Imperial College London

Final Year Individual Project

Robust optimisation approach to pricing electricity swing options

Project Report

Tytus Dobrzynski
e-mail: tytus.dobrzynski06@imperial.ac.uk

Supervised by:
Dr. Daniel Kuhn, Prof. Berç Rustem and Wolfram Wiesemann

June 15, 2010

Contents

Abstract

Electricity swing options are complex, Bermudan-style derivatives on electrical energy. They can be considered as supply contracts for power, which give flexibility in both the timing and amount of delivery. Pricing of such instruments is a challenging task due to the path-dependence of the option, non-storability of electricity and incompleteness of energy markets. We formulate a model for determining a rational (fair) buyer's price of a swing option. In the model, following the robust optimisation approach, we replicate the option by a hedging portfolio consisting of standard energy futures contracts and a risk-free asset in different scenarios. We develop and apply a set of simplifications and approximations which make the problem computationally feasible and validate our valuation scheme by comparing it with existing, well-established methods for pricing derivatives in some limiting cases. In order to prove that the approach is correct, we carry out numerical convergence tests and present the results. Finally, we attempt to determine a set of the algorithm parameters which provide a good balance between the time and accuracy of the solution.

Acknowledgements

I am grateful to Dr. Daniel Kuhn and Wolfram Wiesemann for invaluable help, guidance, motivation and patience with answering all my questions. I would also like to thank Phebe Vayanos for technical help.

Chapter 1

Introduction

Pricing of the swing options for many periods is a large-scale, complex problem and the valuation is done under uncertainty, as the future market movements are unknown and often even unpredictable.

The number of possible scenarios of electricity prices movements is extremely large and the swing option is path-dependent, so it is difficult to use tree-based valuation schemes. Instead, we notice that just a large enough selection of the scenarios can be sampled and out of the set of all possible scenarios and a tractable robust optimisation problem based on polynomial decision rules can be formulated [1]. We are able to efficiently solve such problem by using state-of-the art optimisation software like CPLEX[1] in reasonable time.

In our valuation scheme we assume that a trading strategy of buying a single swing option can be replicated by investing in a portfolio of electricity futures contracts and cash (self-financing trading strategy). Furthermore, we assume that there is no arbitrage in the market. We attempt to price a swing contract by solving a stochastic optimisation problem based on a hedging portfolio of the contracts and cash (or risk-free asset) to find the riskless buyer's price of the option.

As the classical approach to the option pricing problem is to replicate an option with a portfolio of underlying (available) securities in each possible scenario, the robust valuation scheme we propose is natural and justified.

[1]CPLEX is a high-performance solver produced by IBM (http://www-01.ibm.com/software/integration/optimization/cplex/)

1.1 Structure of the report

We begin with presentation of the background of the swing option pricing problem. We provide the overview of the energy markets (Section 2.1), define the derivative itself (Section 2.2) and list some of the methods of pricing such instruments which can be found in the literature (Section 2.3).

In Chapter 3 we give a brief introduction to the pricing model which we use to generate different price movement scenarios (which we interchangeably call *samples*). We also explain in detail how we price the hedging instruments which we use in our pricing model.

Next, we introduce of the exact pricing problem in Chapter 4. This problem is clearly too expensive computationally to solve for large number of periods. Therefore in Section 4.1, we introduce two simplifications which result in significant reduction of the number of decision variables and constraints. In Section 4.2 we describe how we model the decision variables (in form of the polynomial decision rules). Next, we verify that our model is correct by comparing it with the well-known binomial lattice model (Section 4.3) and showing that both valuation schemes yield the same results for simple, path-independent options.

In Chapter 5 we proceed further in simplifying the pricing problem by introducing the *stages aggregation* (grouping individual periods into macro-periods). In Section 5.1 we describe in detail how the exercise patterns during those macro-periods can be found by solving a deterministic linear programming problem. We approximate the constraints from the original problem (Section 5.2) and formulate the final pricing problem (Section 5.3).

Using the approximated model, we proceed to numerical experiments and evaluate their outcomes in Chapter 6. We propose the configuration of the algorithm parameters (sample size, number of stages, number of profiles etc.) which can be used to achieve a reasonable accuracy and we justify this choice. We verify that the results obtained are correct by performing some numerical convergence tests in two cases: for polynomial decision rules of degree 1 and 2.

Final conclusions and suggestions for future work can be found in Chapter 7.

1.2 Contributions

Summary of contributions:

- We formulate the mathematical pricing model for the swing options in form of the robust optimisation linear programming problem based on a dynamic hedging strategy with a portfolio of electricity futures contracts and cash (risk-free asset). The model can be used to find a risk-free bid (buyer's) price of the swing option.

- We establish that our model is correct by carrying out some simple tests and additionally showing that using the binomial lattice method and our model yield the same results in cases of simpler pricing problems (i. e. pricing of path-independent options).

- We introduce stages aggregation into our model in order to make it applicable in terms of computational costs for exotic, path dependent electricity derivatives pricing problems. On a specific numerical example, we show that the approximated model is correct and attempt to find a set of optimal parameters' values for the algorithm which guarantee both reasonable accuracy and solution time.

Chapter 2

Background

2.1 Energy Markets

2.1.1 Historical background

In the past, electricity market was strictly regulated and controlled by governments. Prices were set in advance by the authorities and reflected simply the costs of production of energy which was delivered by state-owned companies (power plants). Therefore, energy consumers were not exposed to any price risk. However, in the last two decades, the process of deregulation started in many countries. Governments decided to liberalise the energy market and free the prices, so that now they depend only on the market forces of supply and demand. Electrical energy became a good which can be traded like any other commodity. In particular, it can now be traded in form of delivery contracts on specialised exchanges, such as European Energy Exchange (EEX) or Amsterdam Power Exchange (APX).

As a consequence of the liberalisation of the energy market, a need for managing the price risk became essential. The electricity prices are mean-reverting, but they exhibit strong seasonalities and temporal jumps and spikes. As in the case of the traditional commodities, the price risk related to electrical energy is hedged away through special financial instruments called derivatives (or options).

Due to high complexity of the energy derivatives and characteristics of the underlying commodity (see Section 2.3.1) new techniques for valuing these options had to be developed in order to make trading possible.

2.1.2 Money Markets vs. Energy Markets

There are a few fundamental differences between energy and other financial markets, presented in detail in [13], section 2. Let us summarise them in brief in this section.

Energy markets are much more immature (the electricity market in particular) and less liquid in comparison to other financial markets. They have more (and complex) fundamental price drivers, such as issues of storage, transfer, weather or advances in technologies of production of energy. However, in the contrary to the traditional financial markets, the influence of economic cycles (another fundamental driver) on prices of energy is low.

Finally, the types of financial contracts required by end users of derivatives are much more complex in the energy market. They involve complex price averaging and take into account customised characteristics of commodity delivery.

2.1.3 European Energy Exchange (EEX)

European Energy Exchange, founded in 2002, based in Leipzig, Germany is a leading European energy exchange with more than 200 members. Power on spot, futures contracts for delivery of electricity and vanilla options (puts and calls) are traded on the exchange.

Later in this report, in our model, we will be developing a trading strategy based on a hedging portfolio of energy futures and cash (or risk-free asset) to find the value of the electricity swing option. As we would like to make the model presented in the paper as close to reality as possible, it is essential to know and take into account the financial products which are indeed available in the market. The following contracts are currently tradeable at European Energy Exchange (EEX) [6]:

- Week futures (starting on Monday and expiring on Sunday on each calendar week)
- Month futures (starting on 1st and expiring on the last day of each calendar month)
- Quarter futures (maturity of 3 months, aligned with calendar quarters)
- Year futures

All the above futures with the following two load profiles are traded on EEX [5]:

- Base load (a constant delivery rate on all delivery days from Monday until Sunday and during all 24 delivery hours of any delivery day during the delivery period)
- Peak load (a constant delivery rate on all delivery days from Monday until Friday and throughout 12 delivery days from 08:00am until 08:00pm of any delivery day during the delivery period)

Contracts cannot be traded once their delivery period has started. Additionally, we assume that one has to pay for a future contract upfront, before the delivery starts.

2.2 Swing Options

The electricity swing options (in other words, the *take-or-pay* options) are Bermudan-style path-dependent derivatives with electrical energy as the underlying commodity. The swing options have multiple exercise rights and have constraints on total volume delivered. A swing contract often comes together with a standard base load future contract for delivery of electrical energy during a given period of time at a predetermined price and specifies the amount of energy to be delivered. The swing contract allows flexibility in delivery process (both in terms of timing and amounts) around the amount of the base load contract.

The swing options are mainly used by risk-averse economic agents who need to hedge against the electricity price risk. The risk could have been also mitigated by using standard European call or put options, or a series of American options, but as pointed out in [9] such solution would often result in overpaying for protection which is not needed. The swing options are much better and cheaper hedging instruments.

More details about the swing options (detailed characteristics, types and mathematical formulation) can be found in [9] and in [13].

2.3 Valuation of the Swing Options

2.3.1 Overview

As the market for electricity derivatives arised, so did the need for valuation of new products. The differences between electricity options and any other commodities' derivatives have been listed by Kuhn and Haarbrücker [7] and include:

- Non-storability. The production of electrical energy has to cover demand instantaneously as it cannot be stored efficiently.
- Markets for electricity derivatives are, in general, illiquid. Financial products are traded *over-the-counter* and as a consequence spreads are large and counterparty risk exists.
- Electricity spot prices are mean-reverting, which means that both low and high levels are temporary and the price will tend to have an average price over time. However, strong seasonalities, jumps and spikes can be observed. For example, weather and power plants outages can be very significant drivers of energy prices.

As a consequence, new sophisticated valuation schemes are required for the electricity swing options.

2.3.2 Previous approaches

A lot of research has been done in the area of valuing the swing options and many pricing techniques have been developed in the recent years. Nearly all of them are variations of the stochastic dynamic programming (SDP) approach.

Monte Carlo simulation

Monte Carlo simulation technique for pricing swing options was presented by Ibanez [8]. They derive theoretical properties of price and the optimal exercise frontier of these derivatives and, by applying the Monte Carlo simulation, compute the optimal exercise frontier recursively to price the option.

Davison and Anderson [4] also proposed a technique for pricing electricity swing options based on the Monte Carlo simulation. Their method works under very simplified assumptions. They take into account weekly average on-peak electricity prices, determine an approximate early exercise boundary and run the Monte Carlo simulation to obtain the price. The drawback of the method is that the resulting prices are underestimate of the true options' prices, because the exercise boundaries that are computed using the techniques shown in the paper are not optimal for the unsimplified models.

Another simulation-based method is the Least Squares Monte Carlo algorithm [10] which uses least squares estimates to approximate conditional expectations within the stochastic dynamic programming scheme.

Furthermore, as pointed out by Pilipovic [13] (section 10.6), using the Monte Carlo simulations on the trading floor has two major concerns: processing time and market-to-market compliance issues. However, techniques based on simulation might be useful for testing methodologies used on trading floors.

Forest of recombining trees

The traditional tree-based approach for valuation of options is based on building a tree for the option settlement price that defines the movements, up and down, from node to node, of the option settlement price from now until the time of option expiration (more details can be found in [13], section 10.5). The technique works well for American-style option, but is in general hard to use it for Asian path-dependent options as there it becomes very time-consuming to arrive at the solution. Another drawback of the method is that it is expensive in terms of resources (memory) needed, especially for longer periods.

The approach developed by Jaillet et al. [9] is based on a multistage tree stochastic dynamic programming method. The procedure starts at the expiration date of the swing option and works backward in time using "backward induction" in three dimensions: price, number of exercise rights left and usage level. At every time period, an action which adds maximum value to the solution is chosen. Each exercise of a swing right causes a jump to a next tree, with fewer exercise rights left. As mentioned earlier, the technique is expensive and both a lot of time and computer memory are needed to build and solve the trees.

Other stochastic programming methods

Carmona and Touzi [3] formulate and analyse the problem of valuation of the swing options as an optimal multiple stopping problem and solve it by methods of the stochastic calculus.

The multistage stochastic programming technique presented by Kuhn and Haarbrücker [7] based on a scenario tree model can be applied to difficult electricity swing option pricing problems which are hard to solve using stochastic dynamic programming methods. It can be applied to more realistic models, where several risk factors exist. What is more, constraints on gradient of consumption rate of energy can be taken into account (in contrary to for example Jaillet et al. [9] approach, where restrictions were imposed only on the consumption rate and its integral). Swing options with many exercise times can be priced efficiently using this technique. This is achieved by introducing a number of simplifying approximations to make the pricing model computationally tractable: reduction of the information process, discretization of the probability space and reduction of the number of decision variables. The resulting stochastic linear programming program is tractable and solvable by LP solvers available in the market (for example CPLEX). This technique seems to be most flexible and precise, as it does not require any special structural requirements on price processes and does not impose unrealistic constraints on the swing options.

Steinbach and Vollbrecht [14] based their pricing model on the one presented in [7]. They constructured scenario trees and used the idea of *value of stochastic solution* (VSS) to help deciding on when to branch and how many branches to use. A high efficiency of this technique has been achieved by decomposing the valuation problem into independently solvable subperiod problems.

Broussev and Pflug [12] develop a game-theoretic model and find the ask (seller's) price of the swing option. They base they approach on a hedging portfolio of futures contracts for the electrical energy. They propose a static hedging strategy (portfolio is defined before the delivery period of the option starts).

Chapter 3

Forward Price Dynamics

In order to formulate a pricing problem for the electricity swing option, we start with construction of a pricing model for the underlying commodity (electrical energy). We assume that the electricity markets are arbitrage-free, frictionless and efficient. This might not be necessary true in the reality, but the liquidity of these markets is likely to increase, so the assumption is justified.

We begin with defining all symbols used later in the report. Let's assume we want to price an electricity swing option with delivery period starting at time 1 and expiring at time I (for some $I \geq 1$). The time between start and expiration is divided into a discrete number of equally-long periods i (i.e. $i \in \mathbb{I}, \mathbb{I} = \{1 \dots I\}$) of length Δ (later in the report w.l.o.g. we assume that $\Delta = 1$). Our model will be based on a virtual hedging portfolio - a collection of electricity futures contracts and cash (risk-free asset). Let $\mathbb{C} = \{0 \dots C\}$ denote the set of all assets which can be used for hedging (there are C types of futures contracts in total and the risk-free asset) and let $c \in \mathbb{C}$ be a single asset with $c = 0$ representing the risk-free asset. Then, let G_i^c be the price of asset c at period i. Similarly, the price of the risk-free asset is denoted by G_i^0.

3.1 Futures contracts

In our model, we attempt to estimate the bid (buyer's) price by valuing a hedging portfolio consisting of the standard (tradeable on the European Energy Exchange) futures contracts. Similarly to the approach presented in [7], we use a logarithmic, probabilistic model for forward price dynamics based on a popular Pilipovic model ([13], Section 6.6). This approach is supported by the fact that forward prices curves are used extensively on trading floors as a tool for supporting decision making process, so it is natural to base our model on them. The meanings of the symbols used to describe the stochastic price process are presented in the Table 3.1.

Symbol	Meaning
$\boldsymbol{\xi}_i$	vector of random variables at time i; $\forall \boldsymbol{\xi} \in \Xi$
$\boldsymbol{\xi}^i$	set of vectors for all periods from 1 up to i
$F_{ij}(\boldsymbol{\xi}^i)$	price at period i of one future contract expiring in period j
σ^n	n^{th} volatility function; there are N functions in total
μ	drift parameter

Table 3.1: Forward price curve symbols

The prices are modelled by the following equation:

$$\log(F_{i+1,j}(\boldsymbol{\xi}^{i+1})) - \log(F_{ij}(\boldsymbol{\xi}^i)) = \sum_{n=1}^{m}(\sigma^n_{j-i}\xi_{i,n}) + \mu_{j-i} \qquad \forall i \in \mathbb{I}, \forall j \geq i \tag{3.1}$$

Where ξ^i_n represents the nth element of the vector $\boldsymbol{\xi}^i \in \mathbb{R}^m$. By removing the recursion, we obtain the following formula:

$$\log(F_{ij}(\boldsymbol{\xi}^i)) = \log(F_{0j}) + \sum_{l=0}^{i-1}[(\sum_{n=1}^{m}\sigma^n_{i-l}\xi_{l,n}) + \mu_{i-l}] \qquad \forall i \in \mathbb{I}, \forall j \geq i \tag{3.2}$$

Note that F_{0j} (today's forward price curve) is assumed to be known in advance. The spot price at time i is defined as $F_{ii}(\boldsymbol{\xi}^i)$.

Then, having all the forward prices curves calculated, we are able to calculate price of any contract using the following formula:

$$G^c_i = \sum_{j=\max\{s_c,i\}}^{t_c} F_{ij}(\boldsymbol{\xi}^i) \qquad \forall i \in \mathbb{I}, \forall c \in \mathbb{C}\backslash\{0\} \tag{3.3}$$

where s_c and t_c mean respectively start and expiration times of a contract.

3.2 The Risk-Free Asset

The price of a unit of the risk free-asset at time period i is simply estimated by a simple compounding formula:

$$G_i^0 = e^{r_f(i-1)} \qquad \forall i \in \mathbb{I} \tag{3.4}$$

where r_f is a risk-free rate.

Chapter 4

The Pricing Model

We assume that the bid (or buyer's) price of the swing option contract is equal to the maximum expected profit that can be made by the option holder from exercising the option. We price the option by replicating it by the hedging portfolio.

Because of the specific features of the electrical energy (non-storability), we cannot hedge using the underlying security (electrical energy) directly, so we have to use derivatives which are available in the market. We construct a hedging portfolio consisting of the electricity futures contracts and the risk-free asset. Just before beginning of each period i, the amounts of hedging instruments can be changed (portfolio rebalancing) and the changes depend on current expectation of the future prices movements (based on the shape of the forward prices curve at period i). Let $x_i^c(\boldsymbol{\xi}^i)$ be the number of futures contracts of type c (or number of units of the risk-free asset) that we own in time period i and let $u_i^c(\boldsymbol{\xi}^i)$ denote a change in the number of contracts between periods $i-1$ and i. Let $\boldsymbol{x}_i = \{x_i^0(\boldsymbol{\xi}^i), x_i^1(\boldsymbol{\xi}^i), \ldots, x_i^C(\boldsymbol{\xi}^i)\}$ be the vector representing the amounts of all hedging instruments at time i, and let $\boldsymbol{x} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_I\}$ be the array of vectors for all periods $i \in \mathbb{I}$ and additionally $\boldsymbol{x}_0$ which denotes the amounts of hedging instruments held initially. Similarly, we define $\boldsymbol{u}_i = \{u_i^0(\boldsymbol{\xi}^i), u_i^1(\boldsymbol{\xi}^i), \ldots, u_i^C(\boldsymbol{\xi}^i)\}$ and $\boldsymbol{u} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_I\}$. Both $x_i^c(\boldsymbol{\xi}^i)$ and $u_i^c(\boldsymbol{\xi}^i)$ are the decision variables in our problem. We assume that initially no futures contracts are held (i.e. $x_0^c = 0, \forall c \in \mathbb{C}\backslash\{0\}$). We need to impose a constraint $x_I^c\left(\boldsymbol{\xi}^I\right) \geq 0$, which ensures that at the maturity $(i = I)$, we have non-negative amounts of all contracts and cash.

The swing option holder has the right to exercise the option at various time periods for chosen amounts of power. The amount of power purchased or sold at each period i is represented by the decision variable $p_i(\boldsymbol{\xi}^i)$. Let $\boldsymbol{p} = \{p_1(\boldsymbol{\xi}^1), p_2(\boldsymbol{\xi}^2), \ldots, p_I\left(\boldsymbol{\xi}^I\right)\}$. In the swing option contracts the limitations on power purchased (or sold) in each time period i are specified. We denote them as $\underline{p}$ (lower limit) and $\overline{p}$ (upper limit). Therefore, in our model, we require:

$$\underline{p} \leq p_i\left(\boldsymbol{\xi}^i\right) \leq \overline{p}, \quad \forall i \in \mathbb{I}. \tag{4.1}$$

Sometimes additional ramping constraints are specified as well. They impose limitations on the change in amounts of the power the option is exercised for in neighbouring periods. This is captured in our model in the form of the following constraint:

$$|p_i\left(\boldsymbol{\xi}^i\right) - p_{i-1}\left(\boldsymbol{\xi}^{i-1}\right)| \leq \rho, \quad \forall i \in \mathbb{I}. \tag{4.2}$$

Where we set the dummy variable $p_{-1} = p_{\text{start}}$.

Moreover, between the start time ($i = 1$) and the expiration ($i = I$) of the delivery, the cumulative energy is required to lie between the given target values $\underline{e}$ and $\overline{e}$.. The strike price of the option is denoted by K.

Symbol	Meaning	Unit
K	Strike price	€/MWh
$\underline{p}$	Lower power limit	MW
$\bar{p}$	Upper power limit	MW
p_{start}	Starting level of power	MW
ρ	Ratchet	MW
$\underline{e}$	Lower energy limit	MWh
$\bar{e}$	Upper energy limit	MWh

Table 4.1: Pricing model parameters

The optimal solution to the robust linear programming problem presented below ($\mathcal{SO}^I_{\text{buyer}}$) represents the (riskless) buyer's price (bid price) for a swing option contract.

$$
\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}} \quad & -x_0^0 && (\mathcal{SO}^I_{\text{buyer}})\\
\text{s.t.} \quad & x_i^c\left(\boldsymbol{\xi}^i\right) = x_{i-1}^c\left(\boldsymbol{\xi}^{i-1}\right) + u_i^c\left(\boldsymbol{\xi}^i\right) && \forall \boldsymbol{\xi} \in \Xi, i \in \mathbb{I}, c \in \mathbb{C}\\
& x_I^c\left(\boldsymbol{\xi}^I\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi, c \in \mathbb{C}\\
& \underline{p}_i \leq p_i\left(\boldsymbol{\xi}^i\right) \leq \overline{p}_i && \forall \boldsymbol{\xi} \in \Xi, i \in \mathbb{I}\\
& \underline{e} \leq \sum_{i\in\mathbb{I}} p_i\left(\boldsymbol{\xi}^i\right) \leq \overline{e} && \forall \boldsymbol{\xi} \in \Xi\\
& |p_i\left(\boldsymbol{\xi}^i\right) - p_{i-1}\left(\boldsymbol{\xi}^{i-1}\right)| \leq \rho && \forall \boldsymbol{\xi} \in \Xi, i \in \mathbb{I}\\
& p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right) - K\right) - \sum_{c\in\mathbb{C}} G_i^c u_i^c\left(\boldsymbol{\xi}^i\right)\\
& \quad + \sum_{\substack{c=1\\ s_c \leq i \leq t_c}}^{C} x_i^c\left(\boldsymbol{\xi}^i\right) F_{ii}\left(\boldsymbol{\xi}^i\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi,\ i \in \mathbb{I}.
\end{aligned}
$$

The last constraint in the problem $\mathcal{SO}^I_{\text{buyer}}$ ensures that the value of the hedging portfolio is non-negative at all times.

Even though the model $\mathcal{SO}^I_{\text{buyer}}$ is realistic, it is too complex to be solved (too expensive computationally). Therefore, we need to introduce some simplifications.

4.1 Simplifications

In the next sections, we will identify and apply two simplifications in order to make the problem $\mathcal{SO}^I_{\text{buyer}}$ easier to solve.

4.1.1 Reduction of the number of variables

Firstly, the pricing model $\mathcal{SO}^I_{\text{buyer}}$ can be simplified by reduction of the number of the decision variables.

By eliminating recursion, we remove all the $x_i^c\left(\boldsymbol{\xi}^i\right), \forall i \in \mathbb{I}$ decision variables from the robust optimisation problem $\mathcal{SO}^I_{\text{buyer}}$, resulting in a new problem $\mathcal{SO}^{II}_{\text{buyer}}$.

$$
\begin{aligned}
\max_{x_0^0,\boldsymbol{u},\boldsymbol{p}} \quad & -x_0^0 && (\mathcal{SO}^{II}_{\text{buyer}})\\
\text{s.t.} \quad & x_0^c + \sum_{j=1}^{I} u_j^c\left(\boldsymbol{\xi}^j\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi,\ c \in \mathbb{C}\\
& \underline{p}_i \leq p_i\left(\boldsymbol{\xi}^i\right) \leq \overline{p}_i && \forall \boldsymbol{\xi} \in \Xi,\ i \in \mathbb{I}\\
& \underline{e} \leq \sum_{i \in \mathbb{I}} p_i\left(\boldsymbol{\xi}^i\right) \leq \overline{e} && \forall \boldsymbol{\xi} \in \Xi\\
& \left|p_i\left(\boldsymbol{\xi}^i\right) - p_{i-1}\left(\boldsymbol{\xi}^{i-1}\right)\right| \leq \rho && \forall \boldsymbol{\xi} \in \Xi,\ i \in \mathbb{I}\\
& \sum_{\substack{c=1\\ s_c \leq i \leq t_c}}^{C} \left(\sum_{j=1}^{i} u_j^c\left(\boldsymbol{\xi}^j\right)\right) F_{ii}\left(\boldsymbol{\xi}^i\right) - \sum_{c \in \mathbb{C}} G_i^c u_i^c\left(\boldsymbol{\xi}^i\right)\\
& \quad + p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right) - K\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi,\ i \in \mathbb{I}
\end{aligned}
$$

4.1.2 Reduction of the number of constraints

Next, we can reduce significantly the number of constraints by modifying the last constraint. Instead of having one constraint for each period, one single inequality constraint for all periods can be defined.

$$
\begin{aligned}
\max_{x_0^0,\boldsymbol{u},\boldsymbol{p}} \quad & -x_0^0 && (\mathcal{SO}^{III}_{\text{buyer}})\\
\text{s.t.} \quad & x_0^c + \sum_{j=1}^{I} u_j^c\left(\boldsymbol{\xi}^j\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi,\ c \in \mathbb{C}\\
& \underline{p}_i \leq p_i\left(\boldsymbol{\xi}^i\right) \leq \overline{p}_i && \forall \boldsymbol{\xi} \in \Xi,\ i \in \mathbb{I}\\
& \underline{e} \leq \sum_{i \in \mathbb{I}} p_i\left(\boldsymbol{\xi}^i\right) \leq \overline{e} && \forall \boldsymbol{\xi} \in \Xi\\
& \left|p_i\left(\boldsymbol{\xi}^i\right) - p_{i-1}\left(\boldsymbol{\xi}^{i-1}\right)\right| \leq \rho && \forall \boldsymbol{\xi} \in \Xi,\ i \in \mathbb{I}\\
& \sum_{i \in \mathbb{I}} \frac{1}{G_i^0} \left(\sum_{\substack{c=1\\ s_c \leq i \leq t_c}}^{C} \left(\sum_{j=1}^{i} u_j^c\left(\boldsymbol{\xi}^j\right)\right) F_{ii}\left(\boldsymbol{\xi}^i\right) - \sum_{c \in \mathbb{C}} G_i^c u_i^c\left(\boldsymbol{\xi}^i\right) \right.\\
& \quad \left. + p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right) - K\right)\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

The new problem $\mathcal{SO}^{III}_{\text{buyer}}$ is much simpler to solve as I constraints from

$\mathcal{SO}^{II}_{\text{buyer}}$ are exchanged with just one, single constraint. Furthermore, we can easily show that the problems $\mathcal{SO}^{II}_{\text{buyer}}$ and $\mathcal{SO}^{III}_{\text{buyer}}$ are in fact equivalent.

Proposition 4.1.1 *The robust problems $\mathcal{SO}^{II}_{buyer}$ and $\mathcal{SO}^{III}_{buyer}$ have the same optimum.*

Proof We need to show that:

1. Any solution which is feasible in $\mathcal{SO}^{II}_{\text{buyer}}$, is also feasible in $\mathcal{SO}^{III}_{\text{buyer}}$. This is trivially satisfied, since the sum of elements which are all non-negative is also non-negative. Therefore, any feasible solution of $\mathcal{SO}^{II}_{\text{buyer}}$ is also feasible in $\mathcal{SO}^{III}_{\text{buyer}}$.

2. Any optimal solution of $\mathcal{SO}^{III}_{\text{buyer}}$, is also the optimal solution of $\mathcal{SO}^{II}_{\text{buyer}}$. Clearly, there is a possibility of finding a solution that is feasible in $\mathcal{SO}^{III}_{\text{buyer}}$, but infeasible in $\mathcal{SO}^{II}_{\text{buyer}}$. Let's assume that such a solution exists and denote it as $(\tilde{x}^0_0, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}})$ (if such a solution does not exist, then the proposition is trivially satisfied). In each period, we can control the amounts of hedging instruments held in the portfolio. We proceed to show that with the same initial capital $x^0_0 = \tilde{x}^0_0$, one can devise a strategy which is feasible for $\mathcal{SO}^{II}_{\text{buyer}}$. For notational convenience, we set

$$\begin{aligned}\kappa_i\left(\boldsymbol{\xi}^i\right) := \sum_{\substack{c=1\\ s_c \le i \le t_c}}^{C} \left(\sum_{j=0}^{i} u^c_j\left(\boldsymbol{\xi}^j\right)\right) F_{ii}\left(\boldsymbol{\xi}^i\right) - \sum_{c=1}^{C} G^c_i u^c_i\left(\boldsymbol{\xi}^i\right) \\ -G^0_i \tilde{u}^0_i\left(\boldsymbol{\xi}^i\right) + p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right) - K\right), \quad \forall i \in \mathbb{I}.\end{aligned} \tag{4.3}$$

Suppose that when rebalancing the portfolio before each period i, one borrows/lends the additional amount $\kappa_i\left(\boldsymbol{\xi}^i\right)/G^0_i$ (respectively, $\kappa_i\left(\boldsymbol{\xi}^i\right)$ is negative/positive), i.e.

$$u^c_i\left(\boldsymbol{\xi}^i\right) = \tilde{u}^c_i\left(\boldsymbol{\xi}^i\right) + \frac{\kappa_i\left(\boldsymbol{\xi}^i\right)}{G^0_i} \tag{4.4}$$

Then, substituting into the last constraint of $\mathcal{SO}^{II}_{\text{buyer}}$ yields:

$$\begin{aligned}
& \sum_{\substack{c=1 \\ s_c \le i \le t_c}}^{C} \left(\sum_{j=0}^{i} u_j^c \left(\boldsymbol{\xi}^j\right) \right) F_{ii}\left(\boldsymbol{\xi}^i\right) - \sum_{c \in \mathbb{C}} G_i^c u_i^c \left(\boldsymbol{\xi}^i\right) + p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right) - K\right) \\
= & \sum_{\substack{c=1 \\ s_c \le i \le t_c}}^{C} \left(\sum_{j=0}^{i} u_j^c \left(\boldsymbol{\xi}^j\right) \right) F_{ii}\left(\boldsymbol{\xi}^i\right) - \sum_{c=1}^{C} G_i^c u_i^c \left(\boldsymbol{\xi}^i\right) - G_i^0 u_i^0 \left(\boldsymbol{\xi}^i\right) \\
& \quad + \quad p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right) - K\right) \\
= & \sum_{\substack{c=1 \\ s_c \le i \le t_c}}^{C} \left(\sum_{j=0}^{i} u_j^c \left(\boldsymbol{\xi}^j\right) \right) F_{ii}\left(\boldsymbol{\xi}^i\right) - \sum_{c=1}^{C} G_i^c u_i^c \left(\boldsymbol{\xi}^i\right) - G_i^0 \tilde{u}_i^c \left(\boldsymbol{\xi}^i\right) - \kappa_i\left(\boldsymbol{\xi}^i\right) \\
& \quad + \quad p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right) - K\right) \\
= & \ 0.
\end{aligned}$$

Such a strategy clearly satisfies the last constraint in $\mathcal{SO}^{II}_{\text{buyer}}$. The only thing left to verify is the first constraint in $\mathcal{SO}^{II}_{\text{buyer}}$ for $c = 0$:

$$\begin{aligned}
x_0^0 + \sum_{j=1}^{I} u_j^0\left(\boldsymbol{\xi}^j\right) &= \underbrace{x_0^0 + \sum_{j=1}^{I} \tilde{u}_j^0\left(\boldsymbol{\xi}^j\right)}_{\ge 0} + \underbrace{\sum_{j=1}^{I} \frac{\kappa_j\left(\boldsymbol{\xi}^j\right)}{G_j^0}}_{\ge 0} \\
&\ge 0
\end{aligned}$$

where both terms are positive since $(\tilde{x}_0^0, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}})$ is feasible in $\mathcal{SO}^{III}_{\text{buyer}}$.

Since any feasible point of $\mathcal{SO}^{II}_{\text{buyer}}$ is feasible for $\mathcal{SO}^{III}_{\text{buyer}}$ and for any feasible point in $\mathcal{SO}^{III}_{\text{buyer}}$, there exist a feasible point in $\mathcal{SO}^{II}_{\text{buyer}}$ with the same objective value, the two problems $\mathcal{SO}^{II}_{\text{buyer}}$ and $\mathcal{SO}^{III}_{\text{buyer}}$ have the same optimum (although the feasible set of the second problem is potentially larger). □

4.2 Polynomial Decision Rules

For modelling the decision variables which values depend on the random variables, we use the polynomial decision rules. Suppose we begin with the vectors of random variables, as defined earlier $\boldsymbol{\xi}_i \in \mathbb{R}^m$:

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{1m} \end{pmatrix}, \boldsymbol{\xi}_2 = \begin{pmatrix} \xi_{21} \\ \vdots \\ \xi_{2m} \end{pmatrix}, \ldots, \boldsymbol{\xi}_i = \begin{pmatrix} \xi_{i1} \\ \vdots \\ \xi_{im} \end{pmatrix}$$

Then, for example, the $u_i^c(\boldsymbol{\xi}^i)$ decision variable can be expressed as:

$$u_i^c\left(\boldsymbol{\xi}^i\right) = \sum_{n \in \mathbb{N}_0^{m \times i}} (v_n^c \prod_{\substack{k=1\ldots i \\ j=1\ldots m}} \xi_{kj}^{n_{kj}}) \tag{4.5}$$

With the following constraint on sum of powers n:

$$\sum_{\substack{k=1\ldots i \\ j=1\ldots m}} n_{kj} \leq d \tag{4.6}$$

Where d is a parameter specifying a degree of the polynomials. One could expect that solving the problem with higher degree of polynomials would result in more accurate solution. On the other hand, this would make the linear programming problem harder to solve, as it the number of variables would increase.

Similarly, we express $p_i(\boldsymbol{\xi}^i)$ as:

$$p_i\left(\boldsymbol{\xi}^i\right) = \sum_{n \in \mathbb{N}_0^{m \times i}} (\pi_n \prod_{\substack{k=1\ldots i \\ j=1\ldots m}} \xi_{kj}^{n_{kj}}) \tag{4.7}$$

The values of the decision variables v_n^c and $\pi_n^c, \forall c \in \mathbb{C}, n \in \mathbb{N}_0^{m \times i}$ are to be found by solving the problem.

A number of samples ($\boldsymbol{\xi}$ vectors are sampled from Ξ) of such polynomials will be generated (to represent different scenarios), and respective number of constraints will be added to the linear programming problem which is being solved. Even though the uncertainty set Ξ is infinite (the number of possible scenarios), we can claim that the solution will be accurate to some degree, which can be precisely estimated as described by Campi and Garatti in [2]. Alternatively, we can also estimate the probability of violation of any of the constraints of a newly generated problem instance by the previously calculated optimal solution (Section 6.3). As more $\boldsymbol{\xi}$ vectors are sampled, more constraints are added to the optimisation problem, increasing its complexity, but also at the same time improving the accuracy of the solution.

4.3 Comparison with Binomial Lattice Model

In this section we compare our pricing model with the well-known binomial lattice model. We can use our optimisation problem formulation to price simple, path-independent call options (with electrical energy as the underlying asset) and compare the results with those obtained by using the binomial lattice.

In the binomial lattice model, if we know the price of the underlying asset at the beginning of a single period (of a fixed length), the price at the beginning of the next period can have only one out of two possible values. These values are estimated by multiplying the initial price by either u or d (respectively, price going up or down). We proceed, step by step, forward in the lattice to find the value of the underlying asset at all periods under all scenarios.

Let the total number of periods in the pricing problem equal T. At the last period (T), assuming that $\underline{p} = 0$, we define the value of the option under some scenario in which price of the underlying equals S_T in that period to be the amount of money which can be gained by exercising the option:

$$C_T = \max\left\{\overline{p}\left(S_T - K\right),\ 0\right\} \tag{4.8}$$

The price S_T differs across scenarios. By the scenarios we mean some combinations of the multiplications of the initial price S_0 by u and d. In the binomial lattice model with T periods, we have T unique scenarios.

We price the option backwards, using the following formula:

$$C_{t-1} = \frac{1}{R}\hat{E}[C_t] + \max\left\{\overline{p}\left(S_{t-1} - K\right),\ 0\right\}, \quad \forall t < T \tag{4.9}$$

where $\hat{E}[C_t]$ denotes the expected value with respect to the *risk-neutral probabilities* q and $(1-q)$ defined as:

$$q = \frac{R-d}{u-d} \tag{4.10}$$

where R is the risk-free rate growth factor.

The final value of the option is computed when reaching the root of the lattice, i.e. C_1. In order to avoid arbitrage, we always require

$$u > R > d \tag{4.11}$$

The details of the risk-neutral pricing and the binomial lattice model can be found in [11].

We can also use our pricing model $\mathcal{SO}^{III}_{\text{buyer}}$ to price the electricity call option. We need to drop the energy constraint and the ramping constraint, as they are not needed in this simple case. We set I to be equal to the number of periods in the binomial lattice and use just one hedging instrument which can be considered as a future contract with maturity date and time equal to the start date and time. For example, a contract starting at time $i = 2$ terminates at $i = 2$ as well. The degree of the polynomials has to be set to $d = T - 1$, as we require that many degrees of freedom to take into account all possible price movements. Instead of the price model described in Chapter 3, we use simple deterministic pricer which multiplies the initial price S_0 by the appropriate sequence of u and d (different and unique for each scenario). We generate as many samples (scenarios), as there are terminal nodes in the binomial lattice. So for 2 periods we generate 2 samples of the constraints, and for 3 periods - 4 samples.

Instead of the $\boldsymbol{\xi}$ vectors with random variables described earlier, we use predefined values, each representing one of the two possible price movements. Each scenario is represented by a unique sequence of such values.

4.3.1 Numerical Example

Our choice of the binomial lattice parameters is shown in the Table 4.2.

Parameter	Value
u	1.2
R	1.1
d	0.9
S_0	20 €/MWh
K	20 €/MWh
$\overline{p}$	60 MW

Table 4.2: Binomial lattice parameters

When evaluating the option using equation (4.9) and the binomial lattice with 2 periods, the price is 145.45 €/MWh. When using larger lattice, with 3 periods, the solution changes to 374.66 €/MWh.

By following the instructions above, we managed to price this simple option using the model $\mathcal{SO}^{III}_{\text{buyer}}$. The linear programming problem for 2 periods, in the form of CPLEX *.lp* file, is presented on Figure 4.2 obtain exactly the same results as when using the binomial lattice model and the recursive pricing formula (both in the case of 2 and 3 periods).

The result presented above is an evidence that our valuation scheme is correct when used for pricing simple call options.

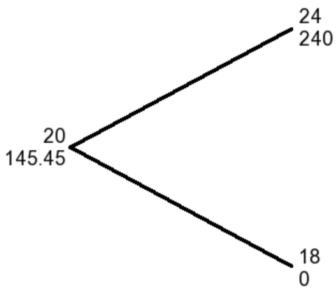

Figure 4.1: The option priced on a 2 period binomial lattice model

```
Maximize
 obj: - x00

Subject To
 c0: uc11 + 1.2 uc21 + uc22 + x00 >= 0.0
 c1: u111 + 1.2 u121 + u122 >= 0.0
 c2: p11 <= 60.0
 c3: p11 >= 0.0
 c4: 1.2 p21 + p22 <= 60.0
 c5: 1.2 p21 + p22 >= 0.0
 c6: - uc11 - 20.0 u111 + 4.363636363636363 p21
+ 3.6363636363636362 p22 - 1.2 uc21 - uc22
- 26.18181818181818 u121 - 21.818181818181817 u122 + 21.818181818181817 u111
+ 26.18181818181818 u121 + 21.818181818181817 u122 >= 0.0

 c7: uc11 + 0.9 uc21 + uc22 + x00 >= 0.0
 c8: u111 + 0.9 u121 + u122 >= 0.0
 c9: p11 <= 60.0
 c10: p11 >= 0.0
 c11: 0.9 p21 + p22 <= 60.0
 c12: 0.9 p21 + p22 >= 0.0
 c13: - uc11 - 20.0 u111 - 1.6363636363636362 p21
- 1.8181818181818181 p22 - 0.9 uc21 - uc22
- 14.727272727272727 u121 - 16.363636363636363 u122 + 16.363636363636363 u111
+ 14.727272727272727 u121 + 16.363636363636363 u122 >= 0.0

Bounds
 - inf <= p11 <= + inf
 - inf <= p21 <= + inf
 - inf <= p22 <= + inf
 - inf <= u111 <= + inf
 - inf <= u121 <= + inf
 - inf <= u122 <= + inf
 - inf <= uc11 <= + inf
 - inf <= uc21 <= + inf
 - inf <= uc22 <= + inf
 - inf <= xc0 <= + inf
End
```

Figure 4.2: CPLEX *.lp* file with the linear programming problem corresponding to pricing the option in 2 period binomial lattice model

Chapter 5

Stages Aggregation

In order to be able to price real electricity swing options with long delivery periods (typically from a few weeks to a couple of months), we have to simplify the model $\mathcal{SO}^{III}_{\text{buyer}}$ further by reducing significantly the number of decision variables. We do this through aggregating the stages, in principle following the method presented in [7].

5.1 Exercise Profiles

In order to make the problem computationally possible to solve, we aggregate individual trading periods ($i \in \mathbb{I}$) into T 'macro periods' and, in a result, reduce further the number of decision variables. Such periods are defined as:

$$\{i_t, i_t + 1, \ldots, i_{t+1} - 1\}, \forall 1 \le t < T$$
$$\{i_t, i_t + 1, \ldots, I\}, t = T$$

Trading decisions $(u_i^c(\boldsymbol{\xi}^i), \forall c \in \mathbb{C})$ are taken only at the beginning of all macro-periods (only at $i = i_t$), at all other single-periods in between the decision variables are set to zero.

Similarly, the exercise patterns $(p_i(\boldsymbol{\xi}^i))$ have to be chosen at the beginning of each macro-period. Therefore, we need to pre-specify exercise profiles.

During each of the periods $i_t \ldots i_{t+1} - 1$, the power we exercise has to be kept within the power constraints limits ($\underline{p}$ and $\bar{p}$). If there was no constraint on total energy exercised, we would like to exercise the option whenever a spot price $(F_{ii}(\boldsymbol{\xi}^i))$ is higher than the strike price (K) for the maximum possible amount of power $(\bar{p})$.

By taking the energy constraint into account, the strike price in each period i should be adjusted by some parameter. Let this parameter be represented as L_i. So the option would be exercised in macro-period t, under some exercise profile α, if the price is higher than $C_t^\alpha = K + L_t$ (cut-off), assuming L_t is constant in some $i_t \dots i_{t+1} - 1$ interval ($L_t = L_i, \forall i : i_t \le i \le i_{t+1} - 1$).

We can find a near-optimal strategy for exercising the option for a certain scenario can be found by solving a simple deterministic linear program:

$$\begin{aligned} \max \quad & \sum_{j=i_t}^{i_{t+1}-1} \left(\left(F_{i_t j} \left(\boldsymbol{\xi}^j \right) - C_t^\alpha \right) p_j \right) \\ \text{s.t.} \quad & \underline{p} \le p_j \le \bar{p}, \forall j = i_t \dots t_{t+1} - 1 \\ & |p_j - p_{j-1}| < \rho, \forall j : i_t \le j \le t_{t+1} - 1 \end{aligned} \tag{5.1}$$

Where ρ is a ratchet to be chosen (as previously) and p_j is no longer a decision variable.

We need to solve (5.1) for α (where $\alpha = 1 \dots A$) different values of the cut-off C_t^α. In general, we require $A \ge 2$, as we need to have at least 2 profiles with the following cut-off values:

$$C_t^1 = \min\{F_{i_t j} \left(\boldsymbol{\xi}^j \right) \mid i_t \le j \le i_{t+1} - 1\}$$

and

$$C_t^A = \max\{F_{i_t j} \left(\boldsymbol{\xi}^j \right) \mid i_t \le j \le i_{t+1} - 1\}$$

However, in order to obtain interesting profiles, we will solve (5.1) for a few more cut-off values, such that $C_t^1 \le C_t^2 \le \dots \le C_t^{A-1} \le C_t^A$.

Solving (5.1) for α (where $\alpha = 1 \dots A$) different values of the cut-off C_t^α results in A exercise profiles for macro-period t (in the form of the vectors filled with values representing amounts of power to be exercised at each atomic period):

$$\mathbf{p}_t^\alpha = (p_{i_t}^\alpha, \dots, p_{i_{t+1}-1}^\alpha) \tag{5.2}$$

5.1.1 Example

To illustrate the idea of pre-defined exercise profiles, we present an example of five exercise patterns found for some exemplary forward prices curve and a period of one week (168 hours).

We found the exercise profiles by solving the deterministic linear programming problem 5.1. The parameters are set to $A = 5$, $\underline{p} = 0\text{MW}$, $\overline{p} = 60\text{MW}$ $\rho = 6\text{MW}$, $K = 20$€/MWh and the dummy variable $p_{-1} = p_{\text{start}} = 0\text{MW}$.

One can notice that in the first profile, where the prices are always non-negative, one would like to exercise as much power as possible in every hour. However, because of the ramping constraint and $p_{\text{start}} = 0\text{MW}$, we observe

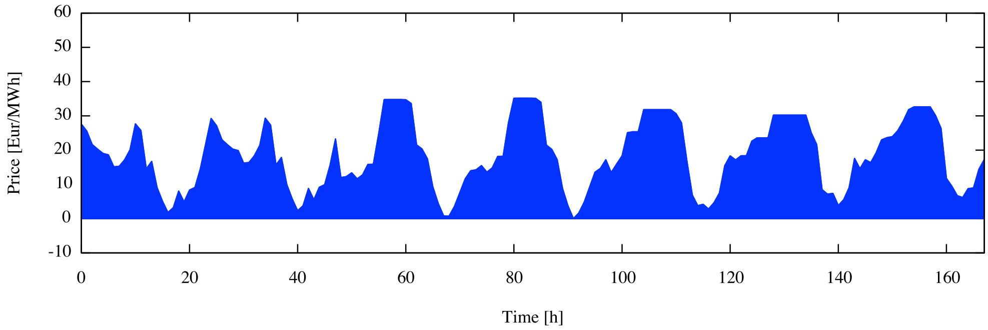

Figure 5.1: Shifted forward price curve for first exercise profile

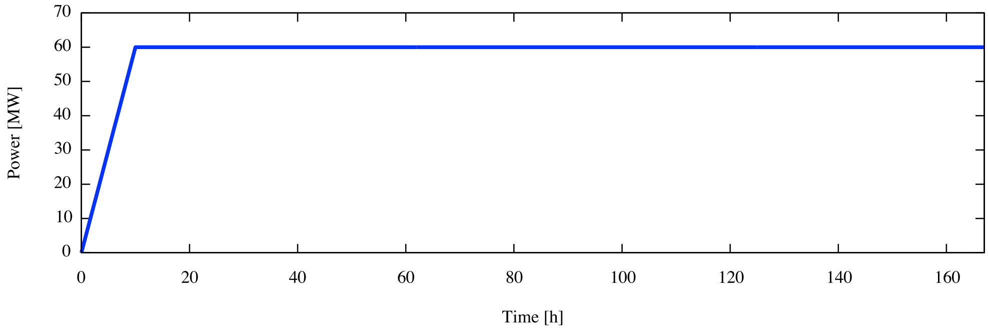

Figure 5.2: First power exercise profile

the delay (in the first 10 hours) in reaching the maximum level of power which can be exercised.

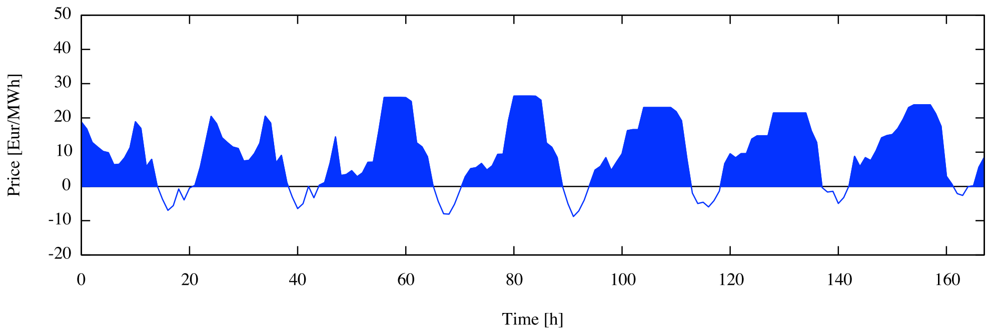

Figure 5.3: Shifted forward price curve for second exercise profile

The second exercise profile is more interesting. The prices at some of the hours shifted with the second cut-off value become negative and it is not profitable to exercise at these times. However, again due to the existence of

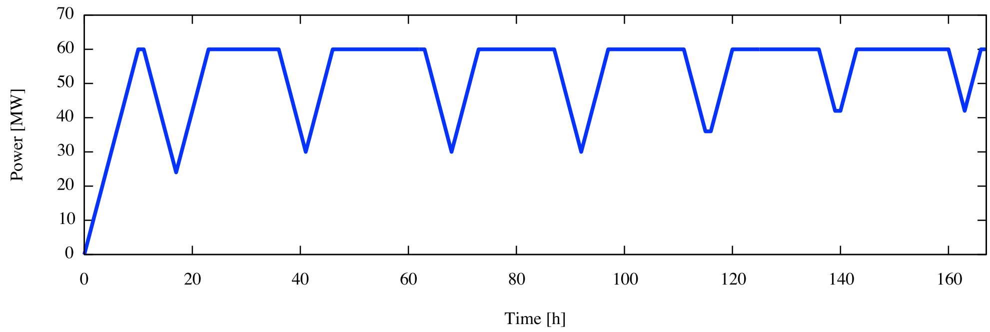

Figure 5.4: Second power exercise profile

the ramping constraint, the option holder is not able to reduce the amount of power to zero immediately, whenever the forward prices curve becomes negative.

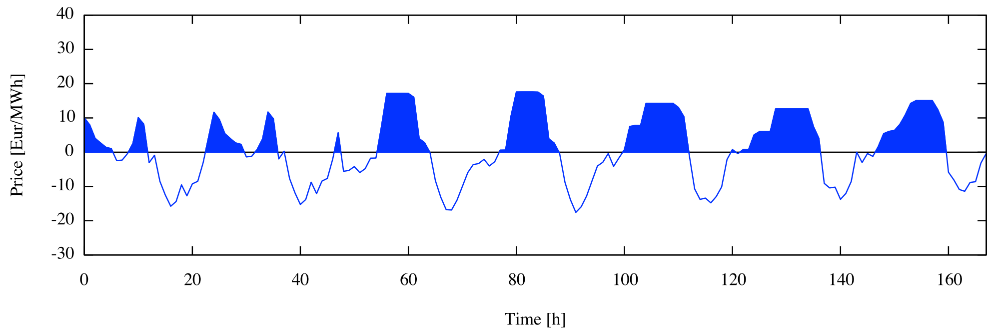

Figure 5.5: Shifted forward price curve for third exercise profile

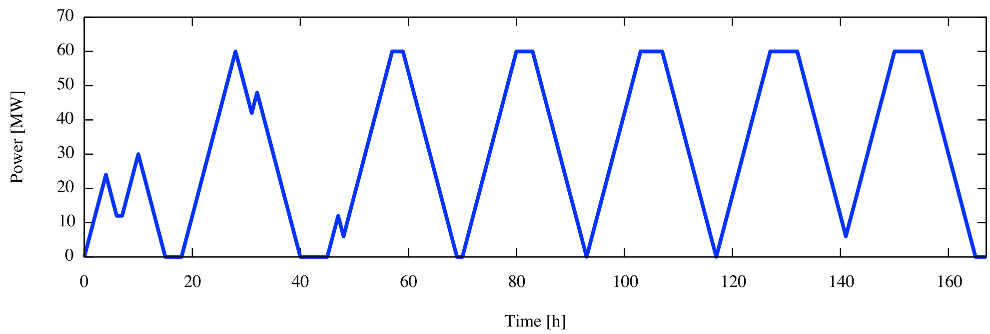

Figure 5.6: Third power exercise profile

In the last exercise profile, all the shifted prices are below zero (as presented on Figure 5.9). In such situation, the option holder does not wish

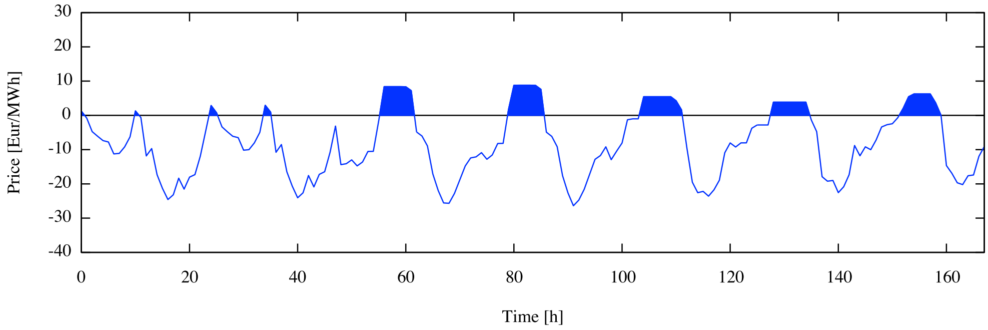

Figure 5.7: Shifted forward price curve for fourth exercise profile

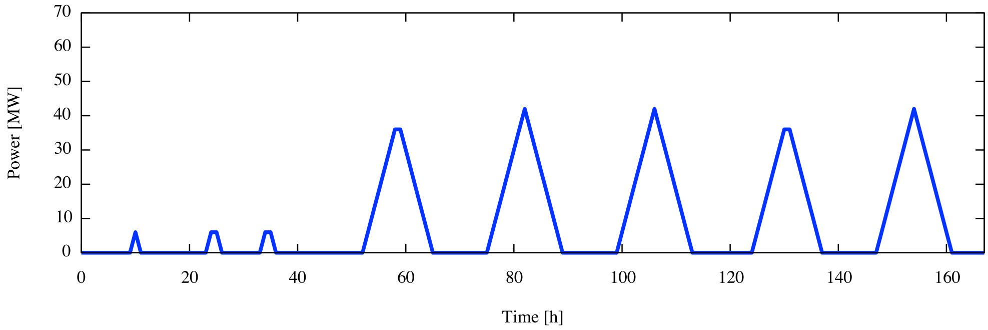

Figure 5.8: Fourth power exercise profile

to exercise at any hour. Therefore, the resulting profile is just a straight, horizontal line aligned with the x axis (Figure 5.10).

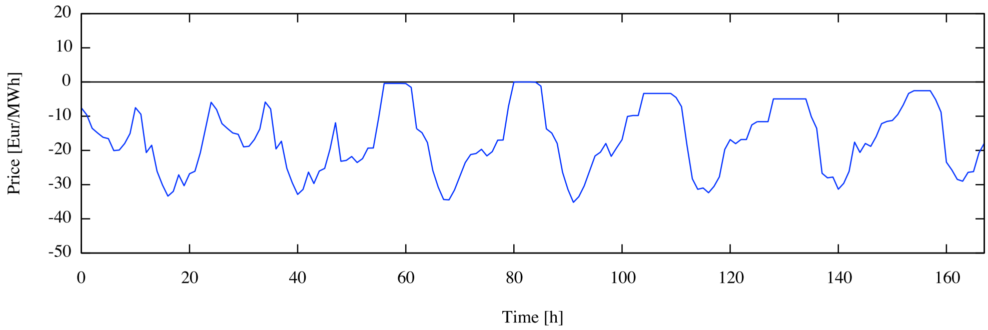

Figure 5.9: Shifted forward price curve for fifth exercise profile

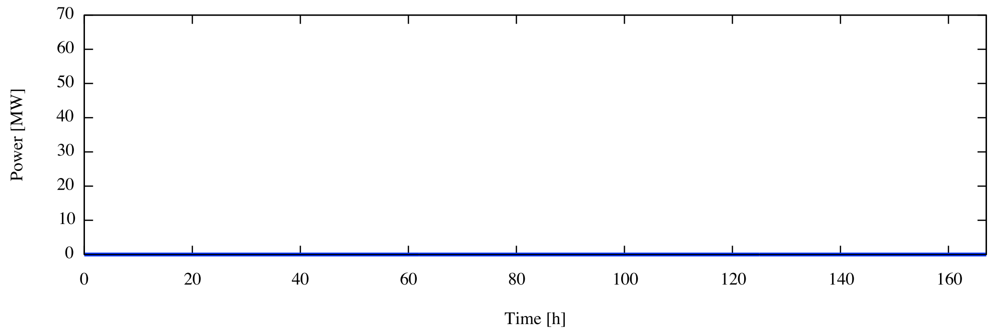

Figure 5.10: Fifth power exercise profile

5.2 Approximation of the Constraints

In this section, we attempt to approximate the original pricing model $\mathcal{SO}^{III}_{\text{buyer}}$ by making use of the exercise profiles and modifying the constraints appropriately.

The only constraints which need to be changed are those which involve power amount decision variables.

Let's define the amounts of power the option is exercised for in each macro-period t using the equation (5.2):

$$\mathbf{p}_t = \sum_{\alpha=1}^{A} (\lambda_t^\alpha \left(\boldsymbol{\xi}^t\right) \mathbf{p}_t^\alpha) \tag{5.3}$$

Where $\lambda_t^\alpha \left(\boldsymbol{\xi}^t\right)$ is unknown and represents a decision variable. We require $\sum_{\alpha=1}^{A} \lambda_t^\alpha \left(\boldsymbol{\xi}^t\right) = 1$ and $\lambda_t^\alpha \left(\boldsymbol{\xi}^t\right) \geq 0$, $\forall \alpha = 1 \ldots A$. All the elements of the $\mathbf{p}_t$ vector need to be bounded by the power limits $\underline{p}$ and $\overline{p}$.

As the energy constraint restricts the aggregate amount of power which can be exercised in all periods, it needs to be changed and related to the chosen exercise profiles. Let e_t^α represent the amount of energy the option is exercised for in macro-period t under some exercise profile α.

$$e_t^\alpha = \mathbf{e}^T \mathbf{p}_t^\alpha = \sum_{i=i_t}^{i_{t+1}-1} p_i^\alpha \tag{5.4}$$

Where $\mathbf{e}$ is a vector of ones, i.e. $\mathbf{e}^T = [1, 1, \ldots, 1]$.

Next, taking into account all exercise profiles, the total energy (e) is represented as:

$$e = \sum_{t=1}^{T} \mathbf{e}^T \mathbf{p}_t = \sum_{t=1}^{T} \sum_{\alpha=1}^{A} \lambda_t^\alpha \left(\boldsymbol{\xi}^t\right) e_t^\alpha \tag{5.5}$$

Hence, we obtain the energy constraint in a certain scenario:

$$\underline{e} \leq \sum_{t=1}^{T} \sum_{\alpha=1}^{A} \lambda_t^\alpha \left(\boldsymbol{\xi}^t\right) e_t^\alpha \leq \bar{e} \tag{5.6}$$

Finally, the last constraint (5.7) from the original problem $\mathcal{SO}_{\text{buyer}}^{III}$ needs to be modified accordingly to depend on the new $\lambda(\boldsymbol{\xi})$ variables instead of the original power decision variables $p(\boldsymbol{\xi})$.

$$\begin{aligned}\sum_{i \in \mathbb{I}} \frac{1}{G_i^0} \Bigg(\sum_{\substack{c=1 \\ s_c \leq i \leq t_c}}^{C} \left(\sum_{j=1}^{i} u_j^c \left(\boldsymbol{\xi}^j\right) \right) F_{ii} \left(\boldsymbol{\xi}^i\right) - \sum_{c \in \mathbb{C}} G_i^c u_i^c \left(\boldsymbol{\xi}^i\right) \\ + p_i \left(\boldsymbol{\xi}^i\right) \left(F_{ii} \left(\boldsymbol{\xi}^i\right) - K \right) \Bigg) \geq 0 \quad \forall \boldsymbol{\xi} \in \Xi\end{aligned} \tag{5.7}$$

The power decision variable appears in the third term of the constraint (5.7):

$$p_i \left(\boldsymbol{\xi}^i\right) \left(F_{ii} \left(\boldsymbol{\xi}^i\right) - K\right) \tag{5.8}$$

By aggregating the stages, (5.8) changes to:

$$\sum_{i=i_t}^{i_{t+1}-1} p_i \left(\boldsymbol{\xi}^i\right) \left(F_{ii} \left(\boldsymbol{\xi}^i\right) - K\right) \tag{5.9}$$

Now let $\boldsymbol{S}_t$ be the vector of all spot prices of single periods within the macro-period t, i.e.:

$$\mathbf{S}_t = \begin{pmatrix} F_{i_t i_t} \left(\boldsymbol{\xi}^{i_t}\right) \\ \vdots \\ F_{i_{t+1}-1 i_{t+1}-1} \left(\boldsymbol{\xi}^{i_{t+1}-1}\right) \end{pmatrix}$$

and $(\mathbf{S}_t - K\mathbf{e})$ is a vector of all differences between the spot and strike prices in all single periods in one macro-period t:

Then, (5.9) can be represented as:

$$\begin{aligned}\sum_{i=i_t}^{i_{t+1}-1} p_i\left(\boldsymbol{\xi}^i\right)\left(F_{ii}\left(\boldsymbol{\xi}^i\right)-K\right) &= \left(\mathbf{S}_t-K\mathbf{e}\right)^T\mathbf{p}_t \\ &= \sum_{\alpha=1}^{A}\lambda_t^{\alpha}\left(\boldsymbol{\xi}^t\right)\left(\mathbf{S}_t-K\mathbf{e}\right)^T\mathbf{p}_t^{\alpha} \quad \forall\boldsymbol{\xi}\in\Xi\end{aligned} \tag{5.10}$$

Substituting into the constraint, yields:

$$\begin{aligned}\sum_{t=1}^{T}\frac{1}{G_t^0}\Bigg(\sum_{\alpha=1}^{A}\lambda_t^{\alpha}\left(\boldsymbol{\xi}^t\right)\left(\mathbf{S}_t-K\mathbf{e}\right)^T\mathbf{p}_t^{\alpha}-\sum_{c\in\mathbb{C}}G_i^c u_i^c\left(\boldsymbol{\xi}^i\right) \\ +\sum_{\substack{c=1\\ s_c\le t\le t_c}}^{C}\left(\sum_{j=1}^{t}u_c^j\left(\boldsymbol{\xi}^j\right)\mathbf{e}^T\mathbf{S}_t\right)\Bigg)\geq 0 \quad \forall\boldsymbol{\xi}\in\Xi\end{aligned} \tag{5.11}$$

5.3 The Final Pricing Model

By introducing all the approximations into the model $\mathcal{SO}^{III}_{\text{buyer}}$, we obtain the final model $\mathcal{SO}^{IV}_{\text{buyer}}$.

$$
\begin{aligned}
\max_{x_0^0,\boldsymbol{u},\boldsymbol{\lambda}} \quad & -x_0^0 && (\mathcal{SO}^{IV}_{\text{buyer}})\\
\text{s.t.} \quad & x_0^c + \sum_{j=1}^{T} u_j^c\left(\boldsymbol{\xi}^j\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi,\ c \in \mathbb{C}\\
& \sum_{\alpha=1}^{A} \lambda_t^\alpha\left(\boldsymbol{\xi}^t\right) = 1 && \forall \boldsymbol{\xi} \in \Xi, t < T\\
& \lambda_t^\alpha\left(\boldsymbol{\xi}^t\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi, t < T\\
& \underline{e} \leq \sum_{t=1}^{T}\left(\sum_{\alpha=1}^{A} \lambda_t^\alpha\left(\boldsymbol{\xi}^t\right) e_t^\alpha\right) \leq \bar{e} && \forall \boldsymbol{\xi} \in \Xi\\
& \sum_{t=1}^{T} \frac{1}{G_t^0}\left(\sum_{\alpha=1}^{A} \lambda_t^\alpha\left(\boldsymbol{\xi}^t\right)(\mathbf{S}_t - K\mathbf{e})^T \mathbf{p}_t^\alpha - \sum_{c\in\mathbb{C}} G_i^c u_i^c\left(\boldsymbol{\xi}^i\right)\right.\\
& \qquad \left. + \sum_{\substack{c=1\\ s_c\leq t\leq t_c}}^{C}\left(\sum_{j=1}^{t} u_c^j\left(\boldsymbol{\xi}^j\right)\mathbf{e}^T\mathbf{S}_t\right)\right) \geq 0 && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

In the next sections we will show how this model can be used to price the electricity swing options in practice and evaluate our pricing scheme.

Chapter 6

Numerical Results and Evaluation

In this section, in order to verify the degree of accuracy of our valuation scheme and to establish whether it can be used in practical situation, we will carry out a series of numerical experiments[1]. Unless otherwise noted, in all computational results are based on the parameters values presented in the Table 6.1.

$t_{\text{start}} = 1$ Aug 2004, 00:00	$\Delta = 1$ h
$t_{\text{end}} = 31$ Oct 2004, 24:00	$I = 2208$ h
$\underline{p} = 0$ MW	$\rho = 6$ MW
$\bar{p} = 60$ MW	$A = 6$
$p_{\text{start}} = 0$ MW	$m = 2$
$\underline{e} = 0$ MWh	$K = 20$€/MWh
$\bar{e} = 52992$ MWh	$r_f = 0$

Table 6.1: Model parameters choice

We denote the beginning and end of the delivery period by t_{start} and t_{end} respectively. The risk-free rate is set to zero for the sake of simplicity.

Our numerical experiments indicate that the pricing model $\mathcal{SO}^{IV}_{\text{buyer}}$ with polynomials of degree 1 ($d = 1$), 5 aggregated stages, 6 power exercise profiles ($A = 6$) and 3000 samples, and quarter, month and week futures as hedging instruments, achieves reasonable accuracy and solution time of approximately 13 seconds. The option value estimated by using these parameters equals 8.21982E€.

[1]All calculations were carried out on a 2.26 GHz Intel Core 2 Duo PC with 4GB of RAM and the instances of the problem $\mathcal{SO}^{IV}_{\text{buyer}}$ were solved with CPLEX 11.2.1.

6.1 Electricity Prices

For our numerical calculations we used a forward prices curve for period of 3 months, from 1 August 2004 (00:00) to 31 October 2004 (24:00). Part of this curve is shown on Figure 6.1.

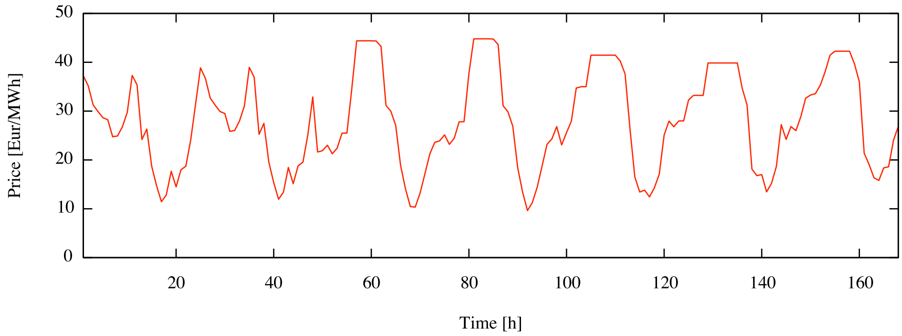

Figure 6.1: Future prices curve used in the first contract week

6.1.1 Forward price model parameters

In the prices generation process we used two volatility functions σ_i^1 and σ_i^2 defined as:

$$\sigma_i^1 = \nu_1 e^{-\alpha_1 (I-i)}$$

and

$$\sigma_i^2 = \nu_2 (e^{-\alpha_2 (I-i)} - e^{-\alpha_1 (I-i)})$$

Where ν_1, ν_2, α_1 and α_2 are parameters.

We decided to use the same values of parameters for the price process as used in [7]. Therefore, we use two volatility functions with parameters set to $\alpha_1 = 6.849\text{E}{-04}$ h^{-1}, $\alpha_2 = 0$, $\nu_1 = 1.068\text{E}{-0.2}$ h$^{-1/2}$ and $\nu_2 = 2.671\text{E}{-03}$ h$^{-1/2}$. You can see the generated expected spot prices curve on Figure 6.2.

6.2 Valuation Under No Uncertainty

We can test our model and assess its accuracy by using it to value the option and comparing the result with outcomes of an analytical computation, which can be performed in some cases. When all sampled forward prices curves coincide (there is no uncertainty), and additionally, there are neither constraints on energy nor ramping constraints on power, we can calculate

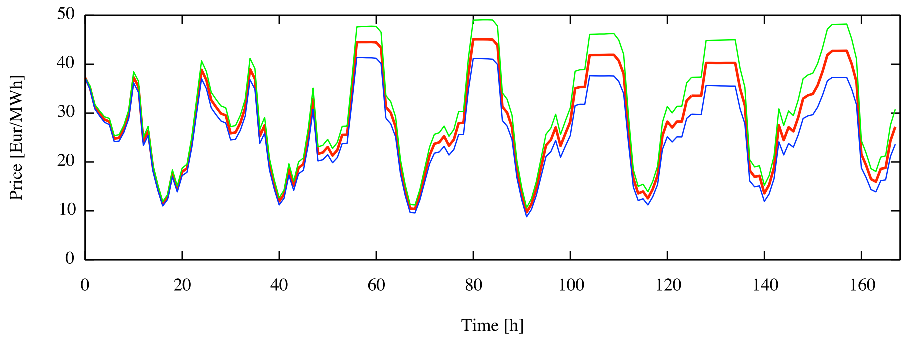

Figure 6.2: The expected value of the spot price distribution in the first contract week estimated on sample of size 1000.

the value of the option analytically in a spreadsheet using the formula 6.1.

$$Price = \bar{p}\sum_{j=1}^{I} max\{F_{0i} - K, 0\} \tag{6.1}$$

That is, the bid price of the option is equal to the maximum amount of money which can be gained by exercising the option (exercise whenever the price is higher than the strike price).

Using the original curve (Figure 6.1) and formula 6.1 we can evaluate the bid price of the swing option, which amounts to 1.38287E6€.

We can verify the correctness of the basic pricing model presented in the chapter 4. By solving the problem $\mathcal{SO}^{III}_{\text{buyer}}$ with energy limits set to $\underline{e} = 0$ MWh, $\overline{e} = I\overline{p} = 132480$ MWh (which is the maximum amount of energy which can be obtained by exercising the option between t_{start} and t_{end}) and $\rho = \overline{p} = 60$ MW, we obtain exactly the same result as by using the formula 6.1 which shows that our problem formulation is correct.

Next, we can verify the accuracy of the approximated model $\mathcal{SO}^{IV}_{\text{buyer}}$ (with aggregated stages) presented in Section 5. The accuracy of the model depends on the number of exercise profiles used. When using 5 profiles, the option value is evaluated to 1.37171E6€, underestimating analytical value by 0.8%. When using 10 and 100 profiles the option values are respectively 1.37922E6€ (underestimation of 0.3%) and 1.38155E6€ (0.1%). We observe that the value of the solution improves when increasing the number of exercise profiles used. However, even the results obtained when using relatively few profiles (5) are satisfactory and show that our valuation scheme is accurate. Therefore, our choice of 6 exercise profiles is justified.

In order to validate our result we can also compare it with the option value calculated analytically by Haarbrücker and Kuhn in [7] which amounts to 1.426E6€. Our price is lower, which is correct, as we estimate the buyer's price (not the equilibrium price).

6.3 Estimation of Violation Probability

We used the polynomial decision rules as described in the Section 4.2, where we mentioned that we needed to estimate the required number of samples (scenarios) we must use in order to minimize the probability of violation of any subsequently generated problem instance.

We proceed by solving the problem $\mathcal{SO}^{IV}_{\text{buyer}}$ several times, each time using different numbers of samples. Then, for each of the problems solved, we generate a large number (1000) of new problem instances (samples), plug in the computed optimal values of the decision variables and check whether any of the constraints is violated. If so, we treat the whole instance (sample) as violated and proceed to next one. The results obtained by following this procedure are presented on Figure 6.3.

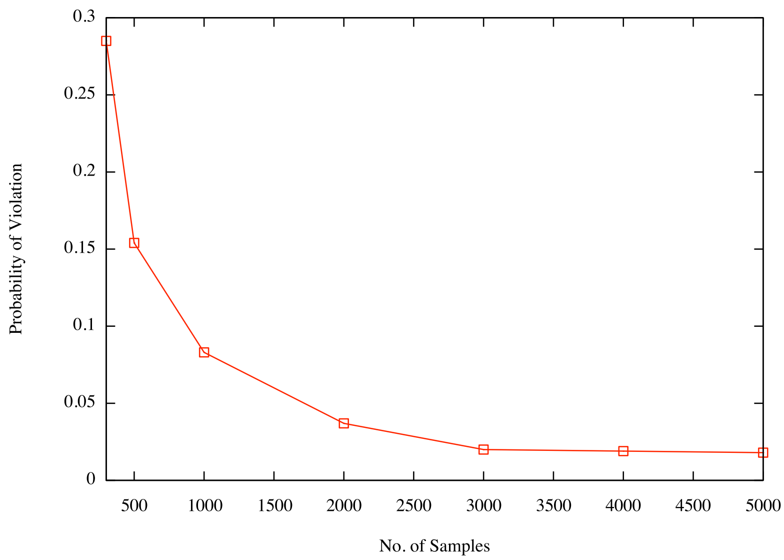

Figure 6.3: Probability of violation of a new sample as a function of number of samples used to compute the optimal values of decision variables. Parameters set as presented in Table 6.1, with $d = 2$, $A = 6$, and only base load month futures used for hedging.

We observe that when we use 3000 samples the violation probability is low (0.02) and does not improve significantly when we increase it further.

The result also confirmes that following the robust optimisation approach yields meaningful results, as a subset of relatively few samples fetched from Ξ includes most of the scenarios possible.

We conclude that using 3000 samples gives acceptable results. Therefore, unless noted otherwise, this number of samples will be used in all the numerical experiments.

6.4 Degree of Polynomials

We need to choose appropriate degree of the polynomial decision rules used to model the decision variables. There is a clear trade-off between the accuracy and time required to reach the solution. As we raise the degree, we expect both the precision and the solution time to increase. We conducted a small numerical experiment which results are presented in the Table 6.2.

Degree (d)	Option Value	Solution Time
0	8.13399E5€	1.33 sec
1	8.21982E5€	13.38 sec
2	8.28744E5€	17 minutes

Table 6.2: Solving problem $\mathcal{SO}^{IV}_{\text{buyer}}$ with various values of d (degree of the polynomial decision rules). $A = 6$, 3000 samples and base load quarterly, monthly and weekly futures used for hedging only.

We observe the dramatic increase in the solution time for degree 2 compared to degree 1. For degree 3, the problem is not computationally feasible at all (the solution has not been found even after 3 hours). At the same time, we observe that the improvement in the option value is not very significant. In the next sections of the report we will investigate some detailed results for both choices of the degree value.

6.5 Valuation Using Polynomials of Degree 1

In order to carry out meaningful numerical experiments, we need to find appropriate number of stages which will be used and choose hedging instruments that we will use.

6.5.1 Choice of hedging instruments

We need to carefully choose the hedging instruments which will be used in our trading strategy. As mentioned before in Section 2.1.3, we have futures contracts with four different maturities tradeable in the market. Additionally, each of them comes in two types: base or peak load.

As we price the option using the forward prices curve of length of three months and thus with delivery period length of three months, yearly contracts are clearly not to be chosen. Once the delivery period of a given contract has started, it cannot be traded anymore. Therefore, using quarterly contracts alone would yield similar results to static hedging (deciding upon amounts of the hedging instruments purchased or sold upfront, before the option delivery period starts), as they could be traded only initially.

Therefore, for valuation of the option, we will mainly use either monthly or weekly futures. In the chosen delivery period of length of three months, starting on t_{start} and commencing on t_{end}, there could be at most three monthly contracts defined and thirteen weekly contracts (starting on Monday and expiring on Sunday in every calendar week). Clearly, using the latter results in substantial increase in the number of random variables and, as a result, increases the solution time. Also, we cannot use the weekly contracts alone, as the delivery period for the swing option is not aligned with the futures' starting day (the option's delivery period starts on Sunday, whereas weekly futures start on Monday). For the same reason, we do not use peak load contracts alone, as prices at some of the periods would be left unhedged.

The results obtained when using different hedging instruments are summarised in the Table 6.3. Using base and peak load contracts together provides more flexibility in comparison to using just the base load ones. Therefore, the option value is always a bit higher. However, the difference in the option value is not as significant as is the increase in the solution time (especially when using all quarterly, weekly and monthly futures).

We see that the solution improves when we increase the number of types of contracts used. Furthermore, we observe that the solution time obtained when using base load quarterly, monthly and weekly futures is relatively low (13.38 seconds) and therefore this choice is optimal. This justifies our choice of the hedging instruments made at the beginning of the chapter.

	Base Load	**Base & Peak Load**
Quarterly	7.85591E5€ *(3.27 sec)*	8.01432E5€ *(3.45 sec)*
Quarterly & Monthly	8.09879E5€ *(4.11 sec)*	8.11185E5€ *(5.06 sec)*
Quarterly, Weekly & Monthly	8.21982E5€ *(13.38 sec)*	8.22914E5€ *(117.29 sec)*

Table 6.3: Results of numerical experiments with various futures contracts used as hedging instruments with polynomials' degree set to 1 ($d = 1$), $A = 6$, 3000 samples and 5 stages. The results are presented in form of the option value computed and the solution time is shown in each case in the brackets.

6.5.2 Number of stages used

In order to determine the influence of the number of stages (*macro-periods*) used on the value of the solution we conduct another numerical experiment. Keeping all other parameters fixed, we modify the number of stages and note the option value obtained. When increasing the number of samples, we reuse already generated constraints and just add newly generated scenarios to existing ones. This reduces computational cost of generation of the problems.

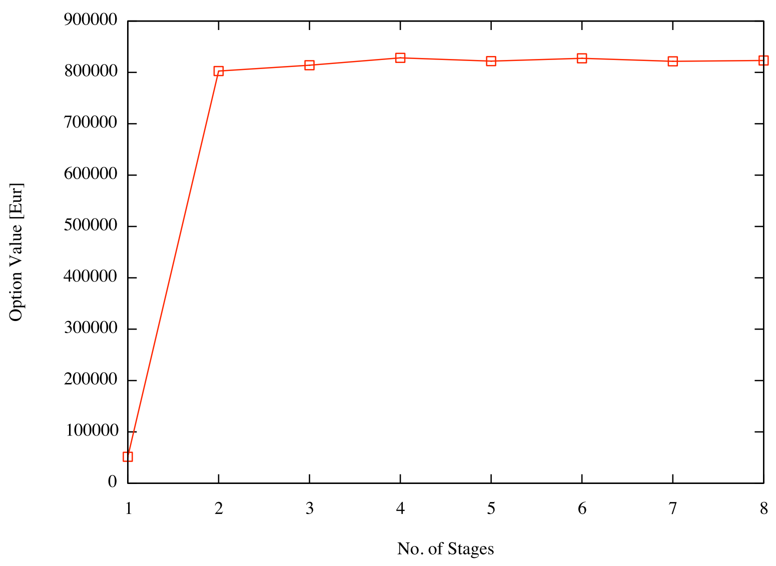

Figure 6.4: Option value as a function of number of stages. Parameters: $d = 1$, $A = 6$, 3000 samples, base load quarterly, monthly and weekly futures used for hedging.

The results are presented on Figure 6.4. We observe that, when using polynomials of degree 1, the option price saturates when we use more than 1 stage. This is a very good result, as adding additional stages to the problem (i.e. increasing the T parameter) is very expensive from the computational point of view. Therefore, we would like to achieve a reasonable accuracy using as few stages as possible. Also, from the practical point of view, the least possible number of aggregated stages is preferred. This is due to the transaction costs, which are not encapsulated by our model. Companies do not want to rebalance their hedging portfolios often due to the costs and administrative constraints.

6.5.3 Results

In order to estimate the accuracy of our method, we perform a numerical convergence analysis. The results can be observed on Figures 6.5 and 6.6.

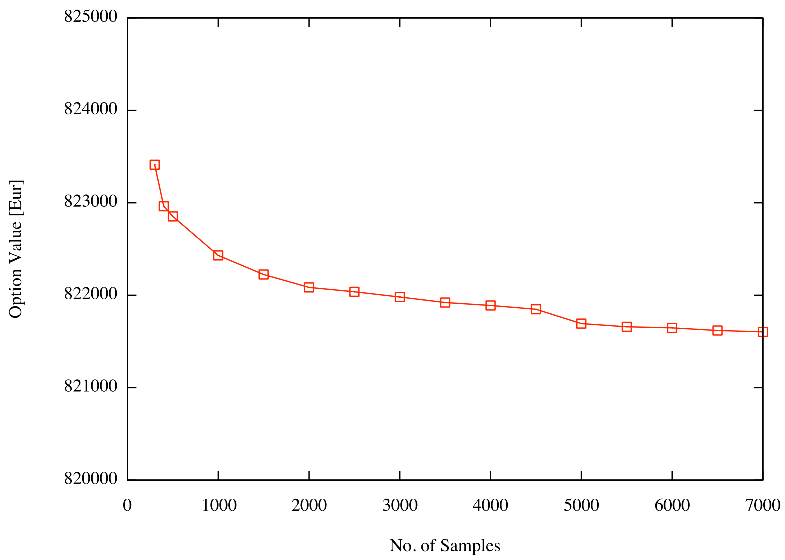

Figure 6.5: Convergence of the option price when increasing the number of scenarios used. Parameters: $d = 1$, $A = 6$, 5 stages, base load quarterly, monthly and weekly futures used for hedging

We observe convergence in both cases. As we increase the computational complexity of the problem (either by increasing number of constraints or number of decision variables), the solution improves, which shows that our approximations are valid and the model is correct.

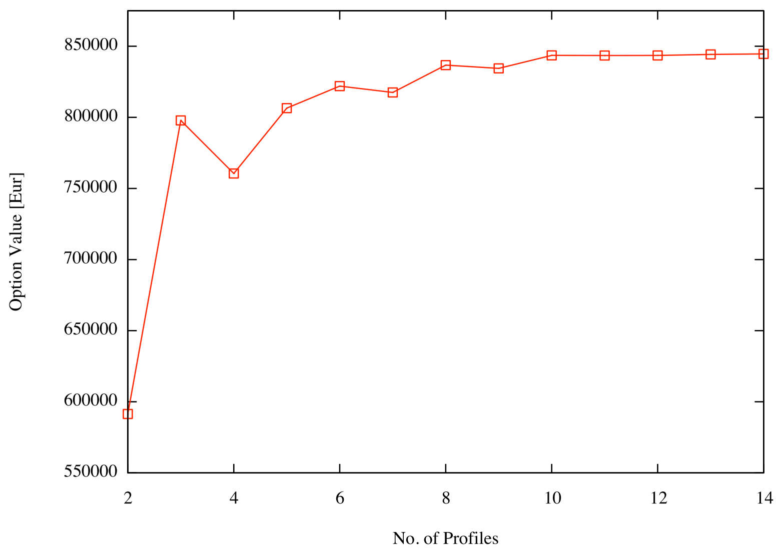

Figure 6.6: Convergence of the option price with increase in number of profiles. Parameters: $d = 1$, 5 stages, 3000 samples, base load quarterly, monthly and weekly futures used for hedging

6.6 Valuation Using Polynomials of Degree 2

6.6.1 Choice of hedging instruments

Similarly to what we did in Section 6.5.1, we need to decide upon the hedging instruments used. However, having the degree of the polynomial decision rules set to 2, the pricing problem becomes much more expensive computationally to solve. We present the solution times for various base load futures in Table 6.4.

Futures Contracts Used	**Option Value**	**Solution Time**
Base Load Quarterly	7.73624E5€	4 minutes
Base Load Monthly	8.16674E5€	11 minutes
Base Load Quarterly & Monthly	8.17863E5€	17 minutes
Base Load Quarterly, Monthly & Weekly	8.28744E5€	2 hours

Table 6.4: Results of numerical experiments with various hedging instruments with polynomials' degree set to 2 ($d = 2$), $A = 6$, 3000 samples and 4 stages

Because of computational reasons, for the numerical experiments carried out using degree of the polynomials set to 2, we based the hedging strategy on monthly futures only (base load contracts only and base and peak load contracts together). This choice is justified by the results in the table, where we observe that the option value computed using both quarterly and monthly futures is very close to that obtained using monthly contracts only.

6.6.2 Number of stages used

When using the polynomials of degree 2, computations involving large number of stages is even more expensive than in the case where we had d set to 1 (Section 6.5.2).

The solution time increases rapidly with the number of stages. It is approximately 600 seconds for four stages, and over 1200 seconds for 5 stages. The problem with 6 stages is not solvable in a reasonable time as it takes more than a couple of hours.

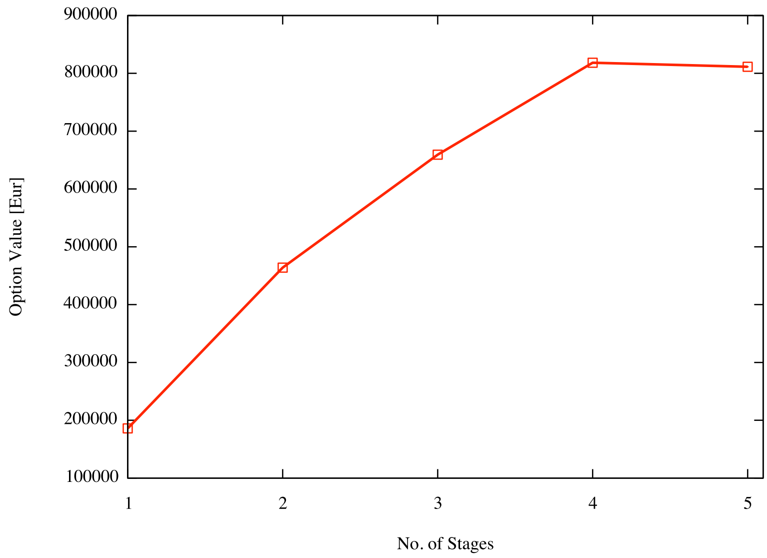

Figure 6.7: Option value as a function of number of stages. Parameters: $d = 2$, $A = 6$, 3000 samples, base load monthly futures used for hedging.

On Figure 6.7, we observe that the option value obtained saturates when we increase the number of stages. However, the convergence is slower than in the case presented in Section 6.5.2, so we have to use at least 4 stages to achieve reasonable accuracy. In fact, this is the only plausible choice, as using 5 stages is impractical from the computational point of view.

6.6.3 Results

As in Section 6.5.3, we perform a numerical convergence analysis for two cases (varying number of samples and profiles used) to validate the results obtained when using the polynomials of degree 2.

The results presented on Figures 6.8 and 6.9 suggest that the option price converges when increasing the size of the problem. However, we were not able to produce results for number of samples higher than 5000 and more than 12 profiles, because of the computational complexity.

As expected, the option price estimated using both base and peak load contracts is always higher (the solution is more accurate).

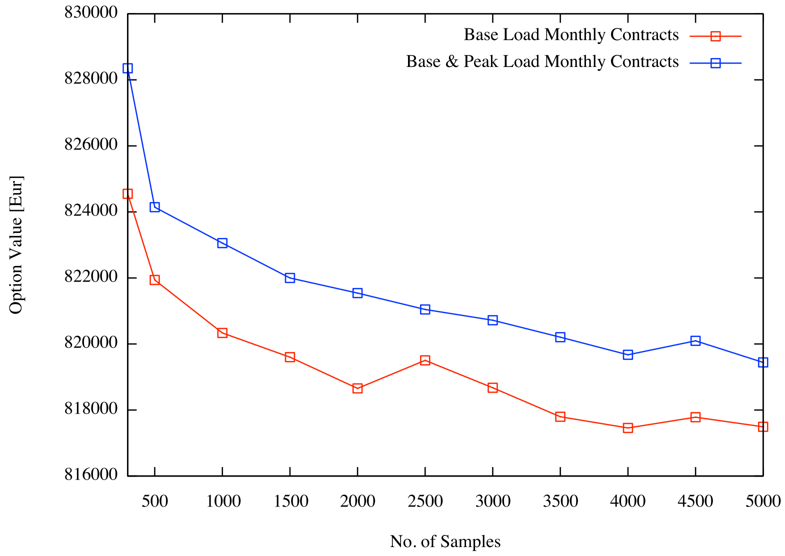

Figure 6.8: Convergence of the option price when increasing the number of scenarios used. Parameters: $d = 2$, $A = 6$.

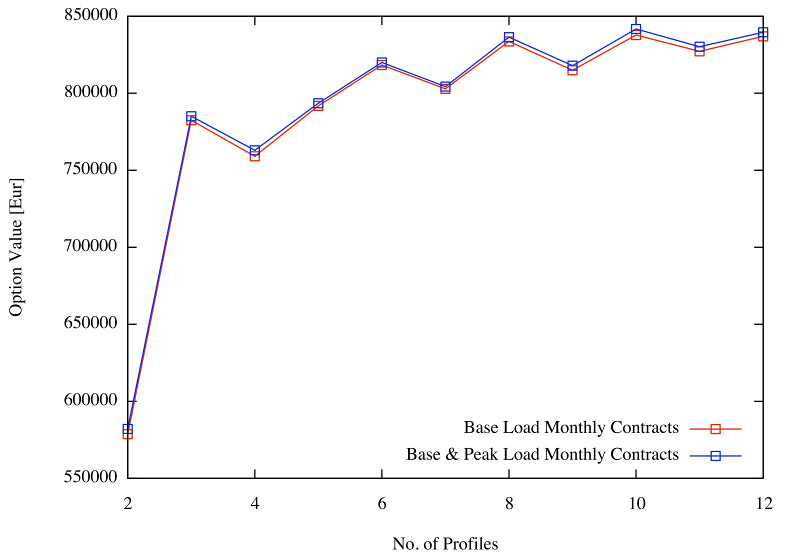

Figure 6.9: Convergence of the option price with increase in number of profiles. Parameters: $d = 2$, 3000 samples.

6.7 Evaluation of the results

The results obtained in our numerical experiments presented in Sections 6.5 and 6.6 suggest that using the polynomial decision rules of degree 1 is preferred, as the solution time is much smaller. In such a set up it is also possible (from computational point of view) to use more types of hedging instruments and more stages. Therefore, the solution can be more accurate.

Graphs presented in Section 6.5 support our assertion that approximations based on 5 stages, 6 exercise profiles, 3000 samples and choice of base load quarterly, monthly and weekly futures contracts are reasonably accurate. The solution time of approximately 13 seconds is acceptable and ensures that our valuation scheme can be used in practice on trading desks. If one needed to decrease the solution time further, we observe from Figure 6.4 that the number of stages can be in fact reduced to just 2 stages without loosing much of the accuracy. Such an action results in decrease of the solution time to approx. 2 seconds, which is a very good result. For example, by using the method presented in [7] one can achieve solution time of approx. 4 seconds.

In Section 6.6 we attempted to use the polynomials of degree 2. Unfortunately, the solution times were highly unsatisfactory.

We managed to solve the problem with up to 8 stages using polynomial decision rules of degree 1 and up to 5 stages with degree equal to 2. This is a significant improvement compared to the static hedging strategy proposed in [12].

We replicated the swing option using futures contracts for delivery of the electrical energy and we experimented with all types of the contracts which are tradeable at European Energy Exchange. We made our valuation scheme realistic by reflecting the real instruments in the model exactly according to their specifications outlined in [6]. We observed that using both base and peak load contracts yielded similar results to using just the base load ones. We expect that if the problem was formulated differently, so that prices at peak hours would behave in a different manner from those at other times, using both types of hedging instruments would give different results.

Chapter 7

Conclusions and Future Work

In this report, we proposed a new scheme for valuation of the electricity swing options, based on the robust optimisation approach. We introduced a few simplifications and approximations into our pricing model to make it computationally possible to solve in a reasonable time. The correctness of the approach based on replicating the option by the hedging portfolio of cash and futures has been validated by comparing it with the binomial lattice model for pricing simple call options.

We presented some numerical results showing how our valuation scheme can be used in practice. The method is efficient and produces meaningful results. It can be used to provide both rough estimates of the option price in a split second or compute the accurate value of the contract, depending on the number of decision variables and constraints used.

Further work in the area could involve formulating a model for finding the seller's price of the option. Then, having both bid and ask prices, the equilibrium price could be estimated. Also, transaction costs were not incorporated into our model. We assumed that we can rebalance our hedging portfolio without being charged with additional costs of purchasing or selling futures contracts. If the scheme was to be used in practice, it would be desirable to take such costs into account.

In the future, one can also try to use more powerful machine for solving the problem instances. Hopefully, if more CPU power was available, solving problem instances with polynomials of degree 2 would become more computationally feasible.

Finally, possibly the solution time of large problems can be reduced by solving them in stages: firstly, generate a small problem instance (with relatively small number of scenarios), solve it, save the optimal basis, and pass it to a larger problem. However, it needs to be investigated whether such method would improve solution times significantly.

References

[1] A. Ben-Tal and A. Nemirovski. Robust optimization - methodology and applications. *Mathematical Programming*, 92(3):453–480, 2002.

[2] M. C. Campi and S. Garatti. The exact feasibility of randomized solutions of uncertain convex programs. *SIAM J. on Optimization*, 19(3):1211–1230, 2008.

[3] Ren Carmona and Nizar Touzi. Optimal multiple stopping and valuation of swing options. *Mathematical Finance*, 18(2):239–268, 2008.

[4] M. Davison and L. Anderson. Approximate recursive valuation of electricity swing options. Technical report, The University of Western Ontario, October 2003.

[5] European Energy Exchange AG, Leipzig, Germany. *EEX Product Information: Power*, 0001d edition, July 2008.

[6] European Energy Exchange AG, Leipzig, Germany. *The products of EEX Group*, 2009.

[7] Gido Haarbrücker and Daniel Kuhn. Valuation of electricity swing options by multistage stochastic programming. *Automatica*, 45(4):889–899, 2009.

[8] A. Ibanez. Valuation by simulation of contingent claims with multiple early exercise opportunities. Computing in Economics and Finance 2002 114, Society for Computational Economics, July 2002.

[9] Patrick Jaillet, Ehud I. Ronn, and Stathis Tompaidis. Valuation of commodity-based swing options. *Manage. Sci.*, 50(7):909–921, 2004.

[10] Francis A Longstaff and Eduardo S Schwartz. Valuing american options by simulation: A simple least-squares approach. *Review of Financial Studies*, 14(1):113–47, 2001.

[11] D. G. Luenberger. *Investment Science*. Oxford University Press, 1998.

[12] Georg C. Pflug and Nikola Broussev. Electricity swing options: Behavioral models and pricing. *European Journal of Operational Research*, 197(3):1041–1050, 2009.

[13] D. Pilipovic. *Energy risk: Valuaing and Managing Energy Derivatives.* McGraw-Hill, second edition, 2007.

[14] M. C. Steinbach and Vollbrecht H.-J. *Efficient Stochastic Programming Techniques for Electricity Swing Options*, chapter 21, pages 485–506. Energy Systems. Springer Berlin Heidelberg, 2009.