D, under the Egli-Milner ordering, a preorder. However, certain technical simplifications come from the partial ordering assumption. We shall now show that in fact we can get the best of both worlds by taking the poset reflection of a preorder. The technical point is that the axiomatization of CUB is so closely constrained by the order that it too respects the poset reflection.

Definition 4.1.5 The theory of (strongly algebraic) *pre-information systems* is defined exactly as in 4.1.1, but with the order allowed to be a preorder: the axiom $s \sqsubseteq t \wedge t \sqsubseteq s \vdash_{s,t:X} s = t$ is omitted.

Proposition 4.1.6 Let (X_0, $\sqsubseteq$, CUB_0) be a pre-information system, and let (X, $\sqsubseteq$) be the poset reflection of (X_0, $\sqsubseteq$) with quotient function f: $X_0 \to X$. Define

$$CUB(S, T) \equiv_{def} \exists S_0, T_0: \mathcal{F}X_0.\ S = f(S_0) \wedge T = f(T_0) \wedge CUB_0(S_0, T_0)$$

Then (X, $\sqsubseteq$, CUB) is an information system.

Proof

Straightforward. Note that if $S = f(S_0) \wedge T = f(T_0)$, then the intuitionistic formula

$$\forall s \in S.\ \forall t \in T.\ s \sqsubseteq t\ \wedge\ \forall u.\ ((\forall s \in S.\ s \sqsubseteq u) \to \exists t \in T.\ t \sqsubseteq u)$$

is equivalent to the corresponding one in X_0 for S_0 and T_0, which is equivalent to $CUB_0(S_0, T_0)$.]

4.2 Products

Given two information systems, X_1 and X_2, their product is defined as follows: the poset is $X_1 \times X_2$ with the product order, $(s_1, s_2) \sqsubseteq (t_1, t_2)$ iff $s_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq t_2$.

We haven't defined CUB yet, but already it is clear that if we can, then this is indeed the product: for an ideal I of $X_1 \times X_2$ is equivalent to a pair of ideals, one from each X_i: $I_1 = \{x_1: \exists x_2. (x_1, x_2) \in I\}$ and I_2 is similar. Of course, one should check that the argument is constructive.

Next, we show that the new information system is strongly algebraic. CUB is defined as in note (v) after 4.1.1 from CUB_0, defined by –

$$CUB_0(S, T) \equiv_{def} \exists T_1{:}\mathcal{F}X_1, T_2{:}\mathcal{F}X_2.\ (CUB(p_1(S), T_1) \wedge CUB(p_2(S), T_2) \wedge T = T_1{\times}T_2)$$

where p_i is the ith projection. The basic reasoning is that (s_1,s_2) is an upper bound for S iff each s_i is an upper bound for $p_i(S)$. If M_i is CUB-closed containing $p_i(S)$, then $M_1{\times}M_2$ is CUB-closed containing S.

The terminal domain is the nullary analogue of this: the poset is $1 = \{*\}$, and CUB(S, T) iff $T = \{*\}$.

4.3 Coproducts

Given two information systems X_1 and X_2, their sum is defined as follows. The poset is the coproduct (disjoint union) $X = X_1 + X_2$ with the sum order: $s \sqsubseteq t$ iff s and t are in the same component X_i, and $s \sqsubseteq t$ in X_i.

To show that this sum gives a coproduct of toposes, we must show that ideals of X are in 1-1 correspondence with points of the coproduct (see Section 2.2). If I is an ideal of X, then we have a complementary pair of propositions $P \equiv \exists x{:}X_1.\ I(x)$ and $\neg P \equiv \exists x{:}X_2.\ I(x)$. Writing $I_i = I \cap X_i$, we get that I_1 or I_2 is an ideal of X_1 or X_2 accordinag as P or $\neg$P, so **if** P **then** I_1 **else** I_2 is a point of $Idl(X_1)+Idl(X_2)$. Conversely, given **if** P **then** I_1 **else** I_2, then $I = I_1+I_2$ is an ideal of X.

Noting that $\mathcal{F}(X_1+X_2) \cong \mathcal{F}X_1{\times}\mathcal{F}X_2$ (the free algebra functor $\mathcal{F}$: **Set** $\to$ **Semilattice** preserves all colimits, and for semilattices Cartesian product is a biproduct), we can define

$$CUB((S_1, S_2), (T_1, T_2)) \text{ iff } CUB(S_i, T_i)\ (i = 1, 2)$$

and if M_i is CUB-closed containing S_i: $\mathcal{F}X_i$ then (M_1, M_2) is CUB-closed containing (S_1, S_2).

The initial domain is the nullary analogue: the poset is Ø, and its unique finite subset Ø is CUB-closed.

4.4 Lifting

If X is an information system, then its lift $X_\perp$ is the poset $\{\perp\}+X$ ordered by $s \sqsubseteq t$ iff $s = \perp$ or $s \sqsubseteq t$ in X.

If S: $\mathcal{F}X$, then we shall write $S = S_1+S_2$ where S_1: $\mathcal{F}\{\perp\}$ and S_2: $\mathcal{F}X$. Then we define –

$$\mathrm{CUB}_0(S, T) \equiv_{def} \exists T':\mathcal{F}X. \ (\mathrm{CUB}(S_2, T') \wedge T = \{\perp: S_2 = \emptyset\} + \{t \in T': S_2 \neq \emptyset\})$$

If M is CUB-closed containing S_2, then $\{\perp\}+M$ is CUB-closed containing S.

The proof that the points are right is somewhat similar to that for coproducts. If I is an ideal of $X_\perp$, then by taking $P = \{* \in 1: \exists x:X.\ I(x)\}$ we get a P-indexed family of ideals of X, where P is a subset of 1.

4.5 Exponentials (function spaces)

Despite the expositional differences, the mathematical substance of this section is that of Abramsky (1991), starting from his Definition 3.4.1. We shall see how the geometric constraints automatically impose the constructivity that Abramsky required.

Let X_s and X_t be two information systems. We wish to define another information system $[X_s \Rightarrow X_t]$ whose points are the approximable mappings from X_s to X_t, and the compact points will be the approximable mappings f that are determined by a finite amount of information $U \subseteq_{fin} f \subseteq X_s \times X_t$. (In terms of the compact open topology, any $U \subseteq_{fin} X_s \times X_t$ corresponds to a basic open, the conjunction of the subbasics $f(\uparrow x) \subseteq \uparrow y$ for $(x,y) \in U$.) Abramsky identified conditions on U (our "fully summarizing") for there to be a least approximable mapping containing it.

Definition 4.5.1 Let $U \subseteq_{fin} X_s \times X_t$.

(i) Suppose $V \subseteq_{fin} U$. We shall say that $W \subseteq_{fin} U$ is a *summary* of V in U iff –

fst(W) is a complete set of upper bounds for fst(V)
$\wedge$ snd(W) is a set of upper bounds for snd(V)

(ii) U is *fully summarizing* iff every $V \subseteq_{fin} U$ has a summary in U. (Note that this is a geometric property of U.)

(iii) The preorder $\sqsubseteq$ on $\mathcal{F}(X_s \times X_t)$ is defined by

$$U \sqsubseteq U' \equiv_{def} \forall (u_s,u_t) \in U.\ \exists (u_s',u_t') \in U'.\ (u_s \sqsupseteq u_s' \wedge u_t' \sqsupseteq u_t)$$

(iv) The (pre-)information system $[X_s \Rightarrow X_t]$ has for its tokens the fully summarizing finite subsets of $X_s \times X_t$, ordered by $\sqsubseteq$.

Let us note immediately the following lemma:

Lemma 4.5.2 Let U, $U' \subseteq_{fin} X_s \times X_t$ with U fully summarizing.

(i) An approximable mapping f_U can be defined by $x\ f_U\ y$ iff $\exists (x', y') \in U$ such that $x \sqsupseteq x'$ and $y' \sqsupseteq y$. It is the least approximable mapping containing U.

(ii) $U' \sqsubseteq U$ iff $U' \subseteq f_U$.

(iii) If U' also is fully summarizing, then $U' \sqsubseteq U$ iff $f_{U'} \subseteq f_U$.

Proof The only part that presents any difficulty is the "ideal" condition of approximable mappings in (i). Suppose $x\ f_U\ y_i$ $(1 \leq i \leq n)$, with $x \sqsupseteq x_i'$, $y_i' \sqsupseteq y_i$ and $(x_i', y_i') \in U$. Let $V = \{(x_i', y_i'): 1 \leq i \leq$

n}, and let W be a summary for it in U. Then since x is an upper bound for fst(V) we have $x \sqsupseteq x''$ for some (x", y") in W, and then $x\ f_U\ y''$ and y" is an upper bound for the y_i's.]

CUB (or rather, as in note (v) after Definition 4.1.1, CUB_0) is defined by what is in effect a description in geometric logic of Abramsky's (1991) normalization algorithm for function spaces (which normalizes expressions representing compact opens of the function space). Of course, we already have an *intuitionistic* formulation of CUB, but we require a geometric one. Because of the positivity of the logic, that will have the flavour of attaining CUB "from below".

Let us consider a preorder < on $\mathcal{FF}(X_s \times X_t)$, defined intuitionistically by $\mathcal{U} < \mathcal{V}$ iff

$$\forall V \in \mathcal{V}.\ \exists U \in \mathcal{U}.\ U \sqsubseteq V$$
$$\wedge\ \forall U \in \mathcal{U}.\ \forall f \subseteq X_s \times X_t.\ (f \text{ an approximable mapping} \wedge U \subseteq f \rightarrow \exists V \in \mathcal{V}.\ U \sqsubseteq V \subseteq f)$$

Lemma 4.5.3 If $\mathcal{U}, \mathcal{V} \subseteq_{fin} [X_s \Rightarrow X_t]$ and $\{\bigcup \mathcal{U}\} < \mathcal{V}$, then $\mathcal{V}$ is a complete set of upper bounds for $\mathcal{U}$.

Proof If $V \in \mathcal{V}$, then $\bigcup \mathcal{U} \sqsubseteq V$ and so $U \sqsubseteq V$ for all U in $\mathcal{U}$.

If $W: [X_s \Rightarrow X_t]$ is an upper bound for $\mathcal{U}$, then $\bigcup \mathcal{U} \subseteq f_W$ and so $\bigcup \mathcal{U} \sqsubseteq V \subseteq f_W$ for some V in $\mathcal{V}$, so $V \sqsubseteq W$.]

Our strategy now is as follows. We define a geometric formula $\Phi(\mathcal{U}_1, \mathcal{U}_2)$ contained in <, which is to represent a single iteration of Abramsky's algorithm (which is non-deterministic). Since < is a preorder, the reflexive transitive closure Φ^* (which is still geometric) is also contained in <. $CUB_0(\mathcal{U}_0, \mathcal{U})$ is then defined as $\Phi^*(\{\bigcup \mathcal{U}_0\}, \mathcal{U})$ and Lemma 4.5.3 gives us everything we need except for existence. (In effect, the algorithm has a loop invariant $\{\bigcup \mathcal{U}_0\} < \mathcal{U}$.) We then show that for every $U_0 \subseteq_{fin} X_s \times X_t$ there is some $\mathcal{U} \subseteq_{fin} [X_s \Rightarrow X_t]$ such that $\Phi^*(\{U_0\}, \mathcal{U})$ and this corresponds to the termination proof of the algorithm (if executed judiciously enough).

Before Φ, we first define a geometric predicate $\Psi(U_0, \mathcal{U}; V, W_s, W_t, M_s, M_t)$ as –

M_i is a CUB-closed finite subset of X_i and $W_i \subseteq_{fin} M_i$ (i = s, t)

$V \subseteq_{fin} U_0 \subseteq_{fin} M_s \times M_t$

W_s and W_t are complete sets of upper bounds for fst(V) and snd(V)

$\mathcal{U} = \{U_0 \cup R$: R a finite, total relation from W_s to $W_t\}$ (finite, by Lemma 2.1.10)

If we are just given $V \subseteq_{fin} U_0 \subseteq_{fin} M_s \times M_t$, then we can certainly find W_s, W_t and $\mathcal{U}$ such that $\Psi(U_0, \mathcal{U}; V, W_s, W_t, M_s, M_t)$. Each finite total relation R summarizes V in $U_0 \cup R$, and $\mathcal{U}$ in effect represents the different possible ways of extending U_0 to summarize V.

Lemma 4.5.4 Suppose $\Psi(U_0, \mathcal{U}; V, W_s, W_t, M_s, M_t)$ and $U_0 \subseteq f$ with f an approximable mapping. Then $U_0 \sqsubseteq U \subseteq f$ for some U in $\mathcal{U}$.

Proof

$V \subseteq_{fin} f$, so for each x in W_s we have x f y' for every (x', y') in V and hence we can find y such that x f y and y is an upper bound for snd(V) and without loss of generality $y \in W_t$. Hence there is a finite total relation R from W_s to W_t such that $R \subseteq f$, which is what we wanted.]

In defining Φ, we shall fix M_s and M_t – this is needed in order to provide finite bounds. $\Phi(\mathcal{U}_1, \mathcal{U}_2)$ ($\mathcal{U}_1, \mathcal{U}_2 \subseteq_{fin} \mathcal{F}(M_s \times M_t)$) shall then mean that there are U_0, $\mathcal{U}'$, V, $\mathcal{U}$, W_s and W_t such that

$\mathcal{U}_1 = \{U_0\} \cup \mathcal{U}'$
$\Psi(U_0, \mathcal{U}; V, W_s, W_t, M_s, M_t)$
$\mathcal{U}_2 = \mathcal{U} \cup \mathcal{U}'$

In other words, we have selected from $\mathcal{U}_1$ an element U_0 and a subset V, found corresponding W_s, W_t and $\mathcal{U}$ for Ψ, and replaced U_0 in $\mathcal{U}_1$ by the elements of $\mathcal{U}$ to get $\mathcal{U}_2$. It is plain that Φ is contained in <, so Φ^* is too. Note also that if $\Phi(\mathcal{U}_1, \mathcal{U}_2)$ then $\Phi(\mathcal{U}_1 \cup \mathcal{V}, \mathcal{U}_2 \cup \mathcal{V})$, and so the same goes for Φ^*. We can deduce that if $\Phi^*(\mathcal{U}_i, \mathcal{V}_i)$ $(1 \le i \le n)$, then $\Phi^*(\bigcup_i \mathcal{U}_i, \bigcup_i \mathcal{V}_i)$.

Finally, we must prove termination. This is quite subtle, for the algorithm is non-deterministic and can easily go into an infinite loop by selecting unintelligent choices. Hence the proof must in effect also show how to find a terminating branch and how to know when to terminate.

Lemma 4.5.5 Let M_s and M_t be finite CUB-closed subsets of X_s and X_t, and suppose $U_0 \subseteq_{fin} M_s \times M_t$. Then there exists $\mathcal{W} \subseteq_{fin} [X_s \Rightarrow X_t]$ such that $\Phi^*(\{U_0\}, \mathcal{W})$.

Proof

Consider the following intuitionistic predicate defined for $A \subseteq_{fin} M_s \times M_t$ and $\mathcal{B} \subseteq_{fin} \mathcal{F}(M_s \times M_t)$:

$$\begin{aligned} P(A, \mathcal{B}) \equiv\ & \forall U \subseteq_{fin} M_s \times M_t. \\ & \quad ((U \cup A = M_s \times M_t \wedge \mathcal{B} \subseteq \mathcal{F}(U) \\ & \qquad \wedge \forall V \subseteq_{fin} U. (V \in \mathcal{B} \vee V \text{ has a summary in } U)) \\ & \quad \to \exists \mathcal{W} \subseteq_{fin} [X_s \Rightarrow X_t].\ \Phi^*(\{U\}, \mathcal{W})) \end{aligned}$$

We shall prove that $\forall A, \mathcal{B}.\ P(A, \mathcal{B})$, using strong $\mathcal{F}$-induction (Theorem 2.1.11) on A and simple $\mathcal{F}$-induction (Theorem 2.1.3) on $\mathcal{B}$. Effectively, the induction on $\mathcal{B}$ is an induction on the number of subsets of U not yet checked to have a summary, while that on A is induction on the number of elements of $M_s \times M_t$ not in U. These "numbers of elements" do not of course exist as genuine cardinalities (for which we would need decidable equality on the elements), but they are there as lengths of lists representing the finite sets and this is seen explicitly in Abramsky's account. We have chosen to work more abstractly, without using explicit list representations, but nonetheless you can see them in the proof of 2.1.11. For a given U_0, the result will follow from $P(M_1 \times M_2, \mathcal{F}(U_0))$.

The outer induction is on A, so let us fix A with the induction hypothesis that

$$\forall a \in A.\ \exists A' \subseteq_{fin} M_1 \times M_2.\ (A = \{a\} \cup A' \wedge \forall \mathcal{B}.\ P(A', \mathcal{B}))$$

We shall prove $\forall \mathcal{B}.\ P(A, \mathcal{B})$ by simple induction on $\mathcal{B}$. First, $P(A, \emptyset)$ is obvious: if U satisfies the conditions to the left of the implication, then it is already fully summarizing, so we can take $\mathcal{W} = \{U\}$. Next, we assume $P(A, \mathcal{B})$ and prove $P(A, \{V\} \cup \mathcal{B})$. Let U satisfy the premisses of the implication. Starting from the given $V \subseteq_{fin} U$, we can find $\mathcal{U}$, W_s and W_t so that $\Psi(U, \mathcal{U}; V, W_s, W_t, M_s, M_t)$. If R is a finite total relation from W_s to W_t, then $R \subseteq M_s \times M_t = U \cup A$, so we can find $R_U \subseteq_{fin} U$ and $R_A \subseteq_{fin} A$ such that $R = R_U \cup R_A$. If $R_A = \emptyset$, then $R \subseteq U$ and so V has a summary in U. Hence U also satisfies the premisses in $P(A, \mathcal{B})$, so by induction we can find $\mathcal{W}$ as required. If $R_A \neq \emptyset$ (remember that emptiness is decidable for finite sets) then take some $a \in R_A$. By the induction hypothesis, we can find $A' \subseteq_{fin} M_s \times M_t$ such that $A = \{a\} \cup A'$ and $\forall \mathcal{B}.\ P(A', \mathcal{B})$. $U \cup R$ satisfies the premisses for $P(A', \mathcal{F}(U \cup R))$ and so we can find suitable $\mathcal{W}$. We have now shown that

$$\forall U' \in \mathcal{U}.\ \exists \mathcal{W} \subseteq_{fin} [M_s \Rightarrow M_t].\ \Phi^*(\{U'\}, \mathcal{W})$$

Now by taking the union of finitely many such $\mathcal{W}$s we can find $\mathcal{W}$ such that $\Phi^*(\mathcal{U}, \mathcal{W})$; and since $\Phi(\{U\}, \mathcal{U})$ the result follows.]

The algorithmic content of this is as follows. A state is a finite set σ of triples $(U_1, A, \mathcal{B})$ such that $U_1 \cup A = M_s \times M_t$, $\mathcal{B} \subseteq \mathcal{F}U_1$ and $\forall V \subseteq_{fin} U_1. (V \in \mathcal{B} \vee V$ has a summary in $U_1)$: hence the induction variables A and $\mathcal{B}$ appear explicitly in the computation. The reason for this is that in order to know when to terminate, we must recognize when our U_1's are fully summarizing and $\mathcal{B}$ contains the subsets V for which we must still check for the existence of summaries or create summaries using Ψ. Using Ψ changes U_1 and so the checking must start all over again, but occurences of this are limited by A which contains the elements not already known to be in U_1. We also have the loop invariant $\{U_0\} < \{U_1 : (U_1, A, \mathcal{B}) \in \sigma\}$. A step in the algorithm is then –

- select $(U_1, A, \mathcal{B})$ from σ with $\mathcal{B} \neq \emptyset$ (if there are none, we can stop)
- select $V \in \mathcal{B}$, leaving $\mathcal{B} = \{V\} \cup \mathcal{B}'$
- find $\mathcal{U}$, W_s and W_t so that $\Psi(U_1, \mathcal{U}; V, W_s, W_t, M_s, M_t)$
- for each finite total relation R from W_s to W_t, decomposed as $R_U \cup R_A$, find a corresponding new state element $(U_1, A, \mathcal{B}')$ if $R_A = \emptyset$, or $(U_1 \cup R, A', \mathcal{F}(U_1 \cup R))$ if $a \in R_A$, where A' is A with a removed – or at least, one occurrence of a is removed from the representation of A.
- The new state is the old state σ with $(U_1, A, \mathcal{B})$ replaced by all the new state elements just found.

We have now proved that if X_s and X_t are strongly algebraic information systems, then so is $[X_s \Rightarrow X_t]$ – so we have defined a geometric morphism EXP: $[IS]^2 \to [IS]$.

Proposition 4.5.6 The points of $[X_s \Rightarrow X_t]$ are equivalent to approximable mappings from X_s to X_t.

Proof If I is an ideal of $[X_s \Rightarrow X_t]$ then we can define an approximable mapping f as the union of the f_U's for U in I. Conversely, if f is an approximable mapping, then let I be the set $\{U \in [X_s \Rightarrow X_t] : U \subseteq f\}$. The only point of difficulty so far is the ideal property of I. Suppose $U_i \in [X_s \Rightarrow X_t]$, $U_i \subseteq f$ $(1 \le i \le n)$. We can find $\mathcal{U} \subseteq_{fin} [X_s \Rightarrow X_t]$ such that $\Phi^*(\{\bigcup_i U_i\}, \mathcal{U})$, and then because $\{\bigcup_i U_i\} < \mathcal{U}$ we have by Lemma 4.5.3 that the U_i's have an upper bound contained in f.

Now suppose we start with I, construct f as above, and then construct I' from f. If $U \in I$ then $U \subseteq f_U \subseteq f$, so $U \in I'$. On the other hand, if $U \in I'$ then for each $u \in U$ we can find U' in I such that $u \in f_{U'}$, and by taking an upper bound U" in I we have each $u \in f_{U''}$ and so $U \subseteq U''$, $U \in I$.

Finally, suppose f is an approximable mapping, let I be defined as above, and then f' from I. If x f' y then x f_U y for some $U \subseteq f$ and so x f y. If x f y then we can find $\mathcal{U}$ such that $\Phi^*(\{\{(x,y)\}\}, \mathcal{U})$, and $\{(x, y)\} \subseteq U \subseteq f$ for some $U \in \mathcal{U}$, so $(x,y) \in f_U$ and x f' y.]

We have now proved –

Theorem 4.5.7 If X_s and X_t are strongly algebraic information sytems then so is $[X_s \Rightarrow X_t]$, and its points are equivalent to approximable mappings from X_s to X_t.]

It follows that, as we wanted, [AM] is a strongly algebraic domain over $[IS]^2$.

4.6 Power domains

Robinson (1986) showed that the well-known Hoare, Smyth and Plotkin (or lower, upper and convex) power domains can be constructed localically, and in fact they are instances of more general powerlocale constructions P_L (lower), P_U (upper) and V (Vietoris). V originated in Johnstone (1982a), while the simpler P_L and P_U are folklore. A constructive account of all three can be found in Vickers (1997).

Definition 4.6.1 Let X be an information system. The *lower, upper* and *convex* power domains, P_LX, P_UX and P_CX, are defined respectively as follows:

- They all have the same tokens, namely $\mathcal{F}X$.
- They have preorders defined as –

$$S \sqsubseteq_L T \equiv \forall s \in S.\ \exists t \in T.\ s \sqsubseteq t$$
$$S \sqsubseteq_U T \equiv \forall t \in T.\ \exists s \in S.\ s \sqsubseteq t$$
$$S \sqsubseteq_C T \equiv S \sqsubseteq_L T \wedge S \sqsubseteq_U T \quad \text{(the \textit{Egli-Milner} ordering)}$$

Note that we do not follow the common convention of excluding the empty set (though there is no constructive problem in doing so if that is what is required). Consequently, each domain includes an "empty" point – in P_L it is bottom, in P_U it is top and in V it is isolated.

The ideal completions of these are homeomorphic to the corresponding powerlocales, and a general proof (covering non-local domains as well as continuous domains) is in Vickers (1993). In what remains, the hard work amounts to a proof that if a domain is spectral algebraic, then so are its power domains.

CUB_L and CUB_U come out from the fact that $\sqsubseteq_L$ and $\sqsubseteq_U$ both make $\mathcal{F}X$ into a join (pre-)semilattice: $S \cup T$ is the join of S and T in P_LX, while in P_UX it is got by taking a union of sets U_{st} such that $CUB(\{s,t\}, U_{st})$ ($s \in S$, $t \in T$). (We have neglected the nullary joins but they are not a problem.) I conjecture that if we are prepared to talk about a topical order-enriched category, then P_LX and P_UX can be characterized as the free join- and meet-semilattices over X. (By a *meet-semilattice* in an order enriched Cartesian category I mean a semilattice S for which the n-ary semilattice operation: $S^n \rightarrow S$ is right adjoint to the diagonal: $S \rightarrow S^n$, in other words a *Cartesian object*, and similarly for join-semilattices but with the adjunction the other way round.)

Let us now concentrate on the convex powerdomain. Just as for the function space, the essential working is already in Abramsky (1991), so this time we shall do no more than sketch the information system theoretic account. If $\mathcal{U}: \mathcal{F}\mathcal{F}X$, then we need to ask when $T: \mathcal{F}X$ is an upper bound for $\mathcal{U}$. For every $U \in \mathcal{U}$ we have $U \sqsubseteq_L T$ and $U \sqsubseteq_U T$. From the former we get that $V \sqsubseteq_L T$ where $V = \bigcup \mathcal{U}$, while from the latter we get that for each t in T there is some choice function ϕ on $\mathcal{U}$ such that t is an upper bound for $\{\phi(U): U \in \mathcal{U}\}$. Hence $W \sqsubseteq_U T$ where $W = \bigcup_\phi W_\phi$ for some W_ϕ with $CUB(\{\phi(U): U \in \mathcal{U}\}, W_\phi)$. Actually, to give a properly constructive account, we need to consider not choice functions but choice *relations* on $\mathcal{U}$, finite total relations R from $\mathcal{U}$ to $\bigcup \mathcal{U}$ such that if U R s then $s \in U$. One can show by techniques similar to those of Lemma 2.1.10 that the set of finite choice relations on $\mathcal{U}$ is finite.

We have thus replaced $\mathcal{U}$ by a pair (V, W) such that the upper bounds (under $\sqsubseteq_C$) of $\mathcal{U}$ are those T such that $V \sqsubseteq_L T$ and $W \sqsubseteq_U T$. If we had V = W, then we'd have $V \sqsubseteq_C T$ and so V would be a least upper bound of $\mathcal{U}$; and if we only had $W \sqsubseteq_C V$ then still $V \cup W$ would be a least upper bound for $\mathcal{U}$. Of course we don't have that in general, but our aim is to work towards a set of pairs (V, W) such that the upper bounds of $\mathcal{U}$ are those T for which $V \sqsubseteq_L T$ and $W \sqsubseteq_U T$ for some (V, W) in the set. For each such pair, if we don't yet have $W \sqsubseteq_C V$ (but of course this negative statement must be treated rather circumspectly just as for the function space) then we can replace it by a set of better pairs.

The two cases, forming the basis for the (simple $\mathcal{F}$) induction, are as follows.

- Consider $(V\cup\{s\}, W)$. For each $w \in W$ we can find U_w such that $CUB(\{s,w\}, U_w)$; let $U = \bigcup_w U_w$. Then we can replace $(V\cup\{s\}, W)$ by $\{(V\cup\{u\}, W)\colon u \in U\}$. For if $V\cup\{s\} \sqsubseteq_L T$ and $W \sqsubseteq_U T$ then $s \sqsubseteq$ some $t \in T$, and $t \sqsupseteq$ some $w \in W$, so $t \sqsupseteq$ some $u \in U_w$ and $V\cup\{u\} \sqsubseteq_L T$. Conversely, if $V\cup\{u\} \sqsubseteq_L T$ for some $u \in U_w$ then $s \sqsubseteq u$ so $V\cup\{s\} \sqsubseteq_L T$. Now if $u \in U_w$ then $u \sqsupseteq w$, and by iterating the process we can ensure $W \sqsubseteq_U V$ in each pair.
- We can replace $(V, W\cup\{s\})$ by $\{(V, W), (V\cup\{s\}, W\cup\{s\})$. For if $W\cup\{s\} \sqsubseteq_U T$ then either $W \sqsubseteq_U T$ or $s \sqsubseteq$ some $t \in T$, in which case $V\cup\{s\} \sqsubseteq_L T$. Both these new pairs help to make $W \sqsubseteq_L V$. We might no longer have $W \sqsubseteq_U V$, but we can restore this by the first case.

To complete the proof of strong algebraicity, if $\mathcal{U}\colon \mathcal{FF}X$ let M be CUB-closed containing $\bigcup \mathcal{U}$. Then $\mathcal{F}M$ is CUB-closed containing $\mathcal{U}$.

4.7 Decidable information systems

The information systems discussed so far have been undecidable in that the order $\sqsubseteq$ did not have a complement. It is interesting that [decIS], the topos classifying decidable information systems, also gives a topical CCC. It is genuinely different from [IS] – it does not happen that by some quirk the general $\sqsubseteq$ has a complement. This is easily seen by considering the information system homomorphisms: if $\sqsubseteq$ did in general have a complement, then it would have to be preserved by homomorphisms, so the homomorphisms would have to correspond to embedding-projection pairs rather than to the more general adjunctions that are clearly possible in the light of Proposition 4.1.4.

In [decIS], the proof of Cartesian closedness can be understood as a use of classical logic: the decidability of $\sqsubseteq$ enables us geometrically to bring $\neg$ and $\rightarrow$ into order-theoretic statements, while $\forall$ is possible because the SFP axioms give finite sets with which $\forall$ can be bounded. Hence one can mimic classical proofs of the Cartesian closedness of the category of posets.

Abramsky (1991) describes his constructions inductively without $\not\sqsubseteq$, and is in effect giving what is needed to show Cartesian closedness for [IS]. However (his theorem 4.2.7), he also proves that $\sqsubseteq$ *is* decidable. This paradox arises because he is considering only the types that arise in his inductive system. Part of his proof shows how $\not\sqsubseteq$ can be defined inductively, and in fact this provides the ingredients for another proof that [decIS] also gives a topical CCC.

Note that if a strongly algebraic information system X has decidable order, then CUB is also decidable: for its negation $\neg CUB(S,T)$ is the geometric formula

$$\exists s\in S,\ t\in T.\ s \not\sqsubseteq t \ \vee\ \exists u.\ (\forall s\in S.\ s \sqsubseteq u \wedge \forall t\in T.\ t \not\sqsubseteq u)$$

5 Solving domain equations

Recall that Theorem 2.3.8 showed the existence of fixpoints (more precisely, initial structures) for endomaps of local toposes (i.e. toposes with initial points). Restricting to the case of locales we get a more elementary case, that locales with bottom points have fixpoints for endomaps. (This fact is so crucial to denotational semantics – in giving meaning to recursion – that it is common to apply Occam's razor rather ruthlessly and define a domain as embodying (as an ω-cpo) the minimal structure needed to make this work: order, bottom, and sups of ω-chains. We do not go this far – not least, because the very assumption that a domain is a partially ordered *set* brings its own problems constructively, certainly if we wish to use the arithmetic universe foundations alluded to in the Conclusions.)

Fixpoints *within* domains are thus covered by the localic case of Theorem 2.3.8. We shall not dwell on this except to note that our domains are not necessarily local and so Corollary 2.3.9 is the appropriate form: if f: $D_\perp \to D$ then up;f has a least fixpoint Yf (find the least fixpoint of f;up in $D_\perp$ in the standard way, and then apply f to it). Y can be internalized by the usual sort of CCC manipulations (and following the techniques of Hyland and Pitts (1989)) as something of (ML-style) polymorphic type $(D_\perp \Rightarrow D) \Rightarrow D$ as follows. That type corresponds to a map from [IS] to itself (map X to $[[X_\perp \Rightarrow X] \Rightarrow X]$). Pulling back the generic domain along this gives a topos E that classifies an information system X equipped with a point of $[[X_\perp \Rightarrow X] \Rightarrow X]$. Y is then a map from [IS] to E over [IS].

For fixpoints *amongst* domains, solving domain equations, we need Theorem 2.3.8 in its topos generality. This raises coherence questions but is conceptually unproblematic and relies on the feature of geometric theories that their classes of models are closed under filtered colimits by quite concrete constructions. We shall apply this to the solution of domain equations, and the machinery is really that already familiar from the information system approach to domains (Larsen and Winskel 1984).

A standard approach – such as Abramsky's (1991) – would solve a domain equation D = F(D) by restricting to local domains, so let us briefly investigate those.

Proposition 5.1 An algebraic dcpo Idl X is local iff X has a least element.
Proof
⇒: If ⊥ is the least element, then {⊥} is an ideal contained in every other.
⇐: Let I be the bottom ideal. It is inhabited, so it has an element x. Now $I \sqsubseteq \downarrow y$ for every y in X, so $x \sqsubseteq y$ and x is a least element of X.]

Amongst the strongly algebraic information systems, the ones whose ideal completion is local form a subtopos [locIS] (an open subtopos, in fact): they are characterized by the additional axiom $\vdash \exists s{:}X.\ \mathrm{CUB}(\emptyset, \{s\})$, for CUB(Ø, {s}) iff s is a bottom element of X. Since we are working with posets, the bottom element is unique, and we write ⊥ for it as usual. (For general algebraic information systems, the local ones do not form a subtopos because the bottom has to be specified as extra structure.)

However, there is a small issue of constructivity here. Such local information systems are closed under all the constructions given except – obviously – coproducts. It is usual to substitute a different sum construction, either the coalesced sum (which identifies the bottoms) or the separated sum (which adjoins a new one). Abramsky uses the coalesced sum, because it is more general – the other can be defined using it. However, in defining CUB for the coalesced sum it turns out that one needs to know when tokens are *not* bottom, for one needs to say that CUB(S, Ø) if S contains non-bottom elements from both summands. This information is not available in a general strongly algebraic information system, for if it were then non-bottomness would have to be preserved by homomorphisms. (For a counterexample, consider {⊥, T} mapping to {*}.) Abramsky solves the problem by introducing a predicate **T** (for Termination) which would correspond in our system to an extra predicate T(s) on X with axioms to make it the complement of CUB(Ø, {s}). He then shows how – in our terms – all the constructions yield information systems with such a predicate. However, one should remember that these are more restricted than the general local, strongly algebraic information sytems.

Proposition 5.2 [locIS] is local.
Proof
The initial local information system is the singleton {⊥}. For any other local information system X, the unique homomorphism maps ⊥ to ⊥ – uniqueness arises because a homomorphism must preserve the bottom-defining property CUB(Ø, {⊥}).]

Therefore, let F: [locIS] → [locIS] be any endomap. By Theorem 2.3.8 [F-Str] is local; its initial point is the canonical solution to D ≡ F(D).

In solving D = F(D), it is a well-known fact that F ought really to have some properties of continuity. In our topical setting it hardly makes any sense for F to be other than a geometric morphism, and then the continuity is automatic – F will preserve filtered colimits, and this is the categorical analogue of Scott continuity. What's more, F does not in this setting have to be functorial with respect to continuous maps between domains, and there is no problem in using examples such as X → [X⇒X], which is a perfectly good geometric morphism. (It is the composite Δ;EXP.) It is, of course, functorial with respect to homomorphisms between information systems, and, following 4.1.4, this is an interesting arrival at the usual trick of using endofunctors on the category of domains and embedding-projection pairs. (We have a slight variant here – the homomorphisms are adjoint pairs rather than embedding-projection pairs. The trick still works, as was pointed out by Taylor (1986). As mentioned in 4.7, the difference corresponds to whether the order ⊑ is decidable or not.) Let us emphasize this. The domain construction F *does not need* to be part of a functor on the topical category. The very act of defining the transformation (geometrically) on objects gives us all the functoriality and continuity that we need.

Let us also describe an approach that works in the context of our non-local domains. We show how to solve a simple form of domain equation, namely

$$D = (F(D))_\perp$$

where F is a construction on our domains without bottom. This is not quite as general as Abramsky's domain construction; it cannot, for instance, be used to construct the lifted natural numbers, because the natural numbers do not constitute a strongly algebraic domain (see note (vi) after 4.1.1). However, the form does cover many important domain equations, and can often be pleasantly simple. For instance, the domain of lists over a finite decidable set A can be described by the domain equation $D = (1+ A\times D)_\perp$ where + and × are the categorical coproduct and product: there is no need to use coalesced sum or smash product.

Theorem 5.3 Let F: [IS] → [IS] be a geometric morphism. Then F;LIFT has an initial structure.
Proof
LIFT: [IS] → [IS] factors via [locIS], which is local, so we can use 2.3.7 and 2.3.8.]

6. Conclusions

From our resetting of domain theory we can draw various conclusions.

First, it serves as a test case for the programme of "geometrizing" mathematics (in the sense of geometric logic). It shows some non-trivial mathematics that can be done in a natural framework that automatically enforces constructivist constraints such as those that Abramsky required for his formal system. Moreover, we see the constructive mathematics going beyond the syntactic systems to the semantic domains (which in Abramsky (1991) were still treated classically).

Second, it shows both the possibility and benefits of putting domain theory into a securely topological setting. It was already understood that domains were not just ordered structures, that the (Scott) topology was important and brought its own insights; we now see that the same goes for categories of domains, albeit that the topology is in Grothendieck's generalized sense. When a set-theoretic category is topologized as a topical category, the continuity needed for the solution of domain equations becomes automatic, and we can see clearly a role of embedding projection pairs as homomorphisms. However, an alternative role appears in the work of Pitts (1996), where domain constructions are treated as having covariant and contravariant parts (with respect to continuous maps), and this decomposition gives rise to an account of induction and coinduction. It would be interesting to understand better the relationship with this work.

Let us now turn to the result of 3.1.1 ("The topos of sets is not Cartesian closed"). Cartesian closedness of **Set** is such a fundamental assumption, central for instance to the theory of elementary toposes, that our negative result calls into question the usefulness of the topical category. However, one can also turn the question on its head and reconsider the exponentials Y^X. One problem with them is that though sets (discrete locales) are exponentiable in **Loc** or **Top**, the exponentials there are not discrete – the natural topology is the compact-open. Hence the exponential Y^X in **Set** could be considered as a mere approximation to a truer exponential in a broader context. This is reflected in the fact that equality between functions cannot be directly evaluated (and functional programming languages will refuse to try).

In view of the beauty of the theory of elementary toposes, it may seem perverse to reject Cartesian closedness; yet there are perhaps grounds for believing that adequate mathematics (including, I conjecture, that presented here) can be done without it. Specifically, although geometric logic includes arbitrary set-indexed disjunctions and coproducts, it seems that in the work here such infinities are restricted to those that can be accessed effectively through free algebra constructions. We conjecture that the full geometric logic is unnecessary, that it suffices to have coherent logic with assorted free algebras, and that the categories corresponding to (what we called) geometric universes would be Joyal's *arithmetic universes* (unpublished notes). It is not immediate that the mathematics here would go through in arithmetic universes; on a number of occasions we use intuitionistic reasoning that would certainly not be interpretable. Nonetheless, the algorithmic flavour of the constructions gives us grounds to feel that it ought to work. If it does, then it could simplify the foundations considerably. For whereas the notion of geometric universes is parametrized by the base category of sets (used to index "set-indexed coproducts"), a notion of arithmetic universe as "pretopos with selected free algebra constructions" would be a finite essentially algebraic theory and therefore self-standing. The essential algebraicity would enable us to present arithmetic universes by generators and relations and hence straightforwardly give classifying categories for the effectively accessible geometric theories.

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