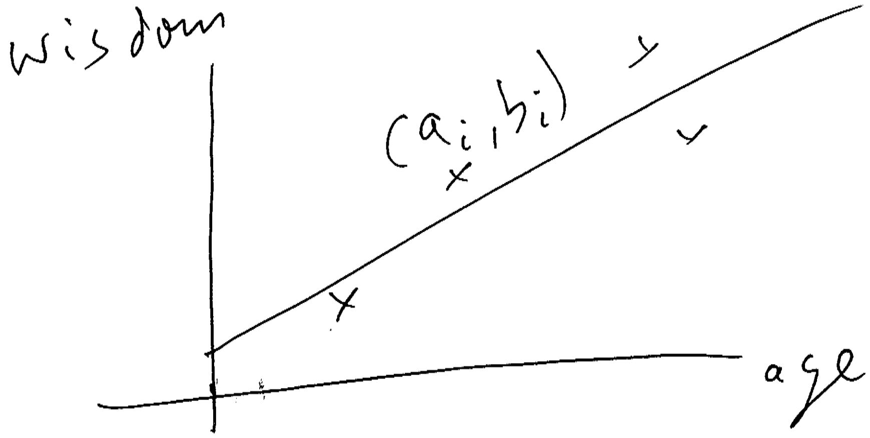
least Square Method

CT Lecture notes 07/02/08



$y = mx + c$

(a_i, b_i)

$$\left.\begin{aligned} b_1 &= m a_1 + c \\ b_2 &= m a_2 + c \\ b_3 &= m a_3 + c \end{aligned}\right\} \text{ Suppose there are only three data points}$$

$$\begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ a_3 & 1 \end{pmatrix} \begin{pmatrix} m \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$Ax = b$

$\|Ax - b\|_2$ minimum if $\|Ax - b\|_2^2$ is minimum

$A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ (very simple example)

Find λ such $\left\| A(\lambda) - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\|^2$ is minim

$$\Rightarrow \left\| \begin{pmatrix} 0 \\ \lambda \\ 2\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} -2 \\ \lambda - 1 \\ 2\lambda \end{pmatrix} \right\|_2^2$$

$$f(\lambda) = 4 + (\lambda - 1)^2 + 4\lambda^2$$

$$f'(\lambda) = 2(\lambda - 1) + 8\lambda = 0 \Rightarrow 10\lambda = 2$$

$$\Rightarrow \lambda = \frac{1}{5}$$

So the best solution is $\lambda = \frac{1}{5}$

$Ax = b$ can only be solved approximately.

So find x such that $\|Ax - b\|_2^2$ is minimized.

$b = b_r + b_n$ $\quad b_r \in \text{range}(A)$ and $b_n \in \text{Null}(A^T)$

$$\|Ax - b_r - b_n\|_2^2 = (Ax - b_r - b_n)\cdot(Ax - b_r - b_n)$$
$$= (Ax - b_r)\cdot(Ax - b_r) + b_n \cdot b_n \cancel{- 2\underbrace{b_n}_{\in \text{Null}(A^T)} \cdot \underbrace{(Ax - b_r)}_{\in \text{range}(A)}} =$$

(Since any vector in Null(A^T) is orthogonal to any vector in range(A))

$$= \|Ax - b_r\|_2^2 + \|b_n\|_2^2$$

will be minimum when $Ax = b_r$ which always has a solution (since $b_r \in \text{range}(A)$)

Theorem

$$Ax = b_r \overset{\text{iff}}{\iff} A^T A x = A^T b$$

Proof Suppose $Ax = b_r$ (we have to show RHS).

then $A^T A x = A^T b_r$ (remember $\boxed{b = \underbrace{b_r}_{\text{range}(A)} + \underbrace{b_n}_{\in \text{Null}(A^T)}}$)

(since $A^T b_n = 0$) $= A^T (b_r + b_n) = A^T b$

Next assume $A^T A x = A^T b$

$A^T (Ax - b) = 0$ (have to show $Ax - b_r = 0$)

$\Rightarrow Ax - b \in \text{Null}(A^T)$

So $Ax - b = b_1 \in \text{Null}(A^T)$

$\Rightarrow Ax - b_1 = b \Rightarrow Ax = b_r$ & $-b_1 = b_n$
by uniqueness of decomposition

$(A^TA = LL^T)$ $\quad L(L^Tx) = A^Tb$ $\qquad \begin{cases} A \in \mathbb{R}^{mn} \\ A^T \in \mathbb{R}^{nm} \end{cases}$

So solve $\boxed{A^TAx = A^Tb}$

$(A^TA)^T = A^T(A^T)^T = A^TA \Rightarrow A^TA$ is symmetric

[remember A^TA is +ve semi-definite i.e.

$x^T\underbrace{A^T}_{\in \mathbb{R}^n}Ax \geq 0$ for $\forall x \in \mathbb{R}^n$ since

$x^TA^TAx = (Ax)^TAx = (Ax)\cdot(Ax) \geq 0$]

Find L (lower triangular matrix) such that

$A^TA = LL^T$ $\qquad$ (then $LL^Tx = \underbrace{A^Tb}_{\text{known}}$)

Cholesky factorization

put $y = L^Tx$ &
solve $Ly = A^Tb$ for y $\Big\}$ then solve $L^Tx = y$ for x

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