1 a A string $x \in \{0, 1\}^*$ is called *random* if $C(x) \geq |x|$ where $C(x)$ is the Kolmogorov complexity of x and $|x|$ is the length of x. Show that for all natural numbers n, there exists some x with $|x| = n$ such that $C(x) \geq n$.

b Consider the family of quadratic maps $F_d : \mathbb{R} \to \mathbb{R}$ given by $F_d(x) = x^2 + d$ where $d \leq 1/4$. When d is just slightly less than $1/4$, find the two fixed points of F_d, determine the type of each of these two fixed points and draw the phase portrait of F_d.

c With respect to the family F_d in part b, show that as d decreases and passes through the parameter value $d = -3/4$ one of the fixed points of F_d changes its type. Explain how a periodic doubling bifurcation takes place at $d = -3/4$.

d Explain how a family of maps like F_d in part b undergoes *periodic doubling route* to chaos and describe the main features of the family as it goes through this scenario.

The four parts carry equal marks.

2 Consider the iterated function system (IFS) defined by the four maps $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ with $i = 1, 2, 3, 4$ as follows

$$\begin{aligned} f_1 : \quad (x,y) &\mapsto (\tfrac{x}{4}, \tfrac{y}{4}) \\ f_2 : \quad (x,y) &\mapsto (\tfrac{x}{4}, \tfrac{y}{4}) + (\tfrac{3}{4}, 0) \\ f_3 : \quad (x,y) &\mapsto (\tfrac{x}{4}, \tfrac{y}{4}) + (0, \tfrac{3}{4}) \\ f_4 : \quad (x,y) &\mapsto (\tfrac{x}{4}, \tfrac{y}{4}) + (\tfrac{3}{4}, \tfrac{3}{4}) \end{aligned}$$

Let $\Sigma = \{1, 2, 3, 4\}$ and let $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the shift map, which takes every infinite sequence to its tail sequence.

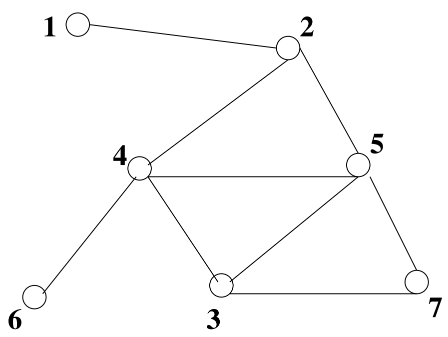
a Find the fixed points of these four maps and show that the unit square $S = [0,1] \times [0,1] \subset \mathbb{R}^2$ is a trapping region for the IFS, i.e., $f_i[S] \subset S$ for each $i = 1, 2, 3, 4$.

b Using the unit square $S = [0,1] \times [0,1]$ as the root of the IFS tree, obtain the nodes of the first and second levels of the tree and sketch them. Describe the attractor A of the IFS and find its similarity dimension. Discuss why the attractor A is a fractal set.

c Construct a one-to-one surjective map $h : \Sigma^{\mathbb{N}} \to A$ from the code space $\Sigma^{\mathbb{N}}$ to the IFS attractor A. Then show that we can construct a map $f : A \to A$ such that $f \circ h = h \circ \sigma$. Explain, without actually providing any definitions or giving the details of any proof, how you would show

- that f is sensitive to initial conditions,
- that f is topologically transitive and,
- that f has a dense orbit.

The three parts carry, respectively, 20%, 35%, and 45% of the marks.



3 a Consider the graph given in the diagram above.

i) Compute the clustering coefficient of the nodes 2, 4, 5, 7.

ii) Find the degree of each node and the probability distribution of the degree of the nodes.

b Describe a Barabási-Albert like algorithm that would start at time $t = 8$ with the graph above in part a and generate a growing graph by adding a single node labelled as t at each point t of discrete time such that the network would satisfy a power law distribution with exponent $\gamma = 3$ for large values of the degree.

c With respect to the algorithm in part b, derive for $t \geq 8$, without specifying any boundary conditions, the recursive equation for the probability $p(k, s, t + 1)$ that the node $s \leq t$ has degree k at time $t + 1$.

The three parts carry, respectively, 45%, 25%, and 30% of the marks.

4 a Consider a Kauffman Network with $K = N$ where N is the size and K is the fixed connectivity of the network. Given the large connectivity, assume before the network enters into a cycle it traverses a random walk in the state space of size $\Omega = 2^N$ starting from a random $\Sigma_0 \in \{0,1\}^N$. Let q_t be the probability that the orbit is still unclosed at time t.

i) Show that the probability p_{t+1} of terminating the excursion at time $t+1$ is

$$p_{t+1} = \frac{t+1}{\Omega} q_t.$$

ii) Find the probability that the closure of the excursion at time t provides a cycle of length $\ell \leq t$.

iii) Show that the probability $P(L)$ that a given initial state is in the basin of attraction of a cycle of length L is given by:

$$P(L) = \sum_{t=L}^{\Omega} \frac{p_t}{t}.$$

b i) For a 2 person game, define the two notions of *Nash equilibrium* and *Pareto optimal*.

ii) Consider the following 2 person game.

$(1_\downarrow, 2 \rightarrow)$	A	B
A	(1,1)	(4,2)
B	(2,4)	(3,3)

Find any Nash equilibrium of the game and any Pareto optimal strategy profile.

The two parts carry, respectively, 70%, and 30% of the marks.