IMPERIAL COLLEGE LONDON
DEPARTMENT OF COMPUTING

FINAL YEAR PROJECT

Exploring Canonical Axiomatisations of Representable Cylindric Algebras

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June 23, 2011

(Project directory: /homes/jb1508/report/)

Abstract

We show that for finite $n \geq 3$ the class of representable cylindric algebras RCA_n cannot be axiomatised by canonical first-order formulas. So, although RCA_n is known to be canonical, which means that it is closed under canonical extensions, there is no axiomatisation where all the formulas are preserved by canonical extensions. In fact, we show that every axiomatisation contains an infinite number of non-canonical formulas.

The proof employs algebras derived from random graphs to construct a cylindric algebra that satisfies any number of axioms we want, while its canonical extension only satisfies a bounded number. We achieve this by relating the chromatic number of a graph to the number of RCA_n axioms satisfied by a cylindric algebra constructed from it.

Finally, we outline a strategy to further generalise the proof to extend the result to variations of cylindric algebras, such as diagonal-free algebras.

Acknowledgements

First and foremost, I would like to thank my supervisor Ian Hodkinson for his continuous support throughout the project. He always found the time to discuss the questions raised during the project, even when they sometimes went beyond it. His enthusiasm helped me to stay motivated. I would also like to thank Philippa Gardner for her helpful feedback. I thank Ka Wai Cheng and David Spreen for their comments.

I am immensely grateful to my parents for enabling me to study at Imperial College. Lastly, I especially want to thank Doerte Letzmann for her invaluable support.

Contents

1 Introduction

"I do believe that the calculus of relations deserves much more attention than it receives. For, aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it."

Alfred Tarski [1941, p. 89]

Algebraic logic is a branch of mathematical logic that studies logic with algebraic means. Within this area, cylindric algebras are an attempt to algebraise first-order logic. They are abstract approximations of algebras of α-ary relations for an ordinal α, that satisfy certain axioms laid down by Tarski. They are equipped with a number of operations: Apart from the boolean operations, these include constants called diagonal elements, which are like equality, and unary operators called cylindrifications, which are like existential quantification. For finite α, these algebras have a close connection to first-order logic with α variables.

An important question within this area is which cylindric algebras are isomorphic to genuine algebras of relations, called cylindric set algebras. This subclass is called the α-dimensional representable cylindric algebras RCA_α. Many researchers have improved the understanding of this class in the last 50 years. The relationship to the canonical extension of an algebra, a cylindric algebra built from the ultrafilters of the elements, is of particular interest. In an unpublished proof, Monk showed that if a cylindric algebra is representable, then so is its canonical extension. In this project we show that this is only barely so, by proving that there is no axiomatisation of RCA_n for finite $n \geq 3$

where all but finitely many formulas that hold on a cylindric algebra, also hold on its canonical extension. Moreover, we attempt to extend this result to the diagonal-free version RDf_n of RCA_n. This adds to the body of evidence that RCA_n is rather difficult to characterise.

In the following we will give a brief overview of the field and explain where the project is placed within it. We will then give motivation and explain why the result is useful. Finally, we will give an intuitive idea of the most important notions needed to understand the result and give a high level sketch of the proof.

1.1 Background to the Field

The project is located in the field of algebraic logic, the study of logic with algebraic means. Algebraisations of logics provide alternative semantics that allow the utilisation of pre-existing mathematical theory for the study of the logic. The field was created by Boole, De Morgan, Peirce and Schröder in the nineteenth century. Best known in this area are most likely boolean algebras, which are algebras of unary relations and correspond to propositional logic. Peirce and Schröder also established the theory of binary relations, which was much later revived by Tarski who studied these as relation algebras. Tarski and Jónsson generalised them into boolean algebras with operators. Cylindric algebras are a special case of these, which were developed by Tarski and his students Louise Chin and Frederick Thompson to algebraise first-order logic.

Representable cylindric algebras have been studied extensively. The two main positive results for RCA_α are the proof of Tarski [1955] that shows that it is a variety, that is it can be axiomatised by equations, and an unpublished proof by Monk that shows RCA_α is canonical. The class RCA_0 is just the class of boolean algebras. For $\alpha = 1, 2$ the class RCA_α is very well behaved, in particular both RCA_1 and RCA_2 are finitely axiomatised. The finite set of axioms for RCA_2 is due to Henkin [Henkin et al., 1985]. However, for $\alpha \geq 3$ there are a number of negative results known about it. Monk [1969] showed that there is no finite axiomatisation of RCA_α. This result was strengthened by Andréka [1997], who showed that the number of variables needed for an equational axiomatisation is unbounded. At the same time Venema [1997] showed, using a result from Hodkinson [1997], that there is no axiomatisation containing only Sahlqvist formulas.[1] So this class seems to be rather hard to characterise.

This project strengthens the previous negative results by showing that RCA_n for finite $n \geq 3$ is only barely canonical. More precisely, we prove that every axiomatisation of RCA_n for finite $n \geq 3$ must contain infinitely many non-canonical formulas. We will see in Section 1.3 what this means.

The representable diagonal-free algebras differ from cylindric algebras only in not having

[1] Sahlqvist formulas are well known formulas from modal logic with interesting properties; in particular they are canonical.

the diagonal elements. They have been introduced by Tarski and correspond to first-order logic without equality [Henkin et al., 1985, pp. 183ff]. We outline a strategy to extend the result to these algebras. An alternative algebraisation of first-order logic without equality are polyadic algebras, that were invented by Halmos [1962]. We believe that similar techniques can be used to extend the result to polyadic algebras.

There are some similar results. Representable relation algebras were shown by Hodkinson and Venema [2005] to be only barely canonical. Furthermore, Goldblatt and Hodkinson [2007] extended the result in Hodkinson and Venema [2005] and proved that the McKinsey–Lemmon logic is only barely canonical as well.

1.2 Motivation

This is primarily a theoretical contribution to the field of algebraic logic. We answer an open question from [Hirsch and Hodkinson, 2009, Remark 7.6] and outline a strategy to prove a conjecture from Kurucz [2010]. The result implies that any equational axiomatisation of RCA_n contains infinitely many equations that are non-Sahlqvist, which strengthens Venema [1997]. It also implies that there is no finite axiomatisation, which was shown by Monk [1969]. Our result enhances the understanding of cylindric algebras within algebraic logic and contributes to the study of canonicity within model theory.

In a more general sense, we hope that the result will inspire new research by pointing out the limits of representable cylindric algebras. In the same way Turing's and Church's negative answer to Hilbert's Entscheidungsproblem did not end all endeavours in first-order logic, but led to the search and study of decidable fragments, we hope that our result will motivate the discovery of interesting subvarieties of representable cylindric algebras. And just as the method of reduction was applied in entirely different areas, such as complexity theory, we believe that a very important contribution of this project is the further development of the employed method, which uses random graphs and relates them to algebras.

Finally, beyond the field of algebraic logic, cylindric algebras have been shown to have applications in many areas of computer science. There are connections to databases [Bussche, 2001] and the semantic web [Goczyła et al., 2009]. There is also a close relationship to the modal logics between K^n and $S5^n$ for $n \geq 3$ [Hirsch et al., 2002]. Moreover, there is a strong connection between cylindric algebras and relation algebras. Representable relation algebras have very similar properties to RCA_n (for $n \geq 3$) and have been shown to have applications in the navigation of XML documents [Marx, 2005], interval algebras used in artificial planning [Allen, 1983] and point based versions of these, interval temporal logics [Hodkinson et al., 2008], and the well known branching time temporal logic CTL* [Bauer et al., 2002].

1.3 Contribution

We explain the contribution by giving a high level overview of the proof. First, we give an intuitive idea of the most important notions needed to understand the result:

- **cylindric algebras (CA)** – abstract approximations of algebras of relations that have a close connection to first-order logic;
- **representable cylindric algebras (RCA)** – a subclass of cylindric algebras that correspond to cylindric set algebras, concrete algebraic structures built from sets;
- **axiomatised** – a class of algebras is axiomatised if it is fully defined by a set of formulas;
- **canonical extension** – each algebra $\mathcal{A}$ embeds into a specific (unique) algebra $\mathcal{A}^\sigma$ with some useful properties built from the ultrafilters of $\mathcal{A}$, called the canonical extension (cf. the canonical model in modal logic);
- **canonical class** – a class is called canonical if it is closed under canonical extensions;
- **canonical formula** – a formula is called canonical if whenever it holds on an algebra $\mathcal{A}$, it also holds on its canonical extension $\mathcal{A}^\sigma$ (e.g. Sahlqvist formulas);
- **chromatic number** – the smallest number of colours needed to colour a graph so that any two adjacent nodes have a different colour;
- **random graph** – Erdös famously constructed graphs with arbitrary minimum cycle length and chromatic number using probabilistic methods. Here we use an enhanced version of these graphs by Hodkinson and Venema [2005] that allows us to fix the chromatic number and have a lower bound for the length of odd cycles in the graph.

In the following we explain the result. As mentioned before, we do know that for $n \geq 3$, the class of representable cylindric algebras of n dimensions RCA_n is a variety, so it can be axiomatised (by equations). We also know that no finite amount of first-order formulas is sufficient. However, RCA_n is canonical, so if an algebra satisfies all the axioms of an axiomatisation, then so does its canonical extension. The open question that this project addresses is whether there is an axiomatisation where each single axiom – by itself – holds on the canonical extension of an algebra if it holds on the algebra. Somewhat surprisingly, we show that such an axiomatisation does not exist. In fact, with a few modifications to the argument outlined here, we will show something stronger: that every axiomatisation must contain infinitely many axioms that are not preserved by the canonical extension. We will furthermore describe an approach, that, by making the proof outlined here more general in certain points, extends the result to variations of cylindric algebras, such as diagonal-free algebras.

We now give a simplified high level overview of the proof. We consider cylindric algebras of finite dimension at least 3, so in the following n is a finite number ≥ 3. We show this by demonstrating that the assumption that such an axiomatisation exists leads to a contradiction. In the following we assume (for a contradiction) that there is a set of canonical axioms T that axiomatises RCA_n, so that all of the axioms of T hold on the canonical extension of an algebra whenever they hold on the algebra itself.

Although we study RCA_n in this report, we will mostly deal with cylindric algebras that are not representable. We are interested in algebras that satisfy some, but not all of the axioms. By considering a set of universal axioms $\Sigma = \{\gamma_0, \gamma_1, \dots\}$ where the axioms gradually get stronger, i.e. γ_i implies all the γ_j with $j \leq i$, we obtain a way to 'measure' representability of an algebra by the number of axioms satisfied by the algebra.[2] This allows us to study what happens if an algebra is not representable.

A cylindric algebra fails to be representable, if one of the (universal) formulas does not hold. This happens if there is a number of 'bad' elements, that, when substituted for the bound variables, makes the formula false. Without loss of generality, this means that we can partition the unit of the algebra into 'bad' elements. The source for this 'bad partition' in our proof is graphs: a graph for our purposes has a 'bad partition' if there is a finite colouring of the vertices so that no two adjacent vertices have the same colour. The smallest number of colours needed is called the chromatic number. The main idea is to construct cylindric algebras from graphs so that an algebra is 'more representable' if the graph it is constructed from has a higher chromatic number and vice versa. This lets us control the 'representability' using the chromatic number, which is easier to handle and allows us to use the whole repertoire of graph theoretic theorems. Most of the hard work of the proof lies in establishing this connection.

To prove the connection between the chromatic number of a graph and the 'representability' of the cylindric algebra constructed from it, we generalise a result from Hirsch and Hodkinson [2009] that shows that the chromatic number is infinite if and only if the algebra is representable. We do this by introducing *algebra-graph-systems*. These are 3-sorted structures that basically capture the relationship between a graph, the powerset boolean algebra of the graph, and the algebra constructed from the graph. We then define a theory $\mathcal{U}$ that collects all the first-order definable properties of such structures built from graphs. A feature of such systems is that they allow us to talk about a relativised 'chromatic number' for them. We can now prove that algebra-graph-systems that have infinite chromatic number are exactly the algebra-graph-systems with representable algebra. This proof is done by generalising some of the steps of the proof in Hirsch and Hodkinson [2009] by showing that they also hold for the algebra-graph-system obtained from a general graph; we call this the *generalisation technique*. Using that for any finite n, the statement 'the chromatic number is greater than n' is expressible in first-order logic, we now have that $\mathcal{U}$ and a set of formulas that expresses infinite chromatic number has the same models as $\mathcal{U}$ with the axiomatisation Σ. As illustrated in Figure 1.1, it now follows by first-order compactness that the chromatic number of a graph Γ and the

[2]We can obtain such an axiomatisation by taking the conjunction of any axiom with the previous axioms or by considering a concrete axiomatisation with that property, e.g. the one from Hirsch and Hodkinson [1997].

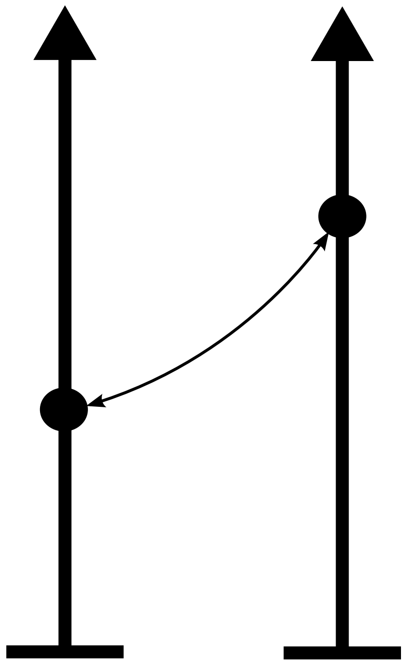

Figure 1.1: The chromatic number of a graph Γ and the 'representability' of the algebra $\mathcal{A}(\Gamma)$ built from it 'drag' each other along.

number of representability axioms satisfied by the algebra $\mathcal{A}(\Gamma)$ from the graph 'drag' each other along, that is, each can be made as large as we want by increasing the other sufficiently.

Using this connection, we will obtain a contradiction by building an algebra that can satisfy an arbitrary number of axioms from Σ, while its canonical extension only satisfies a bounded number. To carry out this construction we use direct and inverse systems of graphs and algebras. We need a sequence of rather eccentric graphs to do this. Erdös [1959] showed the existence of finite graphs of arbitrary chromatic number and minimum cycle length. Defying intuition, the existence of such graphs demonstrated that the chromatic number is a global rather than a local property of a graph. Another important feature of these graphs is that if we consider an inverse system $\Gamma_0, \Gamma_1, \ldots$ of such graphs with fixed (arbitrarily high) chromatic number and increasing minimum cycle length, their inverse limit will have a chromatic number of just two. This is because the inverse limit won't contain a cycle of finite length, so a standard result (2.4.5) tells us it will be two-colourable. We obtain a direct system of algebras, by constructing a cylindric algebra from each of these graphs. The direct limit of this system of algebras will have the same (high) 'chromatic number', in the algebraic sense, as each of the graphs. So this appears to be a good source for a contradiction and we want a connection between the direct limit of the algebras and the algebra from the inverse limit of the graphs.

Luckily, a consequence of our generalisation of a theorem of Goldblatt [1993] gives us exactly the connection we need; it shows that the algebra from the inverse limit of the graphs is isomorphic to the canonical extension of the direct limit of the algebras from the graphs. Recall that we write $\mathcal{A}^\sigma$ for the canonical extension of an algebra $\mathcal{A}$. Figure 1.2 shows the setup for the relationship:

$$\begin{array}{ccccccccc} \Gamma_0 & \leftarrow & \Gamma_1 & \leftarrow & \Gamma_2 & \leftarrow & \dots & \leftarrow & \lim_{\leftarrow} \Gamma_j =: \Gamma \\ \mathcal{A}(\Gamma_0) & \hookrightarrow & \mathcal{A}(\Gamma_1) & \hookrightarrow & \mathcal{A}(\Gamma_2) & \hookrightarrow & \dots & \hookrightarrow & \lim_{\rightarrow} \mathcal{A}(\Gamma_j) =: \mathcal{A} \end{array}$$

$$\Longrightarrow \quad \mathcal{A}(\Gamma) \cong \mathcal{A}^\sigma$$

Figure 1.2: Relationship between algebras built from direct and inverse systems.

Having certain surjective maps between the graphs, we define the algebras and obtain embeddings ‘in the other direction’. The theorem now relates the limits of these inverse and direct systems in the way we need. So we have a way to build an algebra $\mathcal{A}$ of arbitrarily large chromatic number, in the algebraic sense, while its canonical extension A^σ has a chromatic number of just 2. This gives us our contradiction.

In a little more detail, by following the arrows in Figure 1.3 on the next page, we see how we obtain the contradiction: Recall that the inverse limit of an inverse systems of random graphs as above will be two-colourable. Using repeated applications of compactness we get that the sentence that says that the chromatic number is greater than two, θ_2, is implied by one of the axioms γ_s, which is implied by a set of canonical formulas T_s, which is implied by a formula γ_{s+}. Lastly, we can find a k such that the encoding of the chromatic number θ_k implies γ_{s+}. So we set the chromatic number of the graphs to a number greater than k, such as $k+1$. This finally gives us the contradiction: Following the trail of implications with this system of graphs, we get that the chromatic number of the inverse limit is greater than two.

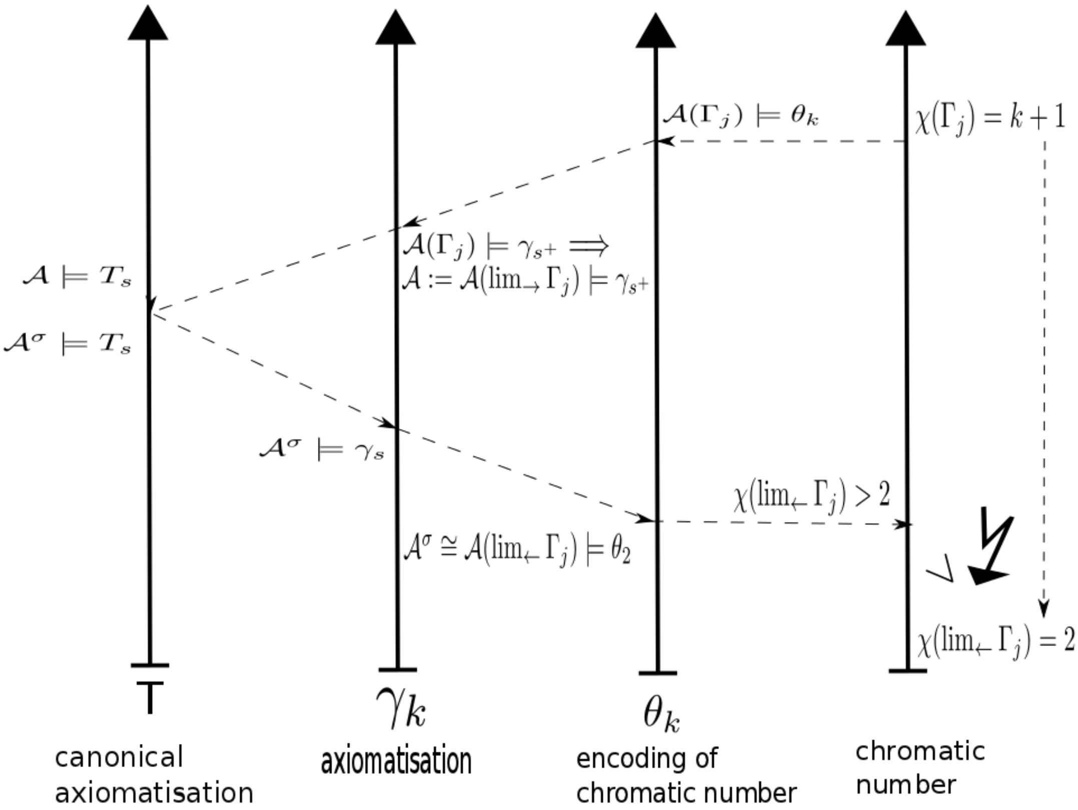

Figure 1.3: Illustration of the last part of the proof.

1.4 Structure of the Report

Chapter 2 The project draws from many areas of mathematics and theoretical computer science. The aim of this chapter is to make the report as self contained as possible by providing the background material needed. The chapter is best used as a reference when reading the proof.

Chapter 3 This chapter contains the proof of the main theorem. It is split into four sections. In *Algebras from Graphs* we show how algebras can be constructed from graphs and introduce the important notion of algebra-graph-system, that allows us to generalise the ideas from Hirsch and Hodkinson [2009]. In *Networks and Patch systems* we introduce further notions that help us study representability and prove some results that we need. In *Chromatic Number & Representability* we prove a relation between the chromatic number of a graph and the number of RCA_n axioms satisfied by the algebra built from it. Finally, in *Direct & Inverse Systems of Algebras and Graphs* we use direct and inverse systems to prove the main result, as sketched above.

Chapter 4 Here we lay out a strategy to generalise the result of Chapter 3 to a wider class of algebras from graphs. We believe that this approach will lead to a generalisation of the result to RDf_n and possibly other variations of cylindric algebras.

Chapter 5 We will discuss the results and some choices made for the proof. Lastly, we will list a number of open questions that this report did not answer.

1.5 Notation

We use the following notational conventions. Throughout the report, n is a fixed finite positive integer and n is at least 3. We identify a non-negative integer m with the set $\{0, 1, \ldots, m-1\}$. If V is a set, we write $[V]^n$ for the set of subsets of size n of V. We write ω for the first infinite ordinal number. We omit the brackets in function applications when we believe it improves readability. With ${}^{\alpha}U$, where α is an ordinal, we denote the set of functions from α to U, so an α-ary relation on U is a subset of ${}^{\alpha}U$. To keep the syntax similar to the finite case, we write x_i for $x(i)$ if $x \in {}^{\alpha}U$ and $i < \alpha$. For definitions we use ':=' to make clear what side is being defined.

2

Background

This chapter provides, except for some very elementary material, all the necessary background needed to understand the statement and proof of the main theorem of this project. Depending on prior knowledge, it may be a good idea to use this chapter as a reference when reading the proof in the next chapter, instead of reading this chapter completely before the next one.

The project draws from many different areas of theoretical computer science and mathematics. We have to assume some knowledge, but the aim is to make this report as self contained as possible. We will state important definitions and results from model theory, algebraic logic, including some universal algebra and graph theory. Some of the results that are important or more unusual will be proved. Moreover, we will state and prove a version of Ramsey's theorem.

2.1 Model Theory

We assume that the reader is familiar with first-order logic. We will first recall some important definitions for later reference, then explain the notions of *elementary*, *variety* and *axiomatisation*, which will be used in the later part. Lastly, we will state the compactness and completeness theorems of first-order logic and prove a consequence that will be very useful for this project.

2.1.1 Syntax and Semantics

We will roughly follow Hodges [1997] in introducing the syntactical notions.

Definition 2.1.1. A *signature* L is a collection of relation symbols, function symbols and constant symbols, each associated with a finite arity. The size of the signature $|L|$ is the smallest infinite cardinal number that is greater than or equal to the number of symbols in L.

Remark. In the following we assume a countably infinite set of variables.

Definition 2.1.2. An *L-term* is either a variable, a constant from L, or, if f is a function symbol with arity n and $t_1, \dots, t_n$ are terms, then $f(t_1, \dots, t_n)$ is a term.

Definition 2.1.3. An *L-formula* is defined as follows.

- If P is an n-ary relation symbol from L and $t, u, t_1, \dots, t_n$ are terms, then $P(t_1, \dots, t_n)$ and $t = u$ are L-formulas. They are said to be *atomic*.
- If φ, ψ are L-formulas and x a variable, then $\neg\varphi$, $\varphi \wedge \psi$ and $\exists x\varphi$ are L-formulas. Furthermore, we define the following abbreviations:
  - $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$;
  - $\varphi \rightarrow \psi := \neg\varphi \vee \psi$;
  - $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$;
  - $\forall x\varphi := \neg\exists x\neg\varphi$;
  - $\top := \forall x(x = x)$
  - $\bot := \neg\top$.

Definition 2.1.4. The *subformulas* of a formula φ are defined as follows.

- If φ is atomic, then φ is the only subformula.
- If φ is of the form $\neg\psi$ or $\exists x\psi$, then φ and the subformulas of ψ are subformulas.
- If φ is of the form $\psi_1 \wedge \psi_2$, then φ and the subformulas of ψ_1 and ψ_2 are subformulas.

Definition 2.1.5. An occurrence of a variable x is *bound* if it is in a subformula of the form $\exists x\varphi$. Otherwise the occurrence is *free*. The *free variables* of a formula are the variables with free occurrences. A formula with no free variables is called a *sentence*. A *theory* is a set of sentences.

Remark. We will assume a formal proof system, such as Hilbert systems or natural deduction, in the following. We do not make this formal here, as it will not be needed for the project.

Definition 2.1.6. If T is a theory and φ a formula, we write $T \vdash \varphi$ if there is a formal proof that given some $\psi_1, \dots, \psi_n \in T$, shows φ. If Σ is a theory, we say $T \vdash \Sigma$ if $T \vdash \sigma$ for all $\sigma \in \Sigma$.

Definition 2.1.7. A theory T is called *consistent* if and only if $T \nvdash \bot$.

We now introduce the central semantic notions: structure and model.

Definition 2.1.8. If L is a signature, an *L-structure* is a tuple $M = (D, I)$, where D is a non-empty set called *domain* and I is an *interpretation* function defined for each symbol in L.

- The interpretation of an n-ary relation symbol R in L is an n-ary relation R^M on D.
- The interpretation of an n-ary function symbol f is a function f^M from D^n to D.
- The interpretation of a constant c is an element c^M in D.

An L-structure is called an *algebra* if L does not contain any relation symbols.

Remark. To keep notation concise we will often write M for the model and for its domain D. If the signature is clear from the context, we will sometimes drop the preceding $L-$.

Definition 2.1.9. Let M be an L-structure and V a set of variables. Then a map $h : V \to M$ is called an *assignment* of the variables in V.

Definition 2.1.10. Let M be a structure, h an assignment and φ be a formula. Extend h, so that it also sends constants $c \in L$ to their interpretation in D. Moreover, if $t_1, \ldots, t_n$ are terms or variables, we define $hf(t_1, \ldots, t_n) = f(ht_1, \ldots, ht_n)$ for function symbols f.
We say $M, h \models \varphi$ (φ is true in M under h):

- If φ is of the form $t_1 = t_2$, then $M, h \models \varphi$ if and only if $ht_1 = ht_2$.
- If φ is of the form $P(t_1, \ldots, t_n)$ for a relation symbol P, then $M, h \models \varphi$ if and only if $(ht_1, \ldots, ht_n)$ is in the interpretation of P.
- If φ is of the form $\neg\psi$, then $M \models \varphi$ if and only if $M, h \not\models \psi$.
- If φ is of the form $\psi_1 \wedge \psi_2$, then $M, h \models \varphi$ if and only if $M, h \models \psi_1$ and $M, h \models \psi_2$.
- If φ is of the form $\exists x \psi$, then $M, h \models \varphi$ if and only if there is some assignment h_x such that $h_x \restriction (V \setminus \{x\}) = h \restriction (V \setminus \{x\})$ and $M, h_x \models \psi$.

Definition 2.1.11. Let T be a theory and M a structure. We say $M \models T$ (M is a *model* of T), if $M \models \varphi$ for all $\varphi \in T$.

Definition 2.1.12. An *elementary* class C is a class of structures such that $M \models T$ if and only if $M \in C$ for some theory T. In this case we say T axiomatises C. In the special case where C is a class of algebras and there is a T that only consists of equations, we call C a *variety*.

Definition 2.1.13. Let L be a signature and M_1, M_2 be L-structures. A *homomorphism* is a function $g : M_1 \to M_2$ with the following properties:

1. If $c \in L$ is a constant, then $g(c^{M_1}) = c^{M_2}$.
2. If $R \in L$ is an n-ary relation symbol and $(a_0, \ldots, a_{n-1}) \in R^{M_1}$ for $a_0, \ldots, a_{n-1} \in M_1$, then $(ga_0, \ldots, ga_{n-1}) \in R^{M_2}$.
3. If $f \in L$ is an n-ary function symbol and $a_0, \ldots, a_{n-1} \in M_1$, then $g(f(a_0, \ldots a_{n-1})) = f(ga_0, \ldots, ga_{n-1})$.

If there is a homomorphism $h : M_2 \to M_1$ such that $h \circ g = id_{M_1}$ and $g \circ h = id_{M_2}$ we call g an *isomorphism.*

The following property will help us simplify some of the proofs later.

Definition 2.1.14. Let $\mathcal{A}$ be an algebra. Then $\mathcal{A}$ is simple if $|\mathcal{A}| > 1$ and for any algebra $\mathcal{A}'$ of the same signature, any homomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ is either trivial or injective.

2.1.2 Completeness and Compactness

In this section we will present some essential results that will be useful later, in particular first-order completeness, compactness and the Downward Löwenheim-Skolem-Tarski Theorem (see e.g. Chang and Keisler [1990] for proofs). Moreover, we will prove a consequence of compactness that is crucial for the main proof.

Theorem 2.1.15 (Completeness Theorem)**.** *Let Σ be a set of sentences of L. Then Σ is consistent if and only if Σ has a model.*

Theorem 2.1.16 (Downward Löwenheim-Skolem-Tarski Theorem)**.** *Every consistent L-theory T has a model of size at most $|L|$.*

Theorem 2.1.17 (Compactness Theorem)**.** *A set of sentences Σ has a model if and only if every finite subset of Σ has a model.*

The following corollary will be used extensively in the proof:

Corollary 2.1.18. *Let T_1, T_2 be theories so that every model of T_2 is a model of T_1. If $S_1 \subseteq T_1$ is finite, then there is a finite subset $S_2 \subseteq T_2$ such that $S_2 \vdash S_1$.*

Proof. Suppose for a contradiction the statement does not hold. Then, for any finite subset U of T_2, we have that $U \cup \{\bigvee_{\varphi \in S_1} \neg\varphi\}$ is consistent and hence by the completeness theorem has a model. By compactness it follows that $T_2 \cup \{\bigvee_{\varphi \in S_1} \neg\varphi\}$ is satisfiable, which means that $T_1 \cup \{\bigvee_{\varphi \in S_1} \neg\varphi\}$ is satisfiable as well. But this is impossible since $S_1 \subseteq T_1$. □

2.2 Ramsey's Theorem

The following theorem was originally stated and proved by Ramsey [1930], but we give a variation of the original statement with our own proof here.

Theorem 2.2.1 (Ramsey's Theorem). *If S is a finite set, $n < \omega$ and $f : [\mathbb{N}]^n \to S$ a function, then there is an infinite subset $M \subseteq \mathbb{N}$ such that $f \restriction [M]^n$ is constant.*

Proof. This is a proof by induction over n. If $n = 1$, this is just the pigeonhole principle.

So let $k \geq 1$ and assume the statement holds for $n = k$. Let $f : [\mathbb{N}]^{k+1} \to S$ be any function. First define $x_0 = 0$ and $f_0(\{r_1, \ldots, r_k\}) = f(\{x_0, r_1, \ldots, r_k\})$ for distinct $x_0 < r_1, \ldots, r_k \in \mathbb{N}$. Then we can apply the induction hypothesis to f_0 and get an infinite subset $M_0 \subseteq \mathbb{N}$ so that f_0 is constant on $[M_0]^k$.

We now choose $x_1 = \min M_0$. Note that $x_1 > x_0$, since f_0 was defined for distinct non-zero numbers. We can now define $f_1(\{r_1, \ldots, r_k\}) = (f \restriction M_0)(\{x_1, r_1, \ldots, r_k\})$ for distinct $x_0 < r_1, \ldots, r_k \in M_0$ and continue in the same way as before, obtaining a sequence of infinite sets

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$$

and the increasing sequence of their minima

$$x_0 < x_1 < x_2 < \ldots,$$

where $x_{i+1} \in M_i$ for $i < \omega$. We also know that f_i is constant on $[M_i]^k$ and thus constant on $[M_j]^k$ for $i \leq j < \omega$. By the pigeonhole principle there will be an infinite subsequence $f_{i_1}, f_{i_2}, \ldots$ so that all the f_{i_j} $(j < \omega)$ map to the same constant value in S. So if we let $M = \{x_{i_1}, x_{i_2}, \ldots\}$, then $f \restriction [M]^{k+1}$ is constant. □

2.3 Algebraic Logic

Algebraic logic is the study of logic with algebraic means and was invented in the nineteenth century by Boole, De Morgan, Peirce and Schröder. Peirce and Schröder also began to develop the theory of binary relations, which was much later expanded by Tarski who studied these as relation algebras. Tarski and Jónsson generalised them into boolean algebras with operators. Cylindric algebras are a special case of these, which were developed by Tarski and his students Louise Chin and Frederick Thompson to algebraise first-order logic [Andréka et al., 1991].

We will develop the theory of cylindric algebras – which are central to this project – beginning by looking at boolean algebras, then studying the general case of boolean algebras with operators. Finally, we will introduce cylindric algebras and their diagonal-free variation.

2.3.1 Boolean Algebras

Boolean algebras are the algebraic counterparts of propositional logic. We will see that they are isomorphic to algebras of unary relations. Here we give model theoretic definitions similar to Hirsch and Hodkinson [2002].

Definition 2.3.1. *We denote by L_{BA} the functional signature with constants $0, 1$, a unary function symbol '$-$' and a binary function symbol '$+$'.*

Remark. We will usually use the same symbol to denote the universe of the structure and the structure itself. The following abbreviations will be convenient:

- $a \leq b$ means $a + b = b$,
- $a < b$ means $a \leq b \wedge \neg(a = b)$ and
- $a \cdot b$ is short for $= -(-a + -b)$.

Definition 2.3.2. An L_{BA} structure $B = (B, 0, 1, +, -)$ is a *boolean algebra* if it satisfies the following for all $a, b, c \in B$:

(B1) $(a + b) + c = a + (b + c)$

(B2) $a + b = b + a$

(B3) $0 + a = a$

(B4) $a + a = a$

(B5) $a + (-a) = 1$

(B6) $-(-a) = a$

(B7) $-1 = 0$

(B8) $a \cdot (b + c) = a \cdot b + a \cdot c$

An important notion is that of an atom.

Definition 2.3.3. Let B be a boolean algebra. An element $0 \neq b \in B$ is called an *atom*, if there is no non-zero element beneath it, i.e. if $a < b \implies a = 0$. If for every element $0 \neq a \in B$ there is an atom b such that $b \leq a$, the algebra B is called *atomic*.

The following notions of filter and ultrafilter are essential for this project.

Definition 2.3.4. Let B be a boolean algebra. A *filter* $F \subseteq B$ is a non-empty subset of the domain such that for any $s, t \in B$:

1. If $t \geq s$ and $s \in F$, then $t \in F$.
2. If $s, t \in F$, then $s \cdot t \in F$.

A filter is called *principal* if it is of the form $\{b \in B \mid b \geq a\}$ for some $a \in B$. An *ultrafilter* U is a filter that is proper, i.e. $U \neq B$, and not strictly contained in any other proper filter. We denote the set of all ultrafilters of B with B_+.

The following property of ultrafilters is very useful.

Definition 2.3.5. Let B be a boolean algebra. We say that a subset $S \subseteq B$ has the *finite intersection property* if for any $s_1, s_2, \ldots, s_n \in S$ we have $s_1 \cdot s_2 \cdots s_n \neq 0$.

Theorem 2.3.6 (Boolean prime ideal theorem)**.** *Let B be a boolean algebra and $S \subseteq B$ a subset with the finite intersection property. Then S is contained in an ultrafilter of B.*

Remark. The Boolean prime ideal theorem cannot be proved directly from the axioms of ZF set theory and is in fact strictly weaker than the axiom of choice.

We will need the following equivalent characterisation of an ultrafilter.

Lemma 2.3.7. *Let B be a boolean algebra and μ a filter of B. Then μ is an ultrafilter of B if and only if for any $b \in B$, either $b \in \mu$ or $-b \in \mu$.*

Proof. Let $\mu \subseteq B$ be an ultrafilter of B and let $b \in B$ such that $b \notin \mu$. Then there is no $a \in \mu$ such that $a \leq b$, because otherwise $b \in \mu$. But this means that for every $a \in \mu$, we have $a \cdot -b \neq 0$, since

$$a \cdot -b = 0 \implies a = a \cdot 1 = a \cdot (b + -b) = a \cdot b \implies a \leq b.$$

So $\mu \cup \{-b\}$ has the finite intersection property and is thus by the boolean prime ideal theorem (2.3.6) contained in an ultrafilter ν. But $\nu \supseteq \mu$, so $-b \in \nu = \mu$.

For the converse, let ν be a filter that contains either b or $-b$ for every $b \in B$. We just have to show that ν is maximal. But this is clearly true: Suppose we add a single element that is not already in the filter, say $a \in B \setminus \nu$ and add it to get ν'. Then ν already contained $-a$, and hence we have $a, -a \in \nu'$. But $a \cdot -a = 0$, and hence $\nu' = B$. So ν' is not a proper filter. □

Lemma 2.3.8. *An ultrafilter $\mu \subseteq B$ is principal if and only if it contains an atom.*

Proof. It is easy to see that if μ contains an atom a, we have $\mu = \{b \in B \mid b \geq a\}$, which is principal.

Conversely, assume μ is principal, i.e. $\mu = \{b \in B \mid b \geq c\}$ for some $c \in B$. Suppose for a contradiction that c is not an atom. Then there is a non-zero element $d \leq c$ that is not contained in μ. So by Lemma 2.3.7 we have that $-d \in \mu$. But by the definition of μ this means $c \leq -d$, and hence $d \leq -d$. So $1 = d + -d = -d$, and thus $d = 0$. This is a contradiction to the assumption that d is non-zero. So μ must contain an atom. □

Lemma 2.3.9. *Let B be a boolean algebra and $\mu \subseteq B$ an ultrafilter. If*

$$b_1 + \cdots + b_n \in \mu \qquad b_i \in B,$$

then μ contains at least one of the b_i.

Proof. If μ contains none of the b_i, it contains all the complements and hence their product, i.e. $-b_1 \cdot \cdots \cdot -b_n \in \mu$. But this is the complement of $b_1 + \cdots + b_n$. □

Fields of sets are a more concrete way to think about boolean algebras, and in fact every boolean algebra is isomorphic to one of these.

Definition 2.3.10. Let X be any set. A *field of sets* over the base X is an L_{BA} algebra $F = (F, \emptyset, X, \cup, \setminus)$, where $\emptyset \neq F \subseteq \wp(X)$ is closed under $\cup$ and $X \setminus \cdot$.

Remark. Note that this is a boolean algebra, with $+$ corresponding to $\cup$, $-$ to $\setminus$, 1 to X and 0 to $\emptyset$.

Definition 2.3.11. A boolean algebra is said to be *representable* if it is isomorphic to a field of sets. The isomorphism is then called a *representation*.

Theorem 2.3.12 (Stone's representation theorem)**.** *Every boolean algebra is representable.*

Proof. Let B be a boolean algebra. We need to show that B is isomorphic to a field of sets. Consider the field of sets

$$F = (\wp(B_+), \emptyset, B_+, \cup, \setminus)$$

and the map

$$h : B \to F, b \mapsto \{\mu \in B_+ \mid b \in \mu\}.$$

We will show that B is isomorphic to a subalgebra of F. So we need to check that h is an injective homomorphism. To check that it is a homomorphism, we need h to preserve $0, 1, +$ and $-$. We have for $a, b \in B$:

$$\begin{aligned} h(0) &= \{\mu \in B_+ \mid 0 \in \mu\} = \emptyset, \\ h(1) &= \{\mu \in B_+ \mid 1 \in \mu\} = B_+, \\ h(a+b) &= \{\mu \in B_+ \mid a + b \in \mu\} \\ &= \{\mu \in B_+ \mid a \in \mu\} \cup \{\mu \in B_+ \mid b \in \mu\} \qquad (\star) \\ &= h(a) + h(b). \end{aligned}$$

The line marked with $(\star)$ follows from Lemma 2.3.9. We also have

$$h(-a) = \{\mu \in B_+ \mid -a \in \mu\} = B_+ \setminus \{\mu \in B_+ \mid a \in \mu\} = -h(a).$$

This follows from Lemma 2.3.7.

It is left to show that h is injective. By Theorem 2.3.6, every non-zero element is contained in an ultrafilter. So we certainly have $\ker h = 0$:

$$h(a) = 0 \implies \{\mu \in B_+ \mid a \in \mu\} = \emptyset \implies a = 0.$$

Now suppose $h(a) = h(b)$ for some $a, b \in B$. Then, using that h is a homomorphism we have

$$h((a \cdot -b) + (b \cdot -a)) = h(a) \cdot -h(b) + h(b) \cdot -h(a) = 0.$$

So $(a \cdot -b) + (b \cdot -a) = 0$. It follows

$$\left.\begin{array}{l} a \cdot -b = 0 \implies a = a \cdot 1 = a \cdot (b + -b) = a \cdot b \\ b \cdot -a = 0 \implies b = b \cdot 1 = b \cdot (a + -a) = a \cdot b \end{array}\right\} \implies a = b.$$ □

From the proof of Theorem 2.3.12 we can extract an important concept.

Definition 2.3.13. Let B be a boolean algebra. Then we call $B^\sigma = (\wp(B_+), \emptyset, B_+, \cup, \setminus)$ the *canonical extension* of B.

Remark. It is easy to see that every finite boolean algebra is isomorphic to its canonical extension.

Example 2.3.14. Let B be a boolean algebra built from three atoms a, b, c. It contains the eight elements $0, 1, a, b, c, -a, -b, -c$. Then it is easy to see that we must have $a + b = -c$, $a + c = -b$ and $b + c = -a$. The ultrafilters are the principal ultrafilters generated by a, b, c:

$$\begin{aligned} \mu_a &= \{a, -c, -b, 1\}, \\ \mu_b &= \{b - a, -c, 1\}, \\ \mu_c &= \{c, -a, -b, 1\}. \end{aligned}$$

So we get the following representation of B:

$$\begin{aligned} 0 &\overset{h}{\mapsto} \emptyset, \\ a &\overset{h}{\mapsto} \{\mu_a\}, \\ b &\overset{h}{\mapsto} \{\mu_b\}, \\ c &\overset{h}{\mapsto} \{\mu_c\}, \\ -a &\overset{h}{\mapsto} \{\mu_b, \mu_c\}, \\ -b &\overset{h}{\mapsto} \{\mu_a, \mu_c\}, \\ -c &\overset{h}{\mapsto} \{\mu_a, \mu_b\}, \\ 1 &\overset{h}{\mapsto} \{\mu_a, \mu_b, \mu_c\}. \end{aligned}$$

The representation is illustrated in Figure 2.1.

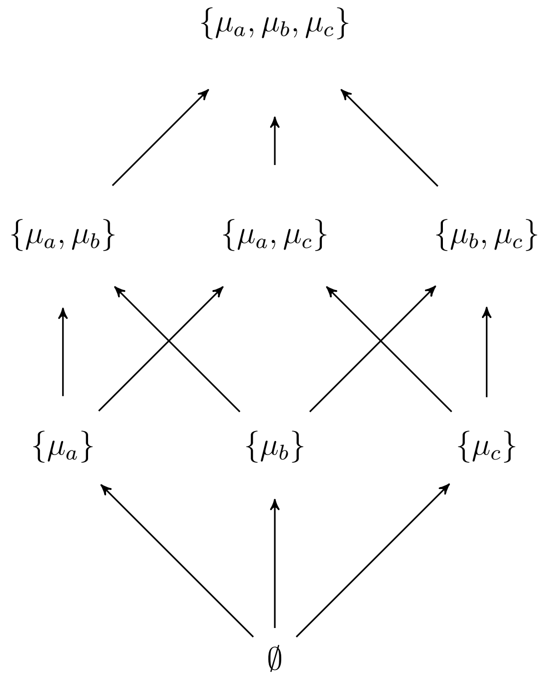

Figure 2.1: The canonical extension of B with the inclusion relation.

2.3.2 Boolean Algebras with Operators

The concept of a boolean algebra can be generalised by adding additional operators. By restricting these operators in certain ways, we can prove many useful things about them [Blackburn et al., 2001].

Definition 2.3.15. Let $B = (B, 0, 1, +, -)$ be a boolean algebra and $n < \omega$. A function $\Omega : B^n \to B$ is called an n-ary *operator* on B if it satisfies the following for any $b_0, \ldots, b_{n-1} \in B$:

1. If $b_i = 0$ for one of the $i < n$, then $\Omega(b_0, \ldots, b_{n-1}) = 0$.
2. For any b, b' and $i < n$ we have

$$\begin{aligned}\Omega(b_0, \ldots, b_{i-1}, (b + b'), b_{i+1}, \ldots b_{n-1}) = &\,\Omega(b_0, \ldots, b_{i-1}, b, b_{i+1}, \ldots b_{n-1}) \\ &+ \Omega(b_0, \ldots, b_{i-1}, b', b_{i+1}, \ldots b_{n-1})\end{aligned}$$

Definition 2.3.16. Let B be a boolean algebra, and O a set of operators. Then we call $(B, 0, 1, +, -, \Omega : \Omega \in O)$ a *boolean algebra with operators* (BAO).

Just as in the BA case, we can work with atoms in BAOs.

Definition 2.3.17. Let L be the signature of a BAO, containing the signature of boolean algebras L_{BA} and function symbols Ω. Define L^a to be the relational signature that contains for each n-ary function symbol $\Omega \in L \setminus L_{BA}$ an $(n+1)$-ary relation symbol R_Ω. A structure of L^a is called *atom structure*.

Definition 2.3.18. Let B be an atomic BAO of signature L. Then L contains the signature of boolean algebras L_{BA} and function symbols Ω. The *atom structure of* B, denoted $At\,B$, is the atom structure that has the atoms of B as a domain and the relations defined by:

$$\text{At}B \models R_\Omega(a_0, \dots, a_{n-1}, b) \iff B \models b \leq \Omega(a_0, \dots, a_{n-1}).$$

for each n, n-ary operator Ω and atoms $a_0, \dots, a_{n-1} \leq b$.

Interestingly, after reducing the algebra to its atom structure, it is possible to gain most of it back.

Definition 2.3.19. Let L be a BAO signature and L^a the corresponding relational signature (as in Definition 2.3.18). Let A be any L^a structure. The *complex algebra* of A is defined to be

$$\mathfrak{Cm}A = (\wp(A), \emptyset, A, \cup, \setminus, \Omega^{\mathfrak{Cm}A} : \Omega \in O)$$

where $\Omega^{\mathfrak{Cm}A}$ is defined in the following way: If $s_0, \dots, s_{n-1} \subseteq A$ and $\Omega \in L$ is an n-ary function, we have

$$\Omega^{\mathfrak{Cm}A}(s_0, \dots, s_{n-1}) = \{a \in A \mid A \models R_\Omega(a_0, \dots, a_{n-1}, a) \text{ for some } a_i \in s_i\}.$$

Just as in the BA case, we can define the canonical extension. We use the ultrafilters to get an atom structure from the BAO.

Definition 2.3.20. Let L be a functional signature containing L_{BA} and B a BAO of L. We define the *ultrafilter structure* B_+ to be the L^a structure which has the ultrafilters of B as domain and, for any n-ary function symbol $\Omega \in L$ an $(n+1)$-ary relation symbol R_Ω such that for any $\mu_0, \dots, \mu_{n-1}, \nu \in B_+$

$$B_+ \models R_\Omega(\mu_0, \dots, \mu_{n-1}, \nu) \iff \Omega(\mu_0, \dots, \mu_{n-1}) \subseteq \nu.$$

Definition 2.3.21. Let B be a BAO. The *canonical extension* B^σ of B is defined to be $\mathfrak{Cm}\, B_+$. A class C of BAOs is said to be *canonical* if it is closed under canonical extensions, that is, $B \in C \implies B^\sigma \in C$. A formula is called *canonical* if its truth value is preserved by canonical extensions, i.e. $A \models \varphi \implies A^\sigma \models \varphi$.

Remark. Stone's representation theorem (2.3.12) can be extended to show that we can embed a BAO B into its canonical extension B^σ.

Direct & Inverse Systems of BAOs

We introduce the notions of direct and inverse system from universal algebra to prove an important relationship between direct limits of algebras and inverse limits of their atom structures, adapting [Grätzer, 2008, pp. 128ff.]. We do not need the definitions in their whole generality, so we simplify them for their application here.

Definition 2.3.22. A *direct system of algebras* S is defined to be a triplet of the following objects:

1. A directed, partially ordered set $(I, \leq)$ called the *carrier* of S; that is for all $i, j \in I$ there is k such that $i \leq k$ and $j \leq k$.
2. An algebra A_i for each $i \in I$.
3. A homomorphism $\varphi_{ij} : A_i \to A_j$ for each $i \leq j$, where φ_{ii} is the identity map for each $i \in I$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for all $i \leq j \leq k$.

Definition 2.3.23. Let S be a direct system of algebras with carrier I. Let $\equiv$ be the equivalence relation over the disjoint union $\bigsqcup_{i \in I} A_i$ such that $x \equiv y$ for $x \in A_i, y \in A_j, i, j \in I$ if and only if there is $k \in I$ such that $i, j \leq k$ and $\varphi_{ik}x = \varphi_{jk}y$.

If f is an r-ary function defined on the algebras and $[x_0], \dots, [x_{r-1}]$ are elements of the direct limit ($[x_i]$ denoting the equivalence class of x_i), we define $f([x_0], \dots, [x_{r-1}]) := [f(x'_0, \dots, x'_{r-1})]$, where the x'_i are all elements of the same A_j for some $j < \omega$ and $x_i \equiv x'_i$ for all $i < r$. This is well defined and gives an algebraic structure on $S/\equiv$.

This algebra $S/\equiv$ is called the *direct limit* of S, denoted $\varinjlim S$.

Remark (1). In the particular case where $I = \omega$, $A_i \subseteq A_j$ for $i \leq j$ and the φ_{ij} are the natural embeddings, the direct limit is essentially the union of the algebras: $\varinjlim S = \bigcup_{i<\omega} A_i$.

Remark (2). Note that the natural embedding $A_i \to \varinjlim S$ is a homomorphism.

Remark (3). We will sometimes write $\varinjlim A_i$ if it is clear which direct system we are talking about.

The dual notion of a direct limit is that of an inverse limit.

Definition 2.3.24. An *inverse system of atom structures* S is defined to be a triplet of the following objects:

1. A directed, partially ordered set $(I, \leq)$; that is for all $i, j \in I$ there is k such that $i \leq k$ and $j \leq k$.
2. An atom structure B_i for each $i \in I$.
3. A homomorphism $\varphi_{ji} : B_j \to B_i$ for each $i \leq j$, where φ_{ii} is the identity map for each $i \in I$ and $\varphi_{ji} \circ \varphi_{kj} = \varphi_{ki}$ for all $i \leq j \leq k$.

Definition 2.3.25. Let S be an inverse system of atom structures with carrier I. Then the *inverse limit* consists of those elements of the product $\prod(B_i \mid i \in I)$ for which we have a connection between the entries via the homomorphisms, that is

$$\varprojlim S = \{x \in \prod_{i \in I} B_i \mid x_i = \varphi_{ji} x_j \text{ for all } i \leq j \text{ in } I\}.$$

The relational structure on $\varprojlim S$ is defined as follows. If R is a relation defined on the atom structures and $x_1 = (x_{1j} : j \in I), x_2 = (x_{2j} : j \in I), \ldots, x_r = (x_{rj} : j \in I)$ are elements of the inverse limit, we define $R(x_1, \ldots, x_r)$ iff $R(x_{ij}, \ldots, x_{ij})$ for all $i = 1, \ldots, r$ and $j \in I$.

Remark. We will sometimes write $\varprojlim A_i$ if it is clear which inverse system we are talking about.

2.3.3 Cylindric Algebras

Cylindric algebras are particular instances of BAOs. They were introduced by Tarski together with his students Louise Chin and Frederick Thompson to algebraise first-order logic. The definitive book on it was published by Henkin et al. [1971]. We will mainly follow this, but also use an introduction by Maddux in Andréka et al. [1991], a survey paper by Monk [2000] and the chapter on cylindric algebras in Hirsch and Hodkinson [2002].

A cylindric algebra is a BAO with two kinds of operators, the cylindrifications c_i that behave like $\exists$ and the diagonals d_{ij} that are like equality.

Definition 2.3.26. A *cylindric algebra* C of dimension α, where α is an ordinal number, is a BAO

$$C = (C, 0, 1, +, -, c_i, d_{ij})_{i,j<\alpha}$$

where d_{ij} are nullary operators and c_i are unary operators, such that the following are satisfied for any $a, b \in C$ and $i, j, k < \alpha$:

(C0) $(C, 0, 1, +, -)$ is a boolean algebra;

(C1) $c_i 0 = 0$;

(C2) $x \leq c_i x$;

(C3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$;

(C4) $c_i c_j x = c_j c_i x$;

(C5) $d_{ii} = 1$;

(C6) if $k \neq i, j$, then $d_{ij} = c_k(d_{ik} \cdot d_{kj})$;

(C7) if $i \neq j$, then $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$.

The class of all cylindric algebras of dimension α is denoted CA_α.

Example 2.3.27. This is a motivational example taken from Henkin et al. [1971]. Let L be a signature, and let Σ be an L-theory. Let $\mathcal{F}$ be the set of all first-order formulas written with variables v_i, $i < \omega$, and define an equivalence relation on $\mathcal{F}$. For any two first-order formulas $\varphi, \psi \in \mathcal{F}$:

$$\varphi \sim \psi \iff \Sigma \vdash \varphi \leftrightarrow \psi.$$

Now define the operations on the equivalence classes. Define 0 to be $[\bot]$, 1 to be $[\top]$, and $+, -, c_i$ in the following way:

$$\begin{aligned} -[\varphi] &= [\neg\varphi], \\ [\varphi] + [\psi] &= [\varphi \vee \psi], \\ c_i[\varphi] &= [\exists v_i \varphi]. \end{aligned}$$

It is easy to see that these are well defined. Then

$$(\mathcal{F}/{\sim}, 0, 1, +, -, c_i, [v_i = v_j])_{i,j<\omega}$$

is a cylindric algebra of dimension ω that corresponds to first-order logic. We can now work on formulas using algebraic operations.

Just as for BAs, we would like to have corresponding concrete structures; cylindric set algebras will play that part. Their theory was developed in Henkin et al. [1985]; Monk [2000] gives a more recent introduction. There is a notion of representation for CAs as well, but – unlike BAs – not all CAs have a representation.

Definition 2.3.28. Let α be an ordinal, U a non-empty set, and $V = {}^\alpha U$. Define

1. $D_{ij} = \{x \in V \mid x_i = x_j\}$ for $i, j < \alpha$.
2. $C_i X = \{x \in V \mid \exists y \in X \forall j < \alpha (j \neq i \rightarrow y_j = x_j)\}$ for $i < \alpha$ and $X \subseteq V$.

If V is closed under intersection, union, complement, C_i and contains all the D_{ij}, then the following is a *cylindric set algebra*:

$$(\wp(V), \emptyset, V, \cup, \setminus, C_i, D_{ij})_{i,j<\alpha}.$$

Remark. It can easily be checked that a cylindric set algebra satisfies (C0)–(C7) and is thus a cylindric algebra.

Definition 2.3.29. A cylindric algebra is said to be *representable* if it is isomorphic to a subalgebra of a product of cylindric set algebras. The isomorphism is then called a *representation.* The class of all representable cylindric algebras of dimension α is called RCA_α.

Example 2.3.30. This example shows that we have a natural representation for the algebras from the previous Example 2.3.27. Let L be a signature and let Σ be a set of

L-formulas containing m free variables $v_1, \ldots, v_m$. Let M be an L-structure. With each first-order formula φ we associate the set X_φ of all the tuples $(x_1, \ldots, x_m) \in M^n$ that are satisfying assignments for φ.

This gives a one-one correspondence between the equivalence classes from the last example $[\varphi]$ and the sets of assignments to the variables X_φ: Given any formula ψ, we have $[\varphi] = [\psi]$ if and only if $X_\varphi = X_\psi$. The operations $+, \cdot, -$ from Example 2.3.27 correspond naturally to the set operations $\cup, \cap, \setminus$, and the constants 0 to $\emptyset$, 1 to M^m. Note that the 'extra' operations $\exists v_i$ correspond to the cylindrification C_i, and that D_{ij} corresponds to the equation $v_i = v_j$. So we have a natural representation for the algebras from the previous example.

The following theorem from [Henkin et al., 1971, Theorem 2.7.40] will allow us to prove that certain algebras we define are cylindric:

Theorem 2.3.31. *Let α be an ordinal and $A = (V, D_{ij}, C_i)_{i,j<\alpha}$ a cylindric atom structure. Then $\mathfrak{Cm}\,A$ is a cylindric algebra if and only if the following five conditions hold for all $i, j, k < \alpha$:*

(i) C_i *is an equivalence relation on* V*;*

(ii) $C_i|C_j = C_j|C_i$*, where* $|$ *denotes composition of the relations;*

(iii) $D_{ii} = V$*;*

(iv) $D_{ij} = C_k(D_{ik} \cap D_{kj})$ *if* $i, j \neq k$*;*

(v) $C_i \cap (D_{ij} \times D_{ij}) \subseteq Id$ *if* $i \neq j$*.*

2.3.4 Diagonal-free Algebras

One aim of this project is to extend our result for cylindric algebras to diagonal-free algebras. They differ from cylindric algebras only in not having the diagonal elements. They have been introduced by Tarski and correspond to first order logic without equality [Henkin et al., 1985, pp. 183ff].

Definition 2.3.32. A *diagonal-free algebra* B of dimension α, where α is an ordinal number, is a BAO

$$B = (B, 0, 1, +, -, c_i)_{i<\alpha}$$

where the c_i are unary operators satisfying (C0)-(C4) of the axioms for cylindric algebras given in Definition 2.3.26. The class of all diagonal-free algebras of dimension α is denoted Df_α.

Definition 2.3.33. Let α be an ordinal, U a non-empty set, and $V = {}^\alpha U$. Define

$$C_i X = \{x \in V \mid \exists y \in X \forall j < \alpha (j \neq i \rightarrow y_j = x_j)\} \text{ for } i < \alpha \text{ and } X \subseteq V.$$

If V is closed under intersection, union, complement and C_i the following is a *diagonal-free set algebra*:

$$(\wp(V), \emptyset, V, \cup, \setminus, C_i,)_{i<\alpha}.$$

Definition 2.3.34. A diagonal algebra is said to be *representable* if it is isomorphic to a subalgebra of a product of diagonal-free set algebras. The isomorphism is then called a *representation*. The class of all representable cylindric algebras of dimension α is called RDf_α.

The following notion of $<n$-dimensional and the theorem by Johnson will aid us in extending our result from cylindric algebras to diagonal-free algebras [Johnson, 1969, Theorem 1.8]:

Definition 2.3.35. An element a of a cylindric algebra $\mathcal{A}$ is called *$<n$-dimensional* if there is some $i < n$ such that $c_i a = a$.

Theorem 2.3.36. *Let $n < \omega$ and $\mathcal{A} \in CA_n$ be an n-dimensional cylindric algebra that is generated by its $<n$-dimensional elements. Then $\mathcal{A}$ is representable if the diagonal-free reduct $\mathcal{A}^-$ of $\mathcal{A}$ is an n-dimensional representable diagonal-free algebra.*

2.4 Relevant Graph Theory

The proof will use graph theory to connect the notion of *chromatic number* to certain properties of cylindric algebras. We will give the relevant definitions here, roughly following Diestel [2006].

Definition 2.4.1. A *graph* $G = (V, E)$ is a pair of a set of *vertices* V and *edges* $E \subseteq [V]^2$, i.e. E is an irreflexive and symmetric binary relation on V.

Definition 2.4.2. Let $G = (V, E)$ be a graph. A subset of vertices $U \subseteq V$ is called *independent* if there are no $x, y \in U$ such that $\{x, y\} \in E$. The *chromatic number* of G, denoted $\chi(G)$, is the smallest number $k < \omega$ such that V can be partitioned into k independent sets. If there is no such number we say $\chi(G) = \infty$.

Definition 2.4.3. Let $G = (V, E)$ be a graph. A *path* of length n in G is a non-empty subgraph $H = (\{v_1, \ldots, v_n\}, E) \subseteq G$ where the v_i are distinct and connected by edges, i.e. $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\} \in E$. If $n \geq 3$ and we also have $\{v_n, v_1\}$, then H is called a *cycle* in G. of length n. We will call a cycle of odd length an *odd cycle* and a cycle of even length an *even cycle*. The minimum cycle length in a graph G is called the *girth*, denoted $g(G)$. If there is no cycle in G, then $g(G)$ is defined to be ∞.

We will make extensive use of products of graphs.

Definition 2.4.4. Let G be a graph and $n < \omega$. Then we write $G \times n$ for the graph that contains n disjoint copies of G with all possible edges between distinct copies.

Remark. It is easy to see that $\chi(G \times n) = \chi(G) \cdot n$ if $\chi(G)$ is finite and ∞ otherwise.

The following proposition is not needed in the actual proof, but is here to help illustrate why Erdös graphs and their refinements are so useful for us. It is a standard result, but we provide our own proof here as it is different from what can usually be found in the literature.

Proposition 2.4.5. *A graph $G = (V, E)$ has chromatic number ≤ 2 if and only if it contains no odd cycle.*

Proof. ($\Longrightarrow$) Suppose $\chi(G) \leq 2$. If $\chi(G) = 1$, there are no edges between any of the vertices, so there is no odd cycle. If $\chi(G) = 2$, there is a partition of V into independent sets A, B, i.e. $V = A \cup B$. Suppose there is an odd cycle $v_1, \ldots, v_{2n+1}$ for some $n \geq 1$. Assume without loss of generality that $v_1 \in A$. Certainly $v_2 \in B$, $v_3 \in A$, etc., so $v_{2n+1} \in A$. But then $\{v_{2n+1}, v_1\} \in E$, although both $v_1, v_{2n+1} \in A$, which contradicts the fact that A is independent.

($\Longleftarrow$) Assume G contains no odd cycles. We will certainly have a chromatic number of ≤ 2 if all the unconnected components of G have a chromatic number of ≤ 2, so we can assume that G is connected, i.e. we can choose a vertex v so there is a path from v to every vertex in $G \setminus \{v\}$.

Now define two sets $A, B \subseteq V$. Firstly, we put $v \in A$. We define the distance $d(v_1, v_2)$ between two distinct vertices v_1, v_2 to be the length of a shortest path between them. and define every vertex in $V \setminus \{v\}$ with an odd distance from v to be in B and with an even distance to be in A. Note that this partitions V, as there cannot be a shortest path from v that is of both even and odd length, and V is connected, so $V = A \cup B$.

We have to show that A and B are both independent. Suppose for a contradiction that we have an edge between two vertices $a_1, a_2 \in A$. Consider shortest paths p_1, p_2 from v to a_1, a_2 respectively. Clearly, the intersection of the paths $p_1 \cap p_2$ is not empty, as it contains at least v. Let $w \in p_1 \cap p_2$ be the vertex with shortest distance to a_1. Since p_1 is a shortest path w is uniquely determined.

Now we show that w is also closer to a_2 than all the other vertices in $p_1 \cap p_2$. Suppose for a contradiction a vertex $w' \in p_1 \cap p_2$ has shortest distance to a_2 and $w \neq w'$. Assume without loss of generality that $d(v, a_1) \geq d(v, a_2)$. Then, since w and w' lie on both shortest paths, we have

$$d(v, w) + d(w, a_1) = d(v, a_1) \geq d(v, a_2) = d(v, w) + d(w, a_2)$$

and therefore

$$\begin{aligned} d(w, a_1) &\geq d(w, a_2) \\ &= d(w, w') + d(w', a_2) \\ &\geq 1 + d(w', a_2) \\ &= d(a_2, a_1) + d(w', a_2) \\ &\geq d(w', a_1). \end{aligned}$$

We obtain $d(w, a_1) \geq d(w', a_1)$, which is not possible because $d(w, a_1) < d(w', a_1)$.

There is a shortest path of either even or odd length from v to w and hence, since w was taken from shortest paths, odd or even shortest paths from w to both a_1 and a_2. Either way, this is an odd cycle, containing w, the path from w to a_1, and the path from a_2 to w. The same argument works for B. Thus $\chi(G) \leq 2$. □

Now we mention some results about particular graphs that will be used for the proof. First we borrow the notion of a p-morphism from modal logic for graphs:

Definition 2.4.6. Let G, G' be graphs and $f : G \to G'$ a surjective homomorphism. If for each edge $\{x', y'\}$ of G' and $x \in f^{-1}(x)$, there is $y \in f^{-1}(y)$ such that $\{x', y'\}$ is an edge of G, then we call f a *surjective p-morphism*.

From the following proposition, an important theorem about the existence of certain graphs is deduced, that we will also state [Hodkinson and Venema, 2005, pp. 4583–4584].

Proposition 2.4.7. *Let $c \geq 3$ and $k \geq 1$ be integers and let β, β^+ be real numbers such that $0 < \beta < \beta^+ < 1$. Let G be a finite graph with n vertices, no independent set of size $> n\beta/k$ and no odd cycles of length $< c$. Then there is a finite graph G^+ with n^+ vertices such that:*

1. *there is a surjective p-morphism $\rho : G^+ \to G$;*
2. *G^+ has no independent set of size $> n^+\beta^+/k$;*
3. *G^+ has no odd cycles of length $\leq c$.*

Theorem 2.4.8. *Let $k \geq 2$. There are finite graphs $G_0, G_1, \ldots$ and surjective p-morphisms $\rho_i : G_{i+1} \to G_i$ for $i < \omega$ such that for each i:*

1. *G_i has no odd cycles of length $\leq i$,*
2. *$\chi(G_i) = k$.*

To illustrate why these graphs are so useful for us, consider an inverse system $G_0, G_1, \ldots$ of such graphs with increasing lower bound for odd cycles $0, 1, \ldots$ and fixed chromatic number $\chi(G_i) = k$. Note that their inverse limit will not have an odd cycle of finite length. So by Proposition 2.4.5, it has a chromatic number of just 2. The chromatic number of the inverse limit can be scaled up by adding complete graphs of a certain size to all the G_i. This feature will be used in the proof of the main theorem.

3

Axiomatisations of Representable Cylindric Algebras

The goal of this chapter is to show that there is no canonical axiomatisation of the class of representable cylindric algebras RCA_n for finite $n \geq 3$. In fact, we will show that there is no axiomatisation containing only finitely many non-canonical sentences.

To do this, we will construct algebras from graphs in Section 3.1 and develop some theory for them in Section 3.2. This will allow us to prove that the number of RCA_n axioms satisfied by these algebras is related to the chromatic number of the graph in Section 3.3. Assuming an axiomatisation with only finitely many non-canonical formulas, we use direct and inverse systems in Section 3.4 to build an algebra that satisfies an arbitrary number of axioms, while its canonical extensions only satisfies a bounded number and thus obtain a contradiction.

3.1 Algebras from Graphs

Here we will describe how to obtain cylindric algebras from graphs. The first step is given by the following definition, which constructs a cylindric atom structure from a graph (cf. [Hirsch and Hodkinson, 2009, Definition 3.5]).

Notation. If $\sim$ is an equivalence relation on n and $i < n$, we will write $\sim_i$ for the restriction of the relation to $n \setminus \{i\}$.

Definition 3.1.1. Let Γ be a graph. Then the atom structure

$$At(\Gamma) = (V, D_{ij}, \equiv_i)_{i,j<n}$$

is defined as follows:

1. V consists of pairs $(K, \sim)$, where $K : n \to \Gamma \times n$ is a partial map and $\sim$ an equivalence relation on n that satisfies the following:
   a) If $|n/{\sim}| = n$, then $dom(K) = n$ and $im(K)$ is not independent.
   b) If $|n/{\sim}| = n - 1$, then there is a unique class $\{i, j\}$ of size 2 with $i < j < n$, $dom(K) = \{i, j\}$ and $K(i) = K(j)$.
   c) Otherwise, i.e. if $|n/{\sim}| < n - 1$, K is nowhere defined.
2. $D_{ij} = \{(K, \sim) \in V \mid i \sim j\}$ for $i, j < n$.
3. $(K, \sim) \equiv_i (K', \sim')$ if and only if $K(i) = K'(i)$ and $\sim_i = \sim'_i$ for $i < n$.

Remark. It is helpful to think of K as a map from sets containing $n - 1$ pairwise non-equivalent elements to vertices in the graph. Note that if two elements i, j are equivalent, then either $K(i)$ and $K(j)$ are both undefined, or $K(i) = K(j)$. We will write $K(i) = K(j)$ in both cases.

Definition 3.1.2. Let $\sim$ be an equivalence relation on n and $i < n$. Then $\sim$ is said to be *i-distinguishing* if $j \not\sim k$ for all distinct $j, k \in n \setminus \{i\}$.

Remark. Note that if $(K, \sim) \in At(\Gamma)$, then K is defined on $i < n$ if and only if $\sim$ is i-distinguishing.

We can now obtain the algebra from the atom structure and show that it is indeed a cylindric algebra, using a proof from [Kurucz, 2010, Claim 3.4 and (4)].

Definition 3.1.3. Let Γ be a graph and $At(\Gamma)$ the atom structure from Γ. Then we write $\mathcal{A}(\Gamma)$ for the algebra $\mathfrak{Cm}\, At(\Gamma)$.

Proposition 3.1.4. *Let Γ be a graph. Then any subalgebra of $\mathcal{A}(\Gamma)$ is an n-dimensional cylindric algebra.*

Proof. Recall that CA_n is a variety. This means that it is closed under subalgebras. So the result follows from Theorem 2.3.31 if we show the following:

(i) for every $i < n$, $\equiv_i$ is an equivalence relation and $D_{ii} = V$;

(ii) for all $i, j < n$, $\equiv_i$ and $\equiv_j$ commute;

(iii) for all $i, j, k < n$ with $i \neq j$, $k \neq i, j$ and for all $(K, \sim) \in V$ we have that $(K, \sim) \in D_{ij}$ if and only if there is $(K', \sim') \in D_{ik} \cap D_{kj}$ such that $(K, \sim) \equiv_k (K', \sim')$;

(iv) for all $i, j < n$ with $i \neq j$ and $(K, \sim), (K', \sim') \in D_{ij}$ we have $(K, \sim) = (K', \sim')$ if $(K, \sim) \equiv_i (K', \sim')$;

The proofs of (i) and (ii) are straightforward, but lengthy. We omit them here.

(iii) Take $i, j, k < n$ such that $k \neq i, j$ and $i \neq j$ and arbitrary $(K, \sim) \in V$. Suppose $(K, \sim) \in D_{ij}$. Then $i \sim j$ and $K(k)$ is undefined. Let $K' = \emptyset$ and $\sim'$ such that $\sim_k = \sim'_k$ and $k \sim' i \sim' j$. Then $(K', \sim') \in D_{ik} \cap D_{kj}$ and $(K, \sim) \equiv_k (K', \sim')$.

Conversely, let $(K', \sim') \in D_{ik} \cap D_{kj}$ such that $(K, \sim) \equiv_k (K', \sim')$. Then $i \sim' k \sim' j$ and $\sim'_k = \sim_k$, so $i \sim j$ and thus $(K, \sim) \in D_{ij}$.

(iv) Take $i, j < n$ such that $i \neq j$ and $(K, \sim), (K', \sim') \in D_{ij}$. Then $i \sim j$, $i \sim' j$ and $\sim_i = \sim'_i$ and thus $\sim = \sim'$. By definition either all of $K(i), K(j), K'(i), K'(j)$ are defined and equal, or none of them is defined. Hence $K = K'$. □

This establishes a relation between graphs and cylindric algebras. However, we need to study this relationship in a more abstract setting.

Definition 3.1.5. We denote by L_{AGS} the signature with three sorts $(\mathcal{A}, \mathcal{G}, \mathcal{B})$ and the following symbols:

1. function symbols $0, 1, +, -, d_{ij}, c_i$ for $i, j < n$ (with the obvious arities that make $\mathcal{A}$ into an algebra with cylindric signature);
2. function symbols $0, 1, +, -$ (that make $\mathcal{B}$ into a boolean algebra);
3. a binary relation symbol $\equiv_i$ on $\mathcal{A}$ for each $i < n$;
4. a binary (edge) relation symbol E on $\mathcal{G}$ (so that $\mathcal{G}$ is a graph);
5. a binary relation symbol H on $\mathcal{G}$;
6. a binary relation symbol $\in$ between the elements of $\mathcal{G}$ and $\mathcal{B}$;
7. a function symbol $R_i : \mathcal{A} \to \mathcal{B}$ for each $i < n$;
8. a function symbol $S_i : \mathcal{B} \to \mathcal{A}$ for each $i < n$.

We need to pick out certain elements, so that all the elements beneath are i-distinguishing and thus have $K(i)$ defined on them.

Definition 3.1.6. Let $\mathcal{A}$ be a cylindric algebra. For $i < n$, define

$$F_i = \prod_{j<k<n, j,k \neq i} -d_{jk}.$$

Remark. Clearly, for a cylindric algebra from a graph $\mathcal{A}(\Gamma)$, F_i is just the element over all the i-distinguishing atoms.

Definition 3.1.7. Let Γ be a graph and $M(\Gamma)$ be the 3-sorted L_{AGS} structure

$$(\mathcal{A}(\Gamma), \Gamma \times n, \wp(\Gamma \times n)).$$

In addition to the usual operations defined on the three sorts, the following operations relate the sorts to each other:

- The relation $\in$ denotes membership of elements of $\Gamma \times n$ in the sets that are elements of $\wp(\Gamma \times n)$.
- We have $H(x, y)$ if and only if there is $\ell < n$, such that $x, y \in \Gamma \times \{\ell\}$.
- For $a, b \in \mathcal{A}(\Gamma)$ we have $a \equiv_i b$ if and only if a and b are both atoms and $\equiv_i$ holds on these atoms in the atom structure $At(\Gamma)$.
- Finally, we have $R_i(a) = \{K(i) \mid (K, \sim) \in F_i \cdot a\}$ and $S_i(B) = \{(K, \sim) \in F_i \mid K(i) \in B\}$.

We now define a theory that helps us talk about the subclass of all the L_{AGS}-structures similar to the ones derived from graphs.

Definition 3.1.8. Define $\mathcal{U}$ to be the set of first-order L_{AGS}-sentences true in all L_{AGS}-structures $M(\Gamma)$ for graphs Γ. An L_{AGS}-structure M that is a model of $\mathcal{U}$ is called an *algebra-graph-system.*

Remark (1)*.* This definition ensures that every first-order statement that holds for algebras from graphs, also holds in any algebra-graph-system. This will allow us to prove first-order statements for algebra-graph-systems, by just showing they hold for algebras from graphs. We will refer to this approach in the following as *generalisation technique.*

Remark (2)*.* Note that in any algebra-graph-system $(\mathcal{A}, \mathcal{G}, \mathcal{B})$, the algebra $\mathcal{A}$ is cylindric. This follows by the generalisation technique, as we know from Proposition 3.1.4 that an arbitrary algebra from a graph will satisfy all the axioms for cylindric algebras.

Recall from Definition 2.3.20 that we can obtain the ultrafilter structure $\mathcal{A}_+$ from a cylindric algebra $\mathcal{A}$. The relation $\equiv_i$ on $\mu, \nu \in \mathcal{A}_+$ is then defined to be

$$\mu \equiv_i \nu \iff c_i\mu = \{c_i a \mid a \in \mu\} \subseteq \nu.$$

It will be convenient to have several equivalent ways to talk about $\equiv_i$ in an ultrafilter structure from the algebra part of an algebra-graph-system.

Lemma 3.1.9. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and $\mu, \nu \in \mathcal{A}_+$. Then the following are equivalent for all $i < n$:*

(i) $\mu \equiv_i \nu$,

(ii) $\{c_i a \mid a \in \mu\} = \{c_i b \mid b \in \nu\}$,

(iii) for each $a \in \mu$ and $b \in \nu$ there are atoms $x, y \in \mathcal{A}$ such that $x \equiv_i y$ and $x \leq a$ and $y \leq b$.

Proof. (**i**) $\Longrightarrow$ (**ii**). Suppose $\mu \equiv_i \nu$. Note that $c_i c_i a = c_i a$ for any $a \in \mathcal{A}$, as this is true for algebras from graphs, first-order definable and thus in $\mathcal{U}$ by the generalisation technique. So for any $a \in \mu$, we have by the definition of $\equiv_i$ for algebras from graphs that $c_i c_i a = c_i a \in \nu$. This shows $\{c_i a \mid a \in \mu\} \subseteq \{c_i b \mid b \in \nu\}$.

Conversely, if $b \in \nu$, then $c_i b \in \nu$ by the generalisation technique, and hence $-c_i b \notin \nu$. Note that we also have $c_i(-c_i b) = -c_i b \notin \nu$, so we have $-c_i b \notin \mu$. But then $c_i c_i b = c_i b \in \mu$ and therefore $\{c_i a \mid a \in \mu\} \supseteq \{c_i b \mid b \in \nu\}$.

(**ii**) $\Longrightarrow$ (**iii**). Suppose $\{c_i a \mid a \in \mu\} = \{c_i b \mid b \in \nu\}$. Choose arbitrary $a \in \mu$ and $b \in \nu$. Then $c_i b \in \mu$, so $c_i b \cdot a \in \mu$ and in particular $c_i b \cdot a \neq 0$. Now it is true in algebras from graphs that for any non-zero element $c \in \mathcal{A}$, there is an atom $x \in \mathcal{A}$ such that $x \leq c$ (that is, the algebra is atomic). So by the generalisation technique, there is an atom $x \in \mathcal{A}$ such that $x \leq c_i b \cdot a$. Then $x \leq a$ and $x \leq c_i b$.

For algebras from graphs the following is certainly true:

$$\forall x : \mathcal{A} \forall b : \mathcal{A}(atom(x) \wedge x \leq c_i b \rightarrow \exists y : \mathcal{A}(atom(y) \wedge y \equiv_i x \wedge y \leq b)).$$

So there is an atom $y \in \mathcal{A}$ such that $y \equiv_i x$ and $y \leq b$ in our model $\mathcal{M}$ as well.

(**iii**) $\Longrightarrow$ (**i**). Assume that $\mu \not\equiv_i \nu$, so there is $a \in \mu$ such that $c_i a \notin \nu$. Then $-c_i a \in \nu$. Take any atoms $x, y \in \mathcal{A}$ such that $x \leq a$ and $y \leq -c_i a$. Then, because the following is true by the definition of c_i for algebras from graphs:

$$\forall a : \mathcal{A} \forall x, y : \mathcal{A}(atom(x) \wedge atom(y) \wedge x \leq a \wedge y \leq -c_i a \rightarrow x \not\equiv_i y),$$

we conclude that $x \not\equiv_i y$. This shows that (iii) does not hold. □

Recall from Definition 2.1.14 that a cylindric algebra $\mathcal{A}$ is simple if $|\mathcal{A}| > 1$ and for any algebra $\mathcal{A}'$ with cylindric signature, any homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ is either trivial or injective. We will see that the algebra part of an algebra-graph-system is simple, which means its representation, if it does exist, is just an embedding into a single cylindric set algebra.

Definition 3.1.10. Let $\mathcal{C}$ be a class of BAOs of the same signature L. Then an L-term d satisfying

$$d(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0. \end{cases}$$

for each $a \in \mathcal{A} \in \mathcal{C}$, is called a *discriminator term.*

Proposition 3.1.11. *The class $\{\mathcal{A}(\Gamma) \mid \Gamma$ a graph$\}$ has a discriminator term.*

Proof. We are going to show that $c_1 \dots c_{n-1} c_{n-1} \dots c_1$ is a discriminator term. Let Γ be a graph and let $\{(K, \sim)\} \in \mathcal{A}(\Gamma)$ be an atom. Recall that

$$c_i\{(K, \sim)\} = \{(K', \sim') \mid K(i) = K'(i), \sim_i = \sim'_i\}.$$

For $1 \leq i < n$, define $K^i : n \rightarrow \Gamma \times n$ to be the partial function given by $K^i(0) = K^i(i) = K(i)$ (possibly undefined) and undefined for $j \neq 0, 1$. Also define $\sim^i$ to be the (unique) equivalence relation on n satisfying $\sim^i_i = \sim_i$ and $0 \sim^i i$. Note that this is a valid atom, so $(K^i, \sim^i) \in At(\Gamma)$, and $(K, \sim) \equiv_i (K^i, \sim^i)$.

This can be done multiple times, and writing K^{ij} for $(K^i)^j$ and $\sim^{ij}$ for $(\sim^i)^j$, we get:

$$(K,\sim) \equiv_1 (K^1,\sim^1) \equiv_2 (K^{12},\sim^{12}) \equiv_3 \cdots \equiv_{n-1} (K^{1\ldots(n-1)},\sim^{1\ldots(n-1)}).$$

Let $(L,\approx) = (K^{1\ldots(n-1)},\sim^{1\ldots(n-1)})$. The above shows that $(L,\approx) \in c_{n-1}\ldots c_1\{(K,\sim)\}$ and $(K,\sim) \in c_1 \ldots c_{n-1}\{(L,\approx)\}$.

Recall that $n \geq 3$, so we have K^1 and K^{12}.[1] By the definition of K^1, the value of $K^1(2)$ is undefined. But this means that $K^{12},\ldots,K^{1\ldots(n-1)}$ are all nowhere defined. So L does not depend on K. Also, $\approx$ is just $n \times n$, so it does not depend on $\sim$. It follows that for any atom $x \in \mathcal{A}(\Gamma)$, we have

$$x \in c_1 \ldots c_{n-1}\{(L,\approx)\} \subseteq c_1 \ldots c_{n-1}c_{n-1}\ldots c_1\{(K,\sim)\}.$$

So $c_1 \ldots c_{n-1}c_{n-1}\ldots c_1\{K,\sim\} = 1$.

Finally, let $0 \neq a \in A(\Gamma)$. We know that $A(\Gamma)$ is atomic, so there is an atom $x \leq a$. Now

$$1 = c_1 \ldots c_{n-1}c_{n-1}\ldots c_1 x \leq c_1 \ldots c_{n-1}c_{n-1}\ldots c_1 a.$$

Also, $c_1 \ldots c_{n-1}c_{n-1}\ldots c_1 0 = 0$. So $d(x) = c_1 \ldots c_{n-1}c_{n-1}\ldots c_1 x$ is a discriminator term for algebras from graphs. □

Corollary 3.1.12. *In every algebra-graph-system $M = (\mathcal{A},\mathcal{G},\mathcal{B})$, the cylindric algebra $\mathcal{A}$ and all its subalgebras are simple.*

Proof. Let $\mathcal{D}$ be a subalgebra of $\mathcal{A}$, $\mathcal{A}'$ an algebra with cylindric signature and $\varphi : \mathcal{D} \to \mathcal{A}'$ a homomorphism. It follows from Proposition 3.1.11, by the generalisation technique, that $\mathcal{A}$ has a discriminator term $d(x)$, and thus $\mathcal{D}$ as well.

Suppose φ is not injective, i.e. there are distinct $a, b \in \mathcal{D}$ such that $\varphi a = \varphi b$. Then $(a-b)+(b-a) \neq 0$ and therefore

$$\begin{aligned}
1 &= \varphi d((a-b)+(b-a)) \\
&= d(\varphi((a-b)+(b-a))) \\
&= d((\varphi a - \varphi b) + (\varphi b - \varphi a)) \\
&= d((\varphi a - \varphi a) + (\varphi a - \varphi a)) \\
&= d(\varphi(a-a) + \varphi(a-a)) \\
&= d(0) \\
&= 0.
\end{aligned}$$

Thus φ is trivial if it is not injective. □

Lemma 3.1.13. *Let $\mathcal{A} \in RCA_n$ be a representable cylindric algebra. If $\mathcal{A}$ is simple, then it has a representation that is an embedding into a single cylindric set algebra.*

[1] This is the only place where we explicitly use the assumption that $n \geq 3$.

Proof. There is a representation $h : \mathcal{A} \to \prod_{k\in K} \mathcal{S}_k$, where K is an index set and for each $k \in K$, S_k is a non-empty base set and

$$\mathcal{S}_k = (\wp(S_k^n), \cup, \setminus, \emptyset, S_k^n, D_{ij}^k, C_i^k)_{i,j<n}$$

Because h is injective and $|\mathcal{A}| > 1$, the index set $K \neq \emptyset$. So choose $\ell \in K$ and let π be the projection of $\prod_{k\in K} \mathcal{S}_k$ onto S_ℓ. Then $\pi \circ h$ is certainly a homomorphism and because

$$\pi \circ h(1) = S_k^n \neq \emptyset = \pi \circ h(0),$$

it is non-trivial. But because $\mathcal{A}$ is simple, $\pi \circ h$ is injective and thus a representation that is an embedding into a single cylindric set algebra. □

3.2 Networks and Patch systems

In this section we will present and adapt some useful tools from Hirsch and Hodkinson [2009], that will help us study representability. This will prepare us for the next section, where results from here will help us to play games on algebras to prove results about representability and the chromatic number.

First recall F_i from Definition 3.1.6. We can show that:

Lemma 3.2.1. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and $i, j < n$. Then $F_i \cdot d_{ij} \leq F_j$.*

Proof. It is enough to show that this is true for algebras from graphs, as this is clearly a set of first-order formulas. Let Γ be a graph and consider $F_i, F_j, d_{ij} \in \mathcal{A}(\Gamma)$. If $(K, \sim) \in F_i \cdot d_{ij}$, then $(K, \sim) \in F_i \cap D_{ij}$. Since $(K, \sim) \in F_i$, we know that it is i-distinguishing and since $(K, \sim) \in D_{ij}$, we know that $i \sim j$. So $(K, \sim)$ is j-distinguishing as well and $(K, \sim) \in F_j$ □

Definition 3.2.2. Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and let $i < n$. An ultrafilter μ of $\mathcal{A}$ is called i-distinguishing if and only if it contains F_i.

Remark. This is clearly equivalent to μ not containing any of the d_{jk} for distinct $j, k \in n \setminus \{i\}$.

Definition 3.2.3. Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system, let μ be an ultrafilter of $\mathcal{A}$ and let $i < n$. For an ultrafilter μ of $\mathcal{A}$, write $\mu(i)$ for the set $\{R_i(a) \mid a \in \mu\} \subseteq \mathcal{B}$.

Lemma 3.2.4. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and $i < n$. Then:*

(i) If $a \in \mathcal{A}$ and $a \leq F_i$, then $S_i(R_i(a)) \geq a$.

(ii) If $B \in \mathcal{B}$, then $R_i(S_i(B)) = B$.

(iii) R_i is surjective for all $i < n$.

Proof. It is again sufficient to show that this is true for algebras from graphs. Let Γ be a graph and consider the structure $M(\Gamma)$ from Definition 3.1.7.

(i) Let $(K,\sim) \in a$ be arbitrary. Recall that

$$R_i(a) = \{K(i) \mid (K,\sim) \in a \cdot F_i\}.$$

So $K(i) \in R_i(a)$ and because $(K,\sim) \in F_i$, we also have

$$(K,\sim) \in \{(K',\sim') \in F_i \mid K'(i) \in R_i(a)\} = S_i(R_i(a)).$$

This shows $a \leq S_i(R_i(a))$.

(ii) Let $B \in \mathcal{B}$. First note that

$$R_i(S_i(B)) = \{K(i) \mid (K,\sim) \in S_i(B)\} = \{K(i) \mid (K,\sim) \in F_i, K(i) \in B\} \subseteq B.$$

For the converse, let $p \in B$ and let $i \neq j < n$. Define $\sim$ to be the (unique) i-distinguishing relation with $i \sim j$ and define K by

$$K(i) = K(j) = p, \qquad K(k) \text{ undefined if } k \neq i,j.$$

Then $(K,\sim)$ is certainly a valid element of $At(\Gamma)$ contained in F_i and therefore $(K,\sim) \in S_i(B)$. But then $p = K(i) \in R_i(S_i(B))$. This shows $B \subseteq R_i(S_i(B))$.

(iii) By (ii), the pre-image of each $B \in \mathcal{B}$ contains $S_i(B) \in \mathcal{A}$. □

Lemma 3.2.5. *Let $M = (\mathcal{A},\mathcal{G},\mathcal{B})$ be an algebra-graph-system and let μ be an i-distinguishing ultrafilter of $\mathcal{A}$. Then:*

(i) The projection $\mu(i)$ is an ultrafilter on $\mathcal{B}$.

(ii) If $j < n$ and $d_{ij} \in \mu$, then μ is also j-distinguishing and $\mu(i) = \mu(j)$.

(iii) If ν is also an ultrafilter of $\mathcal{A}$, then $\mu \equiv_i \nu$ if and only if ν is i-distinguishing and $\mu(i) = \nu(i)$.

Proof. (i) Firstly, $\emptyset \notin \mu(i)$: If $a \in \mu$, then $a \neq 0$, so $R_i(a) \neq \emptyset$, because the following is clearly true for algebras from graphs:

$$\forall a : \mathcal{A}(0 < a \leq F_i \rightarrow R_i(a) \neq 0).$$

Let $R_i(a)$ be an element of $\mu(i)$ such that $a \in \mu$. The following are certainly true for algebras from graphs:

$$\forall a,b : \mathcal{A}(a \leq b \rightarrow R_i(a) \leq R_i(b)), \qquad (\star)$$
$$\forall A,B : \mathcal{B}(A \leq B \rightarrow S_i(A) \leq S_i(B)).$$

So, if $R_i(a) \leq B \in \mathcal{B}$, then, by Lemma 3.2.4, $a \leq S_i(R_i(a)) \leq S_i(B)$. So $S_i(B) \in \mu$ and hence, again by Lemma 3.2.4, $B = R_i(S_i(B)) \in \mu(i)$. This shows that $\mu(i)$ is closed under supersets.

Now let $R_i(a), R_i(b) \in \mu(i)$ be be such that $a, b \in \mu$. So $a \cdot b \in \mu$. By ($\star$), we have $R_i(a) \cdot R_i(b) \geq R_i(a \cdot b) \in \mu(i)$. We showed that $\mu(i)$ is closed under supersets, so $R_i(a) \cdot R_i(b) \in \mu(i)$. This shows that $\mu(i)$ is a filter.

Finally, take any $B \in \mathcal{B}$. The following certainly holds in algebras from graphs:

$$\forall B : \mathcal{B}(S_i(B) + S_i(-B) = F_i).$$

Hence $S_i(B) + S_i(-B) = F_i$, which is contained in μ because μ is i-distinguishing. So one of $S_i(B)$ and $S_i(-B)$ is in μ and by Lemma 3.2.4(ii) one of $B, -B$ is in $\mu(i)$. Thus by Lemma 2.3.7 $\mu(i)$ is an ultrafilter of $\mathcal{B}$.

(ii) This is obvious if $i = j$, so suppose $i \neq j$. Assume $d_{ij} \in \mu$. Then $F_j \in \mu$, because of Lemma 3.2.1 and $d_{ij} \cdot F_i \in \mu$. So μ is also j-distinguishing.

For the second part of the claim, let $R_i(a)$ be an element of $\mu(i)$ for some $a \in \mu$. Define $b = a \cdot d_{ij} \in \mu$. Then, by Lemma 3.2.1, $b \leq F_j$. We need the following statement that is obviously true for algebras from graphs:

$$\forall a : \mathcal{A}(a \leq d_{ij} \rightarrow R_i(a) = R_j(a)).$$

It follows that $R_i(b) = R_i(b)$. Now, again using ($\star$), we have

$$R_i(a) \geq R_i(b) = R_j(b) \in \mu(j).$$

Thus $\mu(i) \subseteq \mu(j)$ and because these are ultrafilters by (i) we have $\mu(i) = \mu(j)$.

(iii) ($\Longrightarrow$) Assume $\mu \equiv_i \nu$. From the definition it follows that $c_i F_i \in \nu$. In algebras from graphs $c_i F_i = F_i$, so we have $F_i \in \nu$ and ν is i-distinguishing. Now let $R_i(a) \in \mu(i)$ such that $a \in \mu$. Moreover, $c_i a \leq c_i F_i = F_i$, and $c_i a \in \nu$. Note the following holds for all algebras from graphs:

$$\forall a : \mathcal{A}(R_i(a) = R_i(c_i a)).$$

So $R_i(a) = R_i(c_i a) \in \nu(i)$. Thus $\mu(i) \subseteq \nu(i)$ and because these are ultrafilters by (i), we have $\mu(i) = \nu(i)$.

($\Longleftarrow$) For the converse let ν be an ultrafilter of $\mathcal{A}$ such that $F_i \in \nu$ and $\mu(i) = \nu(i)$. Let $a \in \mu$, $b \in \nu$. Then

$$R_i(a) \in \mu(i) = \nu(i) \ni R_i(b)$$

and thus $R_i(a) \cdot R_i(b) \neq 0$. Now we need the following statement:

$$\forall a, b : \mathcal{A}(R_i(a) \cdot R_i(b) \neq 0 \rightarrow \exists x, y : \mathcal{A}(atom(x) \wedge atom(y) \wedge x \equiv_i y \wedge x \leq a \wedge y \leq b)).$$

This is true for algebras from graphs because we can just take atoms $x = (K, \sim) \in a$,

$y = (K', \sim') \in b$ such that $(K, \sim), (K', \sim') \in F_i$ and $K(i) = K'(i)$. By definition this gives us $x \leq a$, $y \leq b$ and $x \equiv_i y$. Because this is true for algebras from graphs, it is also true for the structure $\mathcal{M}$. This allows us to apply Lemma 3.1.9(iii) to get $\mu \equiv_i \nu$. □

3.2.1 Ultrafilter Networks

In this section we introduce approximations to representations, so called ultrafilter networks. They will be part of the game to construct representations.

Definition 3.2.6. Let X be a set and $i < n$ and $v \in X^n$.

1. For $w \in X^n$, we say $v \equiv_i w$ if $v_j = w_j$ for all $j < n$, $j \neq i$.
2. Denote by $v[i/j]$ the tuple $w \in X^n$ defined by $w \equiv_i v$ and $w_i = v_j$.
3. If $v_j \neq v_k$ for all distinct $j, k \in n \setminus \{i\}$, then v is called *i-distinguishing*.

Definition 3.2.7. Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system. A *partial ultrafilter network* over $\mathcal{A}$ is a pair $\mathcal{N} = (N_1, N_2)$, where N_1 is a set and $N_2 : N_1^n \to \mathcal{A}_+$ is a partial map that satisfies the following for any $v, w \in N_1^n$:

1. For $i, j < n$, $d_{ij} \in N_2(v)$ if and only if $v_i = v_j$.
2. If $i < n$ and $v \equiv_i w$, then $N_2(v) \equiv_i N_2(w)$.

If N_2 is total, we call $\mathcal{N}$ an *ultrafilter network* over $\mathcal{A}$.

Remark. If $\mathcal{N} = (N_1, N_2)$ and $\mathcal{M} = (M_1, M_2)$ are both partial ultrafilter networks we write $\mathcal{N} \subseteq \mathcal{M}$ to denote $N_1 \subseteq M_1$ and $M_2 \upharpoonright N_1 = N_2$. We will often write $\mathcal{N}$ for both N_1 and N_2.

3.2.2 Patch Systems

The patch systems provide a way to assign ultrafilters of a graph to $(n-1)$-sized subsets, or 'patches', of a set of nodes.

Definition 3.2.8. Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system. A *patch system* for $\mathcal{B}$ is a pair $\mathcal{P} = (P_1, P_2)$, where P_1 is a set and $P_2 : [P_1]^{n-1} \to \mathcal{B}_+$ assigns an ultrafilter of $\mathcal{B}$ to each subset of P_1 of size $n-1$. If $|P_1| < n-1$, then $P_2 = \emptyset$.

Definition 3.2.9. Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and $\mathcal{P} = (P_1, P_2)$ a patch system for $\mathcal{B}$. A set $V = \{v_0, \ldots, v_{n-1}\} \subseteq P_1$ of size n is called *$\mathcal{P}$-coherent* if the following is satisfied: For any $X_i \in P_2(V \setminus \{v_i\})$ $(i < n)$, there are $p_i \in \mathcal{G}$ with $p_i \in X_i$ for each $i < n$, such that $\{p_0, \ldots, p_{n-1}\}$ is not an independent subset of $\mathcal{G}$.

The patch system $\mathcal{P}$ is said to be *coherent* if every set $V \subseteq P_1$ of size n is $\mathcal{P}$-coherent.

Lemma 3.2.10. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and $\mathcal{P} = (P_1, P_2)$ a patch system for $\mathcal{B}$. Let $V = \{v_0, \ldots, v_{n-1}\} \in [P_1]^n$ and for each $i < n$, let $V_i = V \setminus \{v_i\}$. Then V is $\mathcal{P}$-coherent if and only if there exists an ultrafilter μ of $\mathcal{A}$ that is i-distinguishing for all $i < n$ and with $\mu(i) = P_2(V_i)$ for each $i < n$.*

Proof. Let $\mathcal{P} = (P_1, P_2)$ be the patch system for $\mathcal{B}$ and let $V = \{v_0, \ldots, v_{n-1}\} \in [P_1]^n$.

($\Longrightarrow$) Assume V is $\mathcal{P}$-coherent. Write $U_i = P_2(V_i) \in \mathcal{B}_+$ for each $i < n$. Define

$$\mu_0 = \bigcup_{i<n} \{S_i(B) \mid B \in U_i\} \subseteq \mathcal{A}.$$

To show that μ_0 has the finite intersection property, it is sufficient to consider arbitrary $B_i \in U_i$ and prove that $S_0(B_0) \cdot S_1(B_1) \cdots S_{n-1}(B_{n-1}) \neq 0$. By the $\mathcal{P}$-coherence of V, we can find $p_i \in B_i$ for each $i < n$ such that $\{p_1, \ldots, p_{n-1}\}$ is not an independent set. Now the following holds on algebras from graphs because there is an atom that is i-distinguishing for all $i < n$ and that will map to the non-independent set:

$$\forall B_0, \ldots, B_{n-1} : \mathcal{B} \left(\exists p_0, \ldots, p_{n-1} : \mathcal{G} \left(\bigwedge_{i<n} p_i \in B_i \wedge \bigvee_{i,j<n} E(s_i, s_j) \right) \right.$$
$$\left. \rightarrow \exists x : \mathcal{A} \left(atom(x) \wedge \bigwedge_{i<n} x \leq S_i(B_i) \right) \right)$$

We showed that the left hand side of the implication is satisfied, so the right hand side gives us that μ_0 has the finite intersection property. Now we can use Theorem 2.3.6 to extend μ_0 to an ultrafilter μ of $\mathcal{A}$. Since $F_i = S_i(\mathcal{B}) \in \mu$, we have that μ is i-distinguishing for all $i < n$. Moreover, if $B \in U_i$, then $S_i(B) \in \mu$, so $B = R_i(S_i(B)) \in \mu(i)$. Therefore $P_2(V_i) = U_i = \mu(i)$.

($\Longleftarrow$) Assume μ is an ultrafilter of $\mathcal{A}$ that is i-distinguishing for all $i < n$ and with $\mu(i) = P_2(V_i)$ for each $i < n$. Choose arbitrary $X_i \in \mu(i)$ for each $i < n$. For each $i < n$, we can choose $b_i \in \mu$ such that $R_i(b_i) = X_i$. Let $b = \prod_{i<n}(b_i \cdot F_i) \in \mu$. Now the following holds by definition in algebras from graphs:

$$\forall x : \mathcal{A} \left(atom(x) \wedge \bigwedge_{i<n} x \leq F_i \rightarrow \bigvee_{i,j<n} (\exists p, q : \mathcal{G}(E(p,q) \wedge p \in R_i(x) \wedge q \in R_j(x))) \right).$$

So we can choose an atom $x \in \mathcal{A}$ such that $x \leq b$. Then, by the above, we can also choose $v_1, \ldots, v_{n-1}$ with $v_i \in R_i(x)$ such that $\{v_1, \ldots, v_{n-1}\}$ is not independent. So V is coherent. □

We need this lemma to show that the next definition is well defined.

Lemma 3.2.11. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and $\mathcal{N} = (N_1, N_2)$ a partial ultrafilter network over $\mathcal{A}$. Then:*

(i) If $v \in dom(N_2)$ and $i < n$, then $N_2(v)$ is i-distinguishing if and only if v is i-distinguishing.

(ii) If $v \in N_1^n$ is i-distinguishing, then $v[i/j]$ is j-distinguishing and $N_2(v)(i) = N_2(v[i/j])(j)$.

(iii) Let $i, j < n$ and $v, w \in N_1^n$ such that v is i-distinguishing, w is j-distinguishing and $\{v_k \mid i \neq k < n\} = \{w_k \mid j \neq k < n\}$. Then $N_2(v)(i) = N_2(w)(j)$.

(iv) If $\mathcal{P} = (N_1, P_2)$ is a coherent patch system and for each $i < n$ we have $N_2(v)(i) = P_2(\{v_j \mid i \neq j < n\})$ for all i-distinguishing $v \in N_1^n$, then there is a total ultrafilter network $\mathcal{N}^+ = (N_1, N_2^+) \supseteq \mathcal{N}$ such that $\partial\mathcal{N}^+ = P$.

Proof. (i) We have that $N_2(v) \ni F_i$ if and only if it does not contain d_{jk} for $j < k < n$ and $j, k \neq i$. But this is true if and only if v is i-distinguishing by the definition of N_2.

(ii) Let $v \in N_1^n$ be i-distinguishing and $j < n$. Let $w = v[i/j]$. Then $w \equiv_i v$ and $w_i = v_j$. So by the definition of ultrafilter network we have $N_2(v) \equiv_i N_2(w)$ and $d_{ij} \in N_2(w)$. So by Lemma 3.2.5(iii) we have $N_2(v)(i) = N_2(w)(i)$ and by (ii) of the same lemma $N_2(w)$ is j-distinguishing and $N_2(w)(i) = N_2(w)(j)$.

(iii) Let $i, j < n$ and $v, w \in N_1^n$ such that v is i-distinguishing, w is j-distinguishing and $\{v_k \mid i \neq k < n\} = \{w_k \mid j \neq k < n\}$. Assume without loss of generality that $i = j = 0$ (by (ii) we can just replace v by $v[i/0]$ and w by $w[j/0]$).

The proof is by induction on the highest number v, w disagree on: $d(v, w) = \max\{k < n \mid v_k \neq w_k\}$. If they agree on everything or $d(v, w) = 0$, then $v \equiv_0 w$, so $N_2(v) \equiv_i N_2(w)$ and Lemma 3.2.5(iii) gives us $N_2(v)(0) = N_2(w)(0)$.

Assume now that $d(v, w) = k > 0$ and the claim holds if $d(v, w)$ is less than k. Since $\{v_\ell \mid 0 \neq \ell < n\} = \{w_\ell \mid 0 \neq \ell < n\}$, $w_k = v_j$ for some $0 < j < n$. Note that because w is 0-distinguishing we must have $j < k$: otherwise we would have $w_j = v_j = w_k$. Now 'swap' the k and j entries of v, that is define

$$v' = v[0/k][k/j][j/0].$$

By (ii), $N_2(v)(0) = N_2(v')(0)$. Also $v'_k = v_j = w_k$ and $v'_\ell = w_\ell$ for all $\ell > k$. So v' is also 0-distinguishing, $\{v'_\ell \mid 0 \neq \ell < n\} = \{w_\ell \mid 0 \neq \ell < n\}$ and $d(v', w) < k$. So, using the induction hypothesis, we get $N_2(v)(0) = N_2(v')(0) = N_2(w)(0)$.

(iv) We need to define a total function $N_2^+ : N_1^n \to \mathcal{A}_+$ that agrees with N_2 on $dom(N_2)$. So we first put $N_2^+(v) = N_2(v)$ for all $v \in dom(N_2)$.

Now let $v \in (N_1^n \setminus dom(N_2))$. Write $set(v)$ for $\{v_i \mid i < n\}$. We will assign an ultrafilter of $\mathcal{A}$ to v in the following way:

(a) If $|\, set(v)| < n-1$, define

$$D = \prod_{i<j<n, v_i=v_j} d_{ij} \cdot \prod_{i<j<n, v_i \neq v_j} -d_{ij}.$$

By the generalisation technique, D is an atom (in an algebra from a graph it would just be $(\emptyset, \sim)$ where $i \sim j$ if and only if $v_i = v_j$). We define $N_2^+(v)$ to be the principal ultrafilter of $\mathcal{A}$ generated by D. Note that we have $N_2^+(v)(i) = \emptyset = P_2(set(v))$ for all $i < n$.

(b) If $|\, set(v)| = n-1$, there are unique $i < j < n$ such that $v_i = v_j$. Define $\Lambda = F_i \cdot F_j \cdot d_{ij}$ and let

$$N_2^+(v) = \{a \in \mathcal{A} \mid R_i(a \cdot \Lambda) \in P_2(set(v))\} \subseteq \mathcal{A}.$$

It is easy to see that this is an ultrafilter of $\mathcal{A}$, because $P_2(set(v))$ is an ultrafilter of $\mathcal{B}$. By the generalisation technique, $R_i(\Lambda) = \mathcal{B}$, so $\Lambda \in N_2^+(v)$, and therefore $d_{ij} \in N_2^+(v)$ if and only if $v_i = v_j$. Moreover, if $a \in N_2^+(v)$ and $a \leq F_i$, then $R_i(a) \supseteq R_i(a \cdot \Lambda) \in P_2(set(v))$ and hence, as both are ultrafilters, $N_2^+(v)(i) = P_2(set(v))$. By Lemma 3.2.5(ii), $N_2^+(v)(j) = P_2(set(v))$ as well.

(c) If $|\, set(v)| = n$, then by Lemma 3.2.10 there is an ultrafilter μ of $\mathcal{A}$ that is i-distinguishing for all $i < n$ and with $\mu(i) = P_2(\{v_j \mid i \neq j < n\})$ for all $i < n$. We define $N_2^+(v) = \mu$.

We need to check that this defines an ultrafilter network. In all three cases we have for all $i, j < n$:

$$d_{ij} \in N_2^+(v) \text{ if and only if } v_i = v_j. \qquad (\star)$$

Also, v is i-distinguishing if and only if $N_2^+(v) \ni F_i$ for all $i < n$.

Furthermore, in all three cases we constructed the ultrafilters so that for i-distinguishing $v \in N_1^n$ we have

$$N_2^+(v)(i) = P_2(\{v_j \mid i \neq j < n\}). \qquad (\star\star)$$

Now we check the second condition for ultrafilter networks. Assume $v \equiv_i w$. If v is i-distinguishing, then so is b and by $(\star\star)$ we have

$$N_2^+(v)(i) = P_2(\{v_j \mid i \neq j < n\}) = N_2^+(w)(i).$$

So by Lemma 3.2.5(iii) $N_2^+(v) \equiv_i N_2^+(w)$.

Assume now that v is not i-distinguishing. Similar as in (a), we define

$$\Delta = \prod_{j,k \neq i, v_j = v_k} d_{ij} \cdot \prod_{j,k \neq i, v_j \neq v_k} -d_{ij}.$$

By $(\star)$, $\Delta \in N_2^+(v)$ and since $v \equiv_i w$, also $\Delta \in N_2^+(w)$. Now take any $a \in N_2^+(v)$ and

$b \in N_2^+(w)$ and choose atoms $x, y \in \mathcal{A}$ such that $x \leq a \cdot D$ and $y \leq b \cdot D$. Since v is not i-distinguishing, there are distinct $j, k \neq i$ such that $v_j = v_k$. Thus the left product of Δ is non-empty and the following holds for algebras from graphs:

$$\forall x, y : \mathcal{A}(atom(x) \wedge atom(y) \wedge x \leq \Delta \wedge y \leq \Delta \rightarrow x \equiv_i y).$$

(Any atom $(K, \sim)$ in Δ would have K undefined and the equivalence relation on $n \setminus \{i\}$ determined by the d_{jk} in Δ.) So by the generalisation technique, we have $x \equiv_i y$ and by Lemma 3.1.9 $N_2^+(v) \equiv_i N_2^+(w)$. Thus $\mathcal{N}^+ = (N_1, N_2^+)$ is an ultrafilter network.

We also have $\mathcal{N} = (N_1, N_2) \subseteq (N_1, N_2^+) = \mathcal{N}^+$ and because of $(\star\star)$, $\partial\mathcal{N}^+ = \mathcal{P}$. □

The third part in the above lemma says that the ith projection is independent from the ith coordinate and the order of the elements in the vector. So the following is well-defined:

Definition 3.2.12. Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system and $\mathcal{N} = (N_1, N_2)$ an ultrafilter network over $\mathcal{A}$. We define ∂N to be the patch system (N_1, P_2), where

$$P_2 : [N_1]^{n-1} \rightarrow \mathcal{B}_+,$$
$$\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}\} \mapsto N_2(v)(i)$$

for each $i < n$ and i-distinguishing $v \in N_1^n$.

3.3 Chromatic Number & Representability

Here we show that the chromatic number of a graph Γ and the number of representability axioms satisfied by $\mathcal{A}(\Gamma)$ 'drag' each other along. We achieve this by proving that $\mathcal{U} \cup \{\gamma_m \mid m < \omega\}$ and $\mathcal{U} \cup \{\theta_k \mid k < \omega\}$ have very similar models and use compactness to derive the desired result.

Recall from Definition 2.4.2, that the chromatic number of a graph is the size of the smallest partition into independent sets, or ∞ if no such partition exists. Although the chromatic number is in general not first-order definable, we can define it for algebra-graph-systems with the following formula.

Definition 3.3.1. For each $k < \omega$, we define the following L_{AGS}-formula:

$$\theta_k := \forall B_0, \ldots, B_{k-1} : \mathcal{B}\left(\sum_{i<k} B_i = 1 \rightarrow \exists p, q : \mathcal{G}\left(E(p,q) \wedge \bigvee_{i<k} (p \in B_i \wedge q \in B_i)\right)\right).$$

Remark. If $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ is an algebra-graph-system, we will say an element $B \in \mathcal{B}$ is an independent set, if there are no $p, q \in B$ such that $E(p, q)$.

One direction can be proved without further help, apart from some of the machinery from the last section and Ramsey's theorem.

Proposition 3.3.2. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an infinite algebra-graph-system. If $M \not\models \theta_k$ for some $k < \omega$, then $\mathcal{A}$ is not representable.*

Proof. Suppose for a contradiction that $\mathcal{A}$ is representable although $M \not\models \theta_k$. So by Lemma 3.1.13 there is a representation h that embeds $\mathcal{A}$ into a single cylindric set algebra $\mathcal{S} = (\wp(S^n), \cup, \setminus, \emptyset, S^n, D_{ij}, C_i)_{i,j<n}$ with base set S.

Let $\mathcal{N}$ be the ultrafilter network with nodes S and $\mathcal{N}(\bar{s}) = \{a \in \mathcal{A} \mid \bar{s} \in h(a)\} \in \mathcal{A}_+$. This is a well-defined ultrafilter network. Furthermore, by Lemma 3.2.11 we can make this into a well-defined and coherent patch system $\partial\mathcal{N}$.

Now $M \not\models \theta_k$ means

$$M \models \exists B_0, \ldots, B_{k-1} : \mathcal{B}\left(\sum_{i<k} B_i = 1 \wedge \forall p, q \left(\bigwedge_{i<k} p \in B_i \wedge q \in B_i \to \neg E(p, q)\right)\right)$$

which says $\mathcal{G}$ is the union of k independent sets from $\mathcal{B}$, as we certainly have $\forall v : \mathcal{G}(v \in 1)$ in algebras from graphs. So $\mathcal{G}$ has finite chromatic number k and can be partitioned into k independent sets from $\mathcal{B}$, say $B_0, \ldots, B_{k-1}$.

Since $\mathcal{A}$ is infinite and h is injective, $\mathcal{S}$ is infinite and therefore S as well. So we can choose infinitely many distinct elements $s_0, s_1, \ldots$ from S. Now define a map $f : [\omega]^{n-1} \to k$ by letting $f(\{i_0, \ldots, i_{n-1}\})$ be the unique $j < k$ such that $B_j \in \partial\mathcal{N}(\{s_{i_0}, \ldots, s_{a_{i_{n-1}}}\})$. By Ramsey's theorem (2.2.1), we can choose the elements so that f is constant, say $f([\omega]^{n-1}) = c$. Now consider the first n elements that were chosen $\{s_0, \ldots, s_{n-1}\}$. Since f is constant, $B_c \in \partial\mathcal{N}(\{s_j \mid i \neq j < n\})$ for all $i < n$. Because $\partial\mathcal{N}$ is coherent, we can choose n elements $p_0, \ldots, p_{n-1} \in B_c$ so that $\{p_0, \ldots, p_{n-1}\}$ is not an independent set. But this is impossible since B_c is independent. □

For the other direction we define a game that allows us to build a representation for $\mathcal{A}(\Gamma)$ if Γ has infinite chromatic number.

Definition 3.3.3. Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system. A *game* $G(\mathcal{A})$ is an infinite sequence of ultrafilter networks

$$\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \ldots$$

that were build by the following rules: The game begins with the (unique) one-point network $\mathcal{N}_0$. There are ω rounds. In round $t < \omega$, the current network is $\mathcal{N}_t$ and the player $\forall$ chooses an n-tuple $v \in \mathcal{N}_t^n$, a number $i < n$ and an element $a \in \mathcal{A}$ such that $c_i a \in \mathcal{N}_t(v)$. The other player $\exists$ then has to respond with an ultrafilter network $\mathcal{N}_{t+1} \supseteq \mathcal{N}_t$ such that there is $w \in \mathcal{N}_{t+1}^n$ with $w \equiv_i v$ and $a \in \mathcal{N}_{t+1}(w)$. She wins the game if she can play a network that satisfies these constraints in each round.

Lemma 3.3.4. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system. If $\exists$ has a winning strategy in the game $G(\mathcal{A})$, then $\mathcal{A}$ is representable.*

Proof. By the downward Löwenheim-Skolem-Tarski theorem (2.1.16), there is a countable and elementary subalgebra $\mathcal{A}_0$ of $\mathcal{A}$. Let $\mathcal{N}_0 \subseteq \mathcal{N}_1 \ldots$ be the game where $\forall$ plays every possible move in $\mathcal{A}_0$ and $\exists$ uses her winning strategy in $\mathcal{G}_\omega(\mathcal{A})$ to respond. She can do this because she only has to accept or reject at each move.

Define $\mathcal{N} = \bigcup_{t<\omega} \mathcal{N}_t$. This is certainly an ultrafilter network, as all the $\mathcal{N}_t$ are ultrafilter networks.

Now we can use $\mathcal{N}$ to define the following homomorphism for $\mathcal{A}_0$:

$$\begin{aligned} h : \mathcal{A}_0 &\to (\wp(\mathcal{N}^n), \cup, \setminus, \emptyset, \mathcal{N}^n, D^{\mathcal{N}}_{ij}, C^{\mathcal{N}}_i))_{i,j<n} \\ a &\mapsto \{v \in N^n \mid a \in \mathcal{N}(v)\}. \end{aligned}$$

Using that $\mathcal{N}(v)$ is an ultrafilter, it can be checked that this is indeed a homomorphism. Recall from Corollary 3.1.12 that $\mathcal{A}_0$ is simple. So, since $h(1) = N^n \neq \emptyset = h(0)$, the map h is injective. This shows that $\mathcal{A}_0$ is representable and because RCA_n is an elementary class, $\mathcal{A}$ is representable as well. □

Remark. The converse of the lemma also holds, but is not needed here.

Lemma 3.3.5. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system such that $M \models \{\theta_k \mid k < \omega\}$. Then there is an ultrafilter of $\mathcal{B}$ that contains no independent sets.*

Proof. Let $\mu_0 \subseteq \mathcal{B}$ be the set that contains all the B such that $-B$ is independent. Then μ_0 has the finite intersection property: Suppose for a contradiction that for $B_0, \ldots, B_{k-1} \in \mu_0$ we have

$$\begin{aligned} B_0 \cdot B_1 \cdots B_{k-1} = 0 &\implies -(B_0 \cdot B_1 \cdots B_{k-1}) = 1 \\ &\implies (-B_0) + (-B_1) + \cdots + (-B_{k-1}) = 1 \end{aligned}$$

But this means $M \not\models \theta_k$. A contradiction. Thus μ_0 has the finite intersection property and, by the boolean prime ideal theorem (2.3.6), it can be extended to an ultrafilter μ, which does not contain any independent set (because it contains the complement). □

Remark. The converse of Lemma 3.3.5 also holds, but is not needed here.

Proposition 3.3.6. *Let $M = (\mathcal{A}, \mathcal{G}, \mathcal{B})$ be an algebra-graph-system. If $M \models \{\theta_k \mid k < \omega\}$, then $\mathcal{A}$ is representable.*

Proof. By Lemma 3.3.4 it is sufficient to show that $\exists$ has a winning strategy in the game $G(\mathcal{A})$. Suppose we are in round t and the current network is $\mathcal{N}_t$. According to the rules, the player $\forall$ chooses $a \in \mathcal{A}$, $i < n$ and $v \in \mathcal{N}_t^n$. The other player $\exists$ now has to respond with a network $\mathcal{N}_{t+1} \supseteq \mathcal{N}_t$ that contains $w \in \mathcal{N}_{t+1}^n$ such that $v \equiv_i w$ and $a \in \mathcal{N}_{t+1}(w)$. If there is $w \in \mathcal{N}_t^n$ with $v \equiv_i w$ and $a \in \mathcal{N}_{t+1}(w)$, she can just respond with the unchanged network $\mathcal{N}_t$. So we assume in the following that there is no such w.

Step 1. Let $\mathcal{N}_{t+1} = \mathcal{N}_t \cup \{z\}$, where $z \notin \mathcal{N}_t$ is a new node. We will first try to find an ultrafilter for the tuple w, defined by $w \equiv_i v$ and $w_i = z$. To help $\exists$ win the game, the ultrafilter should contain a. We achieve this by showing that the following set has the finite intersection property:

$$\mu_0 = \{a\} \cup \{-d_{ij} \mid i \neq j < n\} \cup \{c_i b \mid b \in \mathcal{N}_t(v)\}.$$

Let $\Delta = \prod_{j \neq i} -d_{ij}$. We can show that $c_i(a \cdot \Delta) \in \mathcal{N}_t(v)$: Clearly, by the generalisation technique, $\Delta + \sum_{j \neq i} d_{ij} = 1$. Therefore, $c_i a = c_i(a \cdot \Delta) + \sum_{j \neq i} c_i(a \cdot d_{ij})$. If $c_i(a \cdot \Delta) \notin \mathcal{N}_t(v)$, then there is $j \neq i$ such that $c_i(a \cdot d_{ij}) \in \mathcal{N}_t(v)$. Now let $v' = v[i/j]$. Then, since $v \equiv_i v'$ and by the definition of ultrafilter networks, $\mathcal{N}_t(v) \equiv_i \mathcal{N}_t(v')$. So by Lemma 3.1.9, $c_i(a \cdot d_{ij}) \in \mathcal{N}_t(v')$ as well. By the construction of v', we have $v_i = v_j$ and therefore $d_{ij} \in \mathcal{N}_t(v')$. Thus $d_{ij} \cdot c_i(a \cdot d_{ij}) \in N_t(v')$. In algebras from graphs we certainly have

$$\forall a : \mathcal{A}(d_{ij} \cdot c_i(a \cdot d_{ij}) = a \cdot d_{ij}).$$

Hence, by the generalisation technique, $a \cdot d_{ij} \in N_t(v')$, and therefore $a \in N_t(v')$. But we assumed that no such tuple exists in $\mathcal{N}_t^n$, so we must have $c_i(a \cdot \Delta) \in \mathcal{N}_t(v)$.

If μ_0 would not have the finite intersection property, then there would be $b_0, \ldots, b_{m-1} \in \mathcal{N}_t(v)$ such that $a \cdot \Delta \cdot c_i b_0 \cdots c_i b_{m-1} = 0$. But then

$$\begin{aligned} a \cdot \Delta &\leq -(c_i b_0 \cdots c_i b_{m-1}) = (-c_i b_0) + \cdots + (-c_i b_{m-1}) \\ \Longrightarrow c_i(a \cdot \Delta) &\leq (c_i - c_i b_0) + \cdots + (c_i - c_i b_{m-1}) = (-c_i b_0) + \cdots + (-c_i b_{m-1}). \end{aligned}$$

But this implies that $(-c_i b_0) + \cdots + (-c_i b_{m-1}) \in \mathcal{N}_t(v)$, which is impossible since all of the b_j, $j < n$ are in $\mathcal{N}_t(v)$. Thus μ_0 has the finite intersection property.

So, by the boolean prime ideal theorem (2.3.6), player $\exists$ can choose an ultrafilter μ of $\mathcal{A}$ that contains μ_0. By construction and Lemma 3.1.9, we have $\mu \equiv_i \mathcal{N}_t(v)$. Moreover, μ satisfies the following

$$d_{jk} \in \mu \iff w_j = w_k \qquad (\star)$$

for all $j, k < n$, because we have $w_i \neq w_j$, $-d_{ij} \in \mu$ and for $j, k \neq i$,

$$w_j = w_k \iff v_j = v_k \iff d_{jk} \in \mathcal{N}_t(v) \iff d_{jk} = c_i d_{jk} \in \mu.$$

Therefore, $\mathcal{N}' \supseteq \mathcal{N}_t$ with nodes $\mathcal{N}_t \cup \{z\}$, $\mathcal{N}'(w) = \mu$ and $\mathcal{N}'(x)$ undefined for tuples containing z other than w, is a valid partial ultrafilter network.

Step 2. $\exists$ also needs to define ultrafilters for all the remaining new tuples containing z. This can be done with the help of the patch system $\mathcal{P} = (\mathcal{N}_{t+1}, P_2)$, defined as follows. We will again write $set(w)$ for $\{w_k \mid k < n\}$.

- For each set of old nodes $V \in [\mathcal{N}_t]^{n-1}$, we define $P_2(V) = \partial\mathcal{N}_t(V)$.

- For each $j < n$, define $W_j = \{w_k \mid j \neq k < n\}$. For the W_j of size $n-1$, she has to define $P_2(W_j)$: If $|W_i| = n-1$, then because $W_i \subseteq \mathcal{N}_t$, she already defined $P_2(W_i) = \mu(i)$.

  Now consider the $j \neq i$ with $|W_j| = n-1$. We showed in ($\star$) that μ is j-distinguishing if w is, so $\mu(j)$ is an ultrafilter in that case. So we define $P_2(W_j) = \mu(j)$. Note that this is well defined, because if there is $k \neq i, j$ such that $W_k = W_j$, then $w_j = w_k$, and thus by ($\star$) $d_{jk} \in \mu$ and by Lemma 3.2.5, $\mu(i) = \mu(j)$.

- For the remaining $W \in [N_{t+1}]^{n-1}$ that contain z, but that are not contained in $set(w)$, we construct the following ultrafilter. Recall that we have an equivalence relation H on G with exactly n equivalence classes, that satisfies the following for algebras from graphs:

  $$\forall x, y : \mathcal{G}(\neg H(x,y) \rightarrow E(x,y)). \qquad (\dagger)$$

  So by the generalisation technique, the same is true for H on $\mathcal{G}$. Furthermore, also by the generalisation technique, each of equivalence classes is contained in $\mathcal{B}$, since we have the following for algebras from graphs:

  $$\forall x : \mathcal{G} \exists B : \mathcal{B}(x \in B \wedge \forall y : \mathcal{G}(H(x,y) \rightarrow y \in B)).$$

  Call these equivalence classes $G_1, \ldots G_n$.

  Now each of the $\mu(j)$ for $j \neq i$ contains exactly one of the G_k, so there must be at least one G_ℓ that is not contained in any of the $\mu(j)$. We are given that $M \models \{\theta_k \mid k < \omega\}$, so by Lemma 3.3.5 there is an ultrafilter ν of $\mathcal{B}$ containing G_ℓ and no independent sets. We define $P_2(W) = \nu$ for all the remaining $W \in [N_{t+1}]^{n-1}$.

For each $j < n$, we certainly have $N_2(u)(j) = P_2(\{u_k \mid j \neq k < n\})$ for all j-distinguishing $u \in N_t^n \cup \{w\}$. So the only thing left to check is that $\mathcal{P}$ is a coherent patch system. Let $U = \{u_0, \ldots, u_{n-1}\} \in [\mathcal{N}_{t+1}]^m$ and write U_j for $U \setminus \{u_j\}$ for each $j < n$. We need to check that U is $\mathcal{P}$-coherent:

- If $z \notin U$, then $U \subseteq \mathcal{N}_t$ and U is $\mathcal{P}$-coherent because $\partial \mathcal{N}_t$ is coherent.

- If $U = set(w)$, then U is $\mathcal{P}$-coherent by Lemma 3.2.10.

- In the case where $z \in U$, and $|U \cup set(w)| = n-1$, we can find $j, k < n$ such that $z \in U_j = U \cap set(w)$, $U_k \not\subseteq \mathcal{N}_t$ and $U_k \not\subseteq set(b)$. Then, by the above, $G_\ell \in P_2(U_k)$. Moreover, by the choice of ℓ, there is $m \neq \ell$, such that $G_m \in P_2(U_j)$.

  Take any $X_r \in P_2(U_r)$ for each $r < n$. Now we can choose $p_r \in X_r$, for each $r < n$, with $p_j \in X_j \cdot G_m$ and $p_k \in X_k \cdot G_\ell$. Since $l \neq m$ and therefore $H(p_j, p_k)$ does not hold, we have $E(p_j, p_k)$ by ($\dagger$). Thus $\{p_0, \ldots, p_{n-1}\}$ is not independent.

- In the remaining cases, $z \in U$ and $|U \cup set(w)| < n-1$. Then there are distinct $j, k < n$ such that $z \in U_j, U_k \not\subseteq \mathcal{N}_t$ and $U_j, U_k \not\subseteq set(w)$. So by the above, we have $P_2(U_j) = P_2(U_k) = \nu$.

> Take any $X_r \in P_2(U_r)$ for each $r < n$. Then $X_j, X_k \in \nu$, and thus $X_j \cdot X_k \in \nu$ and therefore not independent. So there are $p_j, p_j \in X_j \cdot X_k$ such that $E(p_j, p_k)$. For the other $s \neq j, k$ just choose any $p_s \in X_s$. Then $\{p_0, \ldots, p_{n-1}\}$ is not independent.

This shows that $\mathcal{P}$ is coherent. Now Lemma 3.2.11(iv) gives us that there is an ultrafilter network $\mathcal{N}_{t+1} \supseteq \mathcal{N}'$ such that $\partial\mathcal{N}_{t+1} = \mathcal{P}$. So we have $\mathcal{N}_{t+1} \supseteq \mathcal{N}_t$, with $w \equiv_i v$, and $a \in \mu = \mathcal{N}_{t+1}(w)$. So $\exists$ is able to respond to any move made by $\forall$ – she has a winning strategy. $\square$

Recall that RCA_n is a variety. This means that there is an equational axiomatisation of it, which in particular only involves universal quantifiers. Noting that the conjunction of universal statements is universal, we can obtain an axiomatisation $\Sigma = \{\gamma_0, \gamma_1, \ldots\}$ of RCA_n where the axioms gradually get stronger, i.e. γ_i implies all the γ_j with $j \leq i$. Fix such an axiomatisation in the following.[2] We are now ready to prove the main theorem of this section:

Theorem 3.3.7. *The following statements are true:*

(i) For all $k < \omega$ there is $m < \omega$ such that if Γ is an infinite graph and $\mathcal{A}(\Gamma) \models \gamma_m$, then $\chi(\Gamma) > k$.

(ii) For all $m < \omega$ there is $k < \omega$ such that if Γ is a graph and $\chi(\Gamma) > k$, then $\mathcal{A}(\Gamma) \models \gamma_m$.

Proof. First note that if Γ is a graph and $\chi(\Gamma)$ is finite, we have $\chi(\Gamma \times n) = n\chi(\Gamma)$, and that $\chi(\Gamma)$ is infinite iff $\chi(\Gamma \times n)$ is infinite.

(i) Let $k < \omega$. From Proposition 3.3.2 we know that every infinite model of $\mathcal{U} \cup \{\gamma_m \mid m < \omega\}$ is also a model of $\mathcal{U} \cup \{\theta_k \mid k < \omega\}$. Define the following theory to say a model is infinite:

$$T_\infty := \{\phi_i \mid i < \omega\} \qquad \text{where } \phi_m := \exists p_0, \ldots, p_{m-1} : \mathcal{G}\left(\bigwedge_{i<j<m} p_i \neq p_j\right).$$

By Corollary 2.1.18 of the compactness theorem, there is $m < \omega$ such that $\mathcal{U} \cup \{\gamma_m\} \cup T_\infty \vdash \theta_{nk}$. Let Γ be an infinite graph and assume $\mathcal{A}(\Gamma) \models \gamma_m$. Consider the structure $M(\Gamma) = (\mathcal{A}(\Gamma), \Gamma \times n, \wp(\Gamma \times n))$ and note that $M(\Gamma)$ is infinite. Then we have $M(\Gamma) \models \mathcal{U} \cup \gamma_m \cup T_\infty$ and thus $M(\Gamma) \models \theta_{nk}$. This means that $\Gamma \times n$ is not the union of nk independent sets from $\wp(\Gamma \times n)$. Thus $\chi(\Gamma \times n) > nk$.

(ii) Let $m < \omega$. From Proposition 3.3.6 we know that every model of $\mathcal{U} \cup \{\theta_k \mid k < \omega\}$ is also a model of $\mathcal{U} \cup \{\gamma_m \mid m < \omega\}$. So again by Corollary 2.1.18, there is $k < \omega$ such that $\mathcal{U} \cup \{\theta_k\} \vdash \gamma_m$. Let Γ be a graph with $\chi(\Gamma) > k/n$. Then $\chi(\Gamma \times n) > k$. Consider the structure $M(\Gamma) = (\mathcal{A}(\Gamma), \Gamma \times n, \wp(\Gamma \times n))$. Since $\chi(\Gamma \times n) > k$, $\Gamma \times n$ is not the

[2]This only shows the existence of such an axiomatisation. There are also concrete axiomatisations with that property available, e.g. the one from Hirsch and Hodkinson [1997].

union of k independent sets of $\wp(\Gamma \times n)$. So $M(\Gamma) \models \mathcal{U} \cup \{\theta_k\}$ and hence $M(\Gamma) \models \gamma_m$. It follows that $\mathcal{A}(\Gamma) \models \gamma_m$. □

3.4 Direct & Inverse Systems of Algebras and Graphs

We will apply the results from the previous sections to build an algebra, using direct and inverse systems, that satisfies an arbitrary number of axioms, while its canonical extension only satisfies a bounded number.

Lemma 3.4.1. *If there is an axiomatisation T of RCA_n, that contains only finitely many non-canonical formulas $T_{NC} \subseteq T$, then there is $s_0 < \omega$, so that for all $s < \omega$ there is $s^+ < \omega$ such that for all cylindric algebras $\mathcal{A}$ with $\mathcal{A}^\sigma \models \gamma_{s_0}$ we have $\mathcal{A} \models \gamma_{s^+} \implies \mathcal{A}^\sigma \models \gamma_s$.*

Proof. We can use Corollary 2.1.18 of the compactness theorem repeatedly to obtain the result, because T and Σ have the same models. Firstly, there is s_0 so that $\{\gamma_{s_0}\} \vdash T_{NC}$. We also know that for $s < \omega$, we can find a finite subset of canonical formulas $T_s \subseteq T$ such that $T_s \cup T_{NC} \vdash \gamma_s$. Moreover, there is $s^+ < \omega$, so that $\{\gamma_{s^+}\} \vdash T_s$.

Now let $\mathcal{A}$ be a cylindric algebra such that $\mathcal{A}^\sigma \models \gamma_{s_0}$. Then, if $\mathcal{A} \models \gamma_{s^+}$, we have $\mathcal{A} \models T_s$. Because the formulas in T_s are canonical, we have $\mathcal{A}^\sigma \models T_s$. Since $\mathcal{A}^\sigma \models \gamma_{s_0}$, we also have $\mathcal{A}^\sigma \models T_{NC}$. But we know if $\mathcal{A}^\sigma \models T_s \cup T_{NC}$, then $\mathcal{A}^\sigma \models \gamma_s$. □

We are interested in the particular case where we have a direct system of BAOs $A_1 \subseteq A_2 \subseteq \dots$ and an inverse system of the corresponding atom structures built from the ultrafilters $(A_1)_+, (A_2)_+, \dots$. The following generalises a result from [Goldblatt, 1993, p. 46] to BAOs.

Theorem 3.4.2. *Let $A_1 \subseteq A_2 \subseteq \dots$ be a direct system of algebras where the homomorphism is the natural embedding and the carrier is ω. For each $i \le j < \omega$ let φ_{ji} be the following map:*

$$\varphi_{ji} : (A_j)_+ \to (A_i)_+, \mu \mapsto A_i \cap \mu.$$

Then $\{(A_i)_+ \mid i < \omega\}$ with the family of maps φ_{ji} is an inverse system of atom structures and the inverse limit is isomorphic to the atom structure built from the ultrafilters of the direct limit of the algebras:

$$\varprojlim\{(A_i)_+ \mid i < \omega\} \cong \left(\bigcup_{i<\omega} A_i\right)_+$$

Proof. First we remark that $A_i \cap \mu$ certainly is an ultrafilter of A_i, if $j \ge i$ and $\mu \in (A_j)_+$, so the map φ_{ji} is well defined. For $\Omega \in L \setminus L_{BA}$, we will write Ω^k for the interpretation in A_k and Ω^ω for the interpretation in the direct limit. We write R^k_Ω for the interpretation of the corresponding relation in $(A_k)_+$ and $R^{\lim}_\Omega$ for the interpretation in the inverse limit.

We show that $\{(A_i)_+ \mid i < \omega\}$ together with the $\varphi_{ji}, i \leq j < \omega$ is indeed an inverse system of atom structures. Certainly, ω is a directed and partially ordered set. By assumption we have a corresponding atom structure $(A_i)_+$ for each $i \in I$. Fix $i \leq j < \omega$. To check that φ_{ji} is a homomorphism, consider an n-ary operator $\Omega \in L \setminus L_{BA}$. We then have a $(n+1)$-ary relation R^j_Ω on $(A_j)_+$. Suppose $(\mu_0, \ldots, \mu_{n-1}, \nu) \in R^j_\Omega$. We need to check that

$$(\varphi_{ji}\mu_0, \ldots, \varphi_{ji}\mu_{n-1}, \varphi_{ji}\nu) = (A_i \cap \mu_0, \ldots, A_i \cap \mu_{n-1}, A_i \cap \nu)$$

is in R^i_Ω. Suppose for a contradiction that $(A_i \cap \mu_0, \ldots, A_i \cap \mu_{n-1}, A_i \cap \nu) \notin R^i_\Omega$. Then, by definition, $\Omega^i(A_i \cap \mu_0, \ldots, A_i \cap \mu_{n-1}) \not\subseteq A_i \cap \nu$. So there is $x \in \Omega^i(A_i \cap \mu_0, \ldots, A_i \cap \mu_{n-1})$ and $x \notin A_i \cap \nu$ and since x is in A_i we have $x \notin \nu$. But then

$$x \in \Omega^i(\mu_0, \ldots, \mu_{n-1}), \qquad x \notin \nu.$$

But this means by definition that $(\mu_0, \ldots, \mu_{n-1}, \nu) \notin R^j_\Omega$, which contradicts the initial assumption. So φ_{ji} is a homomorphism. Moreover, we have that each ultrafilter $\mu \in (A_i)_+$ is a subset of A_i, so for each $i \in I$, $\varphi_{ii}\mu = A_i \cap \mu = \mu$, and φ_{ii} is the identity. Lastly, we have for an ultrafilter $\mu \in A_k$ that for all $i \leq j \leq k < \omega$ the following holds:

$$\varphi_{ji}(\varphi_{kj}\mu) = \varphi_{ji}(A_j \cap \mu) = A_i \cap A_j \cap \mu = A_i \cap \mu = \varphi_{ki}(\mu).$$

So we have an inverse system of atom structures.

Next we show that for each $i \leq j < \omega$, φ_{ji} is surjective. If $\mu \subseteq A_i$ is an ultrafilter, then, since $A_i \subseteq A_j$, by the boolean prime ideal theorem, it can be extended to an ultrafilter μ' in A_j. Now $\varphi_{ji}\mu' = \mu$.

Finally, we prove that the two limits of the systems are isomorphic. Write A_ω for the direct limit $\left(\bigcup_{i \in I} A_i\right)$ and A_+ for the inverse limit $\varprojlim\{(A_i)_+ \mid i < \omega\}$. Define two maps

$$f : (A_\omega)_+ \to A_+, \mu \mapsto (A_k \cap \mu : k < \omega),$$
$$g : A_+ \to (A_\omega)_+, \bar{\mu} \mapsto \bigcup_{k<\omega} \bar{\mu}_k.$$

We will show that they are both homomorphisms and that $f \circ g = id$ and $g \circ f = id$. They are inverses. For $\mu \in (A_\omega)_+$ we have:

$$\begin{aligned} g(f(\mu)) &= g((A_k \cap \mu : k < \omega)) \\ &= \bigcup_{k<\omega} ((A_k \cap \mu : k < \omega))_k \\ &= \bigcup_{k<\omega} (A_k \cap \mu) \\ &= \mu. \end{aligned}$$

For $\bar{\mu} \in A_+$ we have:

$$\begin{aligned} f(g(\bar{\mu})) &= f\left(\bigcup_{k<\omega} \bar{\mu}_k\right) \\ &= \left(A_k \cap \left(\bigcup_{k<\omega} \bar{\mu}_k\right) : k < \omega\right) \\ &= \bar{\mu}. \end{aligned}$$

It remains to show that f, g are homomorphisms. Let $\Omega \in L \setminus L_{BA}$ be an n-ary operator. Suppose $R^{\omega}_{\Omega}(\mu_0, \ldots, \mu_{n-1}, \nu)$ holds for some ultrafilters $\mu_0, \ldots, \mu_{n-1}, \nu \subseteq A_\omega$. Then, by definition,

$$\begin{aligned} \Omega^{\omega}(\mu_0, \ldots, \mu_{n-1}) \subseteq \nu &\iff \Omega^k(A_k \cap \mu_0, \ldots, A_k \cap \mu_{n-1}) \subseteq A_k \cap \nu \text{ for all } k < \omega \\ &\iff R^k_{\Omega}(A_k \cap \mu_0, \ldots, A_k \cap \mu_{n-1}, A_k \cap \nu) \text{ for all } k < \omega \\ &\iff R^{\lim}_{\Omega}(f\mu_0, \ldots, f\mu_{n-1}, f\nu) \end{aligned}$$

Since g is the inverse of f, this shows that both f and g are homomorphisms. This completes the proof, as we now have an isomorphism between $(A_\omega)_+$ and A_+. □

Recall from Theorem 2.4.8, that for all $2 \leq \ell \leq k < \omega$ we can construct graphs $\Gamma_0, \Gamma_1, \ldots$ in a way such that we have $\chi(\Gamma_s) = k$ and $\chi(\varprojlim \Gamma_s) = \ell$ for finite s, and furthermore

$$\Gamma_0 \overset{f_{10}}{\twoheadleftarrow} \Gamma_1 \overset{f_{21}}{\twoheadleftarrow} \ldots,$$

where the f_{ij} are surjective p-morphisms.

Lemma 3.4.3. *If $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ are graphs and there is a family of surjective p-morphisms $\{f_{ji} \mid j \geq i\}$ such that*

$$\Gamma_0 \overset{f_{10}}{\twoheadleftarrow} \Gamma_1 \overset{f_{21}}{\twoheadleftarrow} \ldots,$$

then there is a family of surjective p-morphisms of atom structures $\{\hat{f}_{ji} \mid j \geq i\}$ such that

$$At(\Gamma_0) \overset{\hat{f}_{10}}{\twoheadleftarrow} At(\Gamma_1) \overset{\hat{f}_{21}}{\twoheadleftarrow} \ldots.$$

Proof. Let $k < \ell < \omega$. For each surjective p-morphism of graphs $f_{\ell k}$ there is a corresponding natural map $\hat{f}_{\ell k}$ defined as follows:

$$\begin{aligned} \hat{f}_{\ell k} : At(\Gamma_\ell) &\to At(\Gamma_k) \\ (K, \sim) &\mapsto (f_{\ell k} \circ K, \sim). \end{aligned}$$

To see that this map is a surjective p-morphism, we need to check the following:

(i) surjectivity;

(ii) the forth property of the cylindrification relations, i.e. if we have $i < n$ and $(K^1, \sim^1) \equiv_i (K^2, \sim^2)$ then $\hat{f}_{\ell k}(K^1, \sim^1) \equiv_i f_{\ell k}(K^1, \sim^2)$;

(iii) the back property of the cylindrification relations, i.e. if we have $i < n$ and $\hat{f}_{\ell k}(K^1, \sim^1) \equiv_i (J^2, \sim^2)$, then there is $(K^2, \sim^2)$ such that $\hat{f}_{\ell k}(K^2, \sim^2) = (J^2, \sim^2)$ and $(K^1, \sim^1) \equiv_i (K^2, \sim^2)$;

(iv) diagonals are preserved, i.e. $(K, \sim) \in D^{\ell}_{ij} \iff f_{\ell k}(K, \sim) \in D^k_{ij}$;

(v) that if $(K, \sim) \in At(\Gamma_\ell)$, then $f_{\ell k}(K, \sim) \in At(\Gamma)$.

To show (i) let $(K', \sim) \in At(\Gamma_k)$. If K' is not defined anywhere, we let K be undefined everywhere as well. If there are $i < j < n$ such $i \sim j$ and $K'(i) = K'(j)$ is defined, there is $p \in \Gamma_\ell \times n$ such that $f_{\ell k}(p) = K'(i)$. Define $K(i) = K(j) = p$ and let K be undefined for the remaining values in that case. Finally, if K' is defined on all values $i < n$, then $im(K')$ is not independent, so there are $i < j < n$ such that there is an edge from $K'(i)$ to $K'(j)$. Since $f_{\ell k}$ is surjective, there is a $p_i \in \Gamma_\ell \times n$ such that $f_{\ell k}(p_i) = K'(i)$. By the back property of $f_{\ell k}$, there is $p_j \in \Gamma_\ell \times n$ such that there is an edge between p_j and p_i and $f_{\ell k}(p_j) = K'(j)$. For the remaining $s \neq i, j$, we can also find vertices $p_s \in \Gamma_\ell \times n$, such that $f_{\ell k}(p_s) = K'(s)$. Now define $K(i) = p_i$. By construction, $(K, \sim) \in At(\Gamma_\ell)$ in all three cases and, moreover, we have $f_{\ell k}(K, \sim) = (K', \sim)$.

For (ii) we have for $(K^1, \sim^1), (K^2, \sim^2) \in At(\Gamma_\ell)$ and $i < \omega$ that

$$\begin{aligned} & (K^1, \sim^1) \equiv_i (K^2, \sim^2) \\ \Longrightarrow\ & K^1(i) = K^2(i) \text{ and } \sim^1_i = \sim^2_i \\ \Longrightarrow\ & f_{\ell k}(K^1(i)) = f_{\ell k}(K^2(i)) \text{ and } \sim^1_i = \sim^2_i \\ \Longrightarrow\ & \hat{f}_{\ell k}(K^1, \sim^1) \equiv_i \hat{f}_{\ell k}(K^2, \sim^2). \end{aligned}$$

For (iii) we have for $(K^1, \sim^1) \in At(\Gamma_\ell), (J^2, \sim^2) \in At(\Gamma_k)$ and $i < \omega$ that

$$\begin{aligned} & \hat{f}_{\ell k}(K^1, \sim^1) \equiv_i (J^2, \sim^2) \\ \Longrightarrow\ & f_{\ell k}(K^1(i)) = J^2(i) \text{ and } \sim^1_i = \sim^2_i. \end{aligned}$$

Now take $(K^2, \sim^2)$ such that $K^2(i) = K^1(i)$ and if $j \neq i$, we choose $K^2(j)$ from the pre-image of $f_{\ell k}(J^2(j))$ if J^2 is defined for j, otherwise we make it undefined. Then we have $\hat{f}_{\ell k}(K^2, \sim^2) = (J^2, \sim^2)$ and $(K^1, \sim^1) \equiv_i (K^2, \sim^2)$.

To see that diagonals are preserved (iv) note that

$$(K, \sim) \in D^{\ell}_{ij} \iff i \sim j \iff \hat{f}_{\ell k}(K, \sim) \in D^k_{ij}.$$

Lastly, for (v), suppose $(K, \sim) \in At(\Gamma_\ell)$ and $|n/{\sim}| = n$. Clearly the domain of K is preserved by $\hat{f}_{\ell k}$. Moreover, since the image of K is not independent, we have by the forth property of $f_{\ell k}$ that the image of K' is neither. The other cases follow directly from the definition of $\hat{f}_{\ell k}$. □

Theorem 3.4.4. *The class of representable cylindric algebras RCA_n has no axiomatisation containing only finitely many non-canonical formulas.*

Proof. Suppose for a contradiction that T is a canonical axiomatisation of RCA_n with only finitely many non-canonical formulas $T_{NC} \subseteq T$. Let s_0 be the value from Lemma 3.4.1. By Theorem 3.3.7, we can find $\ell < \omega$, so that for any algebra $\mathcal{A}(\Gamma)$ from a graph Γ with $\chi(\Gamma) > \ell$ we have $\mathcal{A}(\Gamma) \models \gamma_{s_0}$. Let $m = \ell + 1$. By Theorem 3.3.7 again, there is $s < \omega$ such that for any algebra $\mathcal{A}(\Gamma)$ from an infinite graph Γ, $\mathcal{A}(\Gamma) \models \gamma_s$ implies $\chi(\Gamma) > m$. Let s^+ be the value from Lemma 3.4.1 for this s. Now, again by Theorem 3.3.7, there is $k < \omega$, such that for any algebra $\mathcal{A}(\Gamma)$ from a graph Γ with $\chi(\Gamma) > k$, we have $\mathcal{A}(\Gamma) \models \gamma_{s^+}$.

Now take graphs $\Gamma_0, \Gamma_1, \ldots$ from Theorem 2.4.8 such that $\chi(\Gamma_j) = k + 1$ for all $j < \omega$ and $\chi(\varprojlim \Gamma_j) = m$ and

$$\Gamma_0 \overset{f_{10}}{\twoheadleftarrow} \Gamma_1 \overset{f_{21}}{\twoheadleftarrow} \ldots,$$

where the f_{ij} are surjective p-morphisms. Given these graphs, using Lemma 3.4.3 it is now easy to construct embeddings:

$$\mathcal{A}(\Gamma_0) \hookrightarrow \mathcal{A}(\Gamma_1) \hookrightarrow \ldots.$$

Define $\mathcal{A} = \varinjlim \mathcal{A}(\Gamma_s)$. Then, because $\chi(\Gamma_j) > k$, we have $\mathcal{A}(\Gamma_j) \models \gamma_{s^+}$ for all $j < \omega$. As these are are universal formulas, they are preserved by direct limits, and we therefore have $\mathcal{A} \models \gamma_{s^+}$.

Moreover, from Theorem 3.4.2 we get

$$At(\varprojlim \Gamma_j) \cong \varprojlim At(\Gamma_j) \cong \mathcal{A}_+$$

and thus $\mathcal{A}(\varprojlim \Gamma_j) \cong \mathcal{A}^\sigma$. We chose the graphs so that $\chi(\varprojlim \Gamma_j) = m > \ell$. So $\mathcal{A}^\sigma \models \gamma_{s_0}$, and therefore by Lemma 3.4.1 $\mathcal{A}^\sigma \models \gamma_s$. But then, since $\mathcal{A}^\sigma$ is clearly infinite, $m = \chi(\varprojlim \Gamma_j) > m$, a contradiction. □

4

Axiomatisations of Representable Diagonal-free Algebras

In this chapter we will outline a strategy to generalise the result from the previous chapter for representable cylindric algebras to representable diagonal-free algebras, the diagonal-free version of RCA. There are several conceivable ways to achieve this extension. The most obvious way would be to define diagonal-free algebras from graphs, and then try to discover a modified route through the proof from the last chapter that does not require the diagonals. We believe this is possible, but it would essentially duplicate the effort, as we would need to re-prove most results. We think there is a more elegant way that will not only be a unified approach for the cylindric and the diagonal-free cases, but possibly simplify the proof for other variations of cylindric algebras as well.

The idea is to generalise the proof from the last chapter, so that it can deal with both cylindric and diagonal-free algebras. The key result that we want to utilise for this is Theorem 2.3.36 (by Johnson [1969]) that relates the representability of cylindric and diagonal-free algebras. However, in order to apply the theorem we need to restrict ourselves to cylindric algebras that are generated by their $<n$-dimensional elements.

In the following section we will provide some arguments in favour of our conjecture. We think that all the machinery from the previous chapter will still work for models of a well chosen subset of the theory $\mathcal{U}$, that will give us generalised algebra-graph-systems. The generalised algebras from graphs will be generated by their $<n$-dimensional elements, and have the same connection to the chromatic number as the algebras in the previous chapter. In the last section we will explain how these results can be used to prove the theorem for representable diagonal-free algebras.

4.1 Generalised Algebra-Graph-Systems

The algebras from graphs $\mathcal{A}(\Gamma)$ do not seem to be generated by their $<n$-dimensional elements. So, using an idea from Kurucz [2010], we define a subalgebra that is generated by its $<n$-dimensional elements.

Definition 4.1.1. Let Γ be a graph. Then we define $\overline{\mathcal{A}}(\Gamma)$ to be the subalgebra of $\mathcal{A}(\Gamma)$ generated by the elements $\{S_i(B) \mid i < n, B \subseteq \Gamma \times n\}$.

Proposition 4.1.2. $\overline{\mathcal{A}}(\Gamma)$ *is an n-dimensional cylindric algebra generated by its $<n$-dimensional elements.*

Proof. Firstly, by Proposition 3.1.4, $\overline{\mathcal{A}}(\Gamma)$ is an n-dimensional cylindric algebra. Let $S_i(B)$, for some $i < n$ and $B \in \Gamma \times n$, be an arbitrary element of $\overline{\mathcal{A}}(\Gamma)$. Take $(K, \sim) \in S_i(B)$ and $(K', \sim')$ such that $(K, \sim) \equiv_i (K', \sim')$. Then, as $K(i)$ is defined and $K(i) = K'(i)$, the value at $K'(i)$ is defined as well. Thus $(K', \sim') \in F_i$ and therefore $(K', \sim') \in S_i(B)$. This shows that $c_i S_i(B) = S_i(B)$. □

This gives us the following:

Notation. We will write $\mathcal{A}^-$ for the diagonal-free reduct of a cylindric algebra $\mathcal{A}$.

Lemma 4.1.3. *Let Γ be a graph. Then $\overline{\mathcal{A}}(\Gamma)$ is a representable cylindric algebra if and only if the diagonal-free reduct $(\overline{\mathcal{A}}(\Gamma))^-$ is a representable diagonal-free algebra.*

Proof. If $\overline{\mathcal{A}}(\Gamma)$ is a representable cylindric algebra, then we can obtain a representation for $(\overline{\mathcal{A}}(\Gamma))^-$ by dropping the diagonals from the representation.

Conversely, suppose $(\overline{\mathcal{A}}(\Gamma))^-$ is a representable diagonal-free algebra. By Proposition 4.1.2 we know that $\overline{\mathcal{A}}(\Gamma)$ is generated by its $<n$-dimensional elements. So $\overline{\mathcal{A}}(\Gamma)$ is representable by Theorem 2.3.36. □

To continue with the proof we want $(\overline{\mathcal{A}}(\Gamma), \Gamma \times n, \wp(\Gamma \times n))$ to be an algebra-graph-system, but this is not the case. However, we could define a subset $\overline{\mathcal{U}}$ of the theory of $\mathcal{U}$ to obtain generalised a algebra-graph-systems that would make it one. A first idea would be to take all the universal sentences in $\mathcal{U}$ and 'manually' add all the sentences involving existential quantifiers that we need. The rationale of this is that most of the sentences needed in the proof were universal and that universal sentences would automatically hold on subalgebras. However, we also required some sentences involving existential quantifiers, so this approach does need some careful consideration. This problem remains open.

We think that by using this more general approach, we would be able to prove the main theorem of the previous chapter, Theorem 3.4.4, in the same way as done there. Moreover, we think that we could prove the theorem for RDf_n as well, giving us a unified approach.

4.2 Towards a Proof for RDf_n

Here we will argue how the analogue of Theorem 3.4.4 for RDf_n can be proved using the setup from the previous section. We assume a universal axiomatisation $\overline{\Sigma} = \{\delta_1, \delta_2, \dots\}$ of RDf_n, similar to the axiomatisation Σ of RCA_n. Unfortunately, we cannot use compactness here as the signatures do not match. However, in this particular case we believe that it may be possible to show the following:

Lemma 4.2.1. *The following statements are true:*

(i) For all $k < \omega$ there is $m < \omega$ such that if Γ is a graph and $(\overline{\mathcal{A}}(\Gamma))^- \models \delta_m$, then $\overline{\mathcal{A}}(\Gamma) \models \gamma_k$.

(ii) For all $m < \omega$ there is $k < \omega$ such that if Γ is a graph and $\overline{\mathcal{A}}(\Gamma) \models \gamma_k$, then $(\overline{\mathcal{A}}(\Gamma))^- \models \delta_m$.

Recall that one of the central results of the previous chapter was Theorem 3.3.7. With all the above we believe that it should be possible to prove the following analogue for diagonal-free algebras:

Theorem 4.2.2. *The following statements are true:*

(i) For all $k < \omega$ there is $m < \omega$ such that if Γ is an infinite graph and $(\overline{\mathcal{A}}(\Gamma))^- \models \delta_m$, then $\chi(\Gamma) > k$.

(ii) For all $m < \omega$ there is $k < \omega$ such that if Γ is a graph and $\chi(\Gamma) > k$, then $(\overline{\mathcal{A}}(\Gamma))^- \models \delta_m$.

Another important ingredient in the main proof is the result that we cannot have an arbitrary gap between the number of axioms satisfied by the algebra and its canonical extension. The analogue of 3.4.1 should also hold for diagonal-free algebras and can be proved in the same way as it is done in the previous chapter:

Lemma 4.2.3. *If there is an axiomatisation S of RDf_n, that contains only finitely many non-canonical formulas $S_{NC} \subseteq T$, then there is $s_0 < \omega$, so that for all $s < \omega$ there is $s^+ < \omega$ such that for all diagonal-free algebras $\mathcal{A}^-$ with $(\mathcal{A}^-)^\sigma \models \delta_{s_0}$ we have $\mathcal{A}^- \models \delta_{s^+} \implies (\mathcal{A}^-)^\sigma \models \delta_s$.*

These results should be enough to prove the analogue of the main theorem (3.4.4):

Theorem 4.2.4. *The class of representable diagonal-free algebras RDf_n has no axiomatisation containing only finitely many non-canonical formulas.*

We believe that it is possible to fill in the gaps in the argument above and to provide a full proof. Using the well known connection to modal logic, this would confirm the following conjecture from Kurucz [2010]:

Conjecture. Any axiomatisation of a logic L in the interval between K^n and $S5^n$ must contain infinitely many non-canonical formulas.

5
Further Remarks and Open Questions

We have shown that there is no axiomatisation of representable cylindric algebras containing only finitely many non-canonical formulas. There were already a number of results known, that revealed RCA_n is not so easy to grasp for finite $n \geq 3$. In particular, it had been established that there is no finite axiomatisation and no axiomatisation consisting of Sahlqvist formulas. However, our result was still surprising as RCA_n is known to be canonical. It shows that RCA_n is only barely canonical, and provides further evidence that it is not easy to work with this class.

One of the main choices made for this report was to define the first-order theory $\mathcal{U}$ to strengthen the results in Hirsch and Hodkinson [2009] using the generalisation technique. At the time several other approaches were investigated: the modification of the game used in Hirsch and Hodkinson [2009], the use of non-principal ultraproducts of algebras from graphs, and lastly the use of an explicit theory and compactness. The chosen approach turned out to be the best. Somewhat ironically this means that we used a technique from logic (compactness) to prove a result about cylindric algebras, an algebraisation of first-order logic that was invented to help prove results in logic.

We believe that similar techniques can be used to extend the result to polyadic algebras, and possibly other algebraisations of logics. We have already presented a strategy to further generalise the proof to deal with the representable diagonal-free algebras RDf and hope to give a complete proof in a future publication. The result would also have implications for modal logic: It would show that any axiomatisation of a logic L in the interval between K^n and $S5^n$ must contain infinitely many non-canonical formulas.

More generally, the proof techniques used here, in particular the random graph construction and their correspondence to algebras, are likely useful in many areas of algebraic logic and can also be applied elsewhere. As mentioned in the introduction, we exploit graphs as a source of 'bad partitions' for the algebras. The rich theory of graphs can most likely provide more useful results, now that the link is established. We hope that this report will further propagate the use of these methods.

A question that was not answered in this report is whether the result also holds for RCA_α with infinite $\alpha \geq \omega$. This is very likely the case, but would require a different approach.

Another open question is whether it is possible to change the signature of cylindric algebras to obtain a class that is canonically axiomatised. This would immediately imply that the class is canonical and might entail more positive properties. The existence of a canonical axiomatisation is therefore a good characteristic to test for in new algebras. We hope to inspire new research in this direction and to motivate the discovery of interesting subvarieties of representable cylindric algebras.

Bibliography

J. F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.

Hajnal Andréka. Complexity of equations valid in algebras of relations part I: Strong non-finitizability. *Annals of Pure and Applied Logic*, 89:149–209, 1997.

Hajnal Andréka, J. Donald Monk, and I. Németi. *Algebraic Logic*. János Bolyai Mathematical Society and Elsevier Science, Budapest, 2nd edition, 1991.

Sebastian Bauer, Ian Hodkinson, Frank Wolter, and Michael Zakharyaschev. On non-local propositional and local one-variable quantified CTL*. In *Proc. 9th International Symposium on Temporal Representation and reasoning*, pages 2–9, 2002.

Patrick Blackburn, Maarten de Rijke, and Yde Vedema. *Modal Logic*. Cambridge University Press, Cambridge, 2001.

Jan Van Den Bussche. Applications of Alfred Tarski's ideas in database theory. In *Proceedings of the 15th International Workshop on Computer Science Logic.*, 2001.

Chen Chung Chang and H. Gerome Keisler. *Model Theory*. Elsevier Science, Amsterdam, 3rd edition, 1990.

Reinhard Diestel. *Graph Theory*. Springer, Berlin, Heidelberg, New York, 3rd edition, 2006.

Paul Erdös. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

Krzysztof Goczyła, Aleksander Waloszek, and Wojciech Waloszek. Algebra of ontology modules for semantic agents. In Ngoc Nguyen, Ryszard Kowalczyk, and Shyi-Ming Chen, editors, *Computational Collective Intelligence. Semantic Web, Social Networks and Multiagent Systems*, volume 5796 of *Lecture Notes in Computer Science*, pages 492–503. Springer Berlin / Heidelberg, 2009.

Robert Goldblatt. *Mathematics of Modality*, volume 43 of *Lecture notes*. CSLI Publications, Stanford, CA, 1993.

Robert Goldblatt and Ian Hodkinson. The McKinsey–Lemmon logic is barely canonical. *The Australasian Journal of Logic*, 5:1–19, 2007.

George Grätzer. *Universal Algebra*. Springer, New York, 2nd edition, 2008.

Paul R. Halmos. *Algebraic logic*. Chelsea, New York, 1962.

Leon Henkin, J. Donald Monk, and Alfred Tarski. *Cylindric Algebras, Part I.* North-Holland, Amsterdam, London, 1971.

Leon Henkin, J. Donald Monk, and Alfred Tarski. *Cylindric Algebras, Part II.* North-Holland, Amsterdam, London, 1985.

Robin Hirsch and Ian Hodkinson. Axiomatising various classes of relation and cylindric algebras. *Logic Journal of the IGPL*, 5:209–229, 1997.

Robin Hirsch and Ian Hodkinson. *Relation Algebras by Games*. Elsevier Science, Amsterdam, 2002.

Robin Hirsch and Ian Hodkinson. Strongly representable atom structures of cylindric algebras. *Journal of Symbolic Logic*, 74:811–828, 2009.

Robin Hirsch, Ian Hodkinson, and Agi Kurucz. On modal logics between $K\times K\times K$ and $S5\times S5\times S5$. *Journal of Symbolic Logic*, 67:221–234, 2002.

Wilfrid Hodges. *A Shorter Model Theory.* Cambridge University Press, Cambridge, 1997.

Ian Hodkinson. Atom structures of cylindric algebras and relation algebras. *Annals of Pure and Applied Logic*, 89:117–148, 1997.

Ian Hodkinson and Yde Venema. Canonical varieties with no canonical axiomatisation. *Trans. Amer. Math. Soc*, 357:4579–4605, 2005.

Ian Hodkinson, Angelo Montanari, and Guido Sciavicco. Non-finite axiomatizability and undecidability of interval temporal logics with C, D, and T. In *Computer Science Logic*, pages 308–322, 2008.

James S. Johnson. Nonfinitizability of classes of representable polyadic algebras. *Journal of Symbolic Logic*, 34:344–352, 1969.

Agi Kurucz. On the complexity of modal axiomatisations over many-dimensional structures. In L. Beklemishev, V.Goranko, and V.Shehtman, editors, *Advances in Modal Logic, Volume 8*, pages 241–254. College Publications, 2010.

Maarten Marx. XML navigation and Tarski's relation algebras. In C.-H. Luke Ong, editor, *CSL*, volume 3634 of *Lecture Notes in Computer Science*, pages 1–2. Springer, 2005.

J. Donald Monk. Nonfinitizability of classes of representable cylindric algebras. *Journal of Symbolic Logic*, 34:331–343, 1969.

J. Donald Monk. An introduction to cylindric set algebras. *Logic Journal of the IGPL*, 8:451–496, 2000.

F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30(1):264–286, 1930.

Alfred Tarski. On the calculus of relations. *Journal of Symbolic Logic*, 6:73–89, 1941.

Alfred Tarski. Contributions to the theory of models, III. *Indagationes Mathematicae*, 58:56–64, 1955.

Yde Venema. Atom structures and Sahlqvist equations. *Algebra Universalis*, 38, 1997.

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