$$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$$A = (a_1, a_2, \cdots, a_n)$$

$$\text{range}(A) = \{ Ax \mid x \in \mathbb{R}^n \}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}$$

$\text{range}(A) = \{ \alpha v \mid \alpha \in \mathbb{R} \}$

where

$$v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\text{Null}(A^T) = \{ z \mid A^T z = 0 \}$$

$$\left\{ \begin{array}{l} \forall z_1, z_2 \in \text{Null}(A^T) \\ \forall \lambda_1, \lambda_2 \in \mathbb{R} \end{array} \right\} \text{ Prove:}$$

$$\lambda_1 z_1 + \lambda_2 z_2 \in \text{Null}(A^T)$$

$$A^T(\lambda_1 z_1 + \lambda_2 z_2) =$$

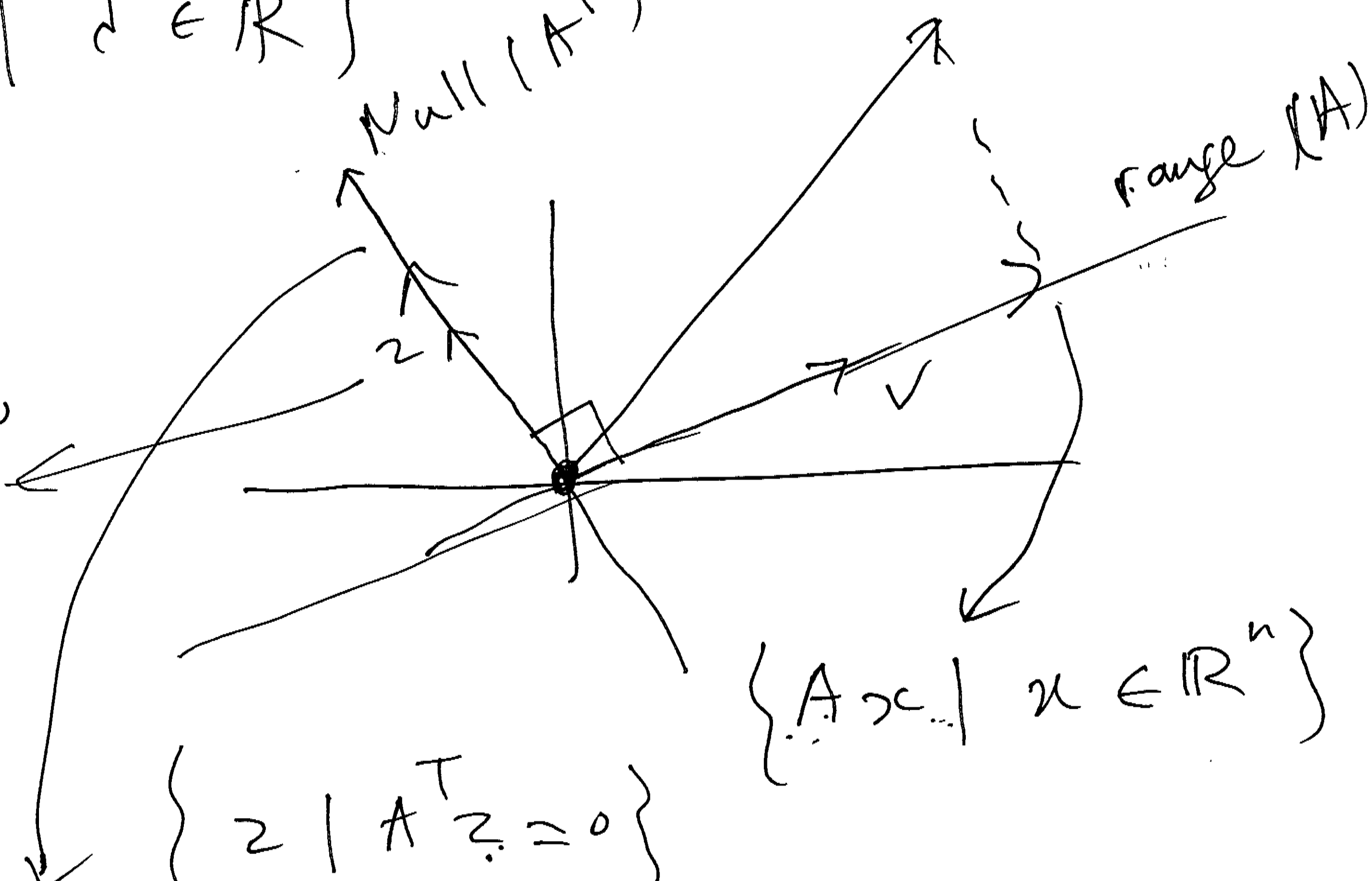
$$\lambda_1 \underbrace{A^T z_1}_{0} + \lambda_2 \underbrace{A^T z_2}_{0} = 0$$

$A \quad 2 \times 2$

$\text{range}(A) = \{\alpha v \mid \alpha \in \mathbb{R}\}$

$v \in \mathbb{R}^2$

$\vec{x} = \alpha v + \alpha' z$



$A^T z = 0$

$\{Ax \mid x \in \mathbb{R}^n\}$

$\{z \mid A^T z = 0\}$

$\cancel{v \in \text{Range}(A)} \wedge$

$v \in \text{Range}(A) \wedge \text{Null}(A^T)$

$A^T v = 0$

$\exists x.\ v = Ax$

$v \cdot v = 0$

$\|v\| = 0 \Rightarrow v = 0$

Prove:

A non-singular $\Rightarrow$ A^{-1} exists

$$AA^{-1} = A^{-1}A = I$$

$$AB = \begin{pmatrix} 1 & & O \\ & 1 & \\ & & \ddots & \\ O & & & 1 \end{pmatrix}$$

$$A(b_1, \cdots, b_m) = \begin{pmatrix} 1 & & O \\ & 1 & \\ & & \ddots & \\ O & & & 1 \end{pmatrix}$$

$$Ab_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad Ab_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \cdots Ab_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$b_1, \cdots b_m$ exist.

So $\exists B$. $AB = I$. Similarly $CA = I$ for some C.

$$B = IB = (CA)B = C(AB) = CI = C$$

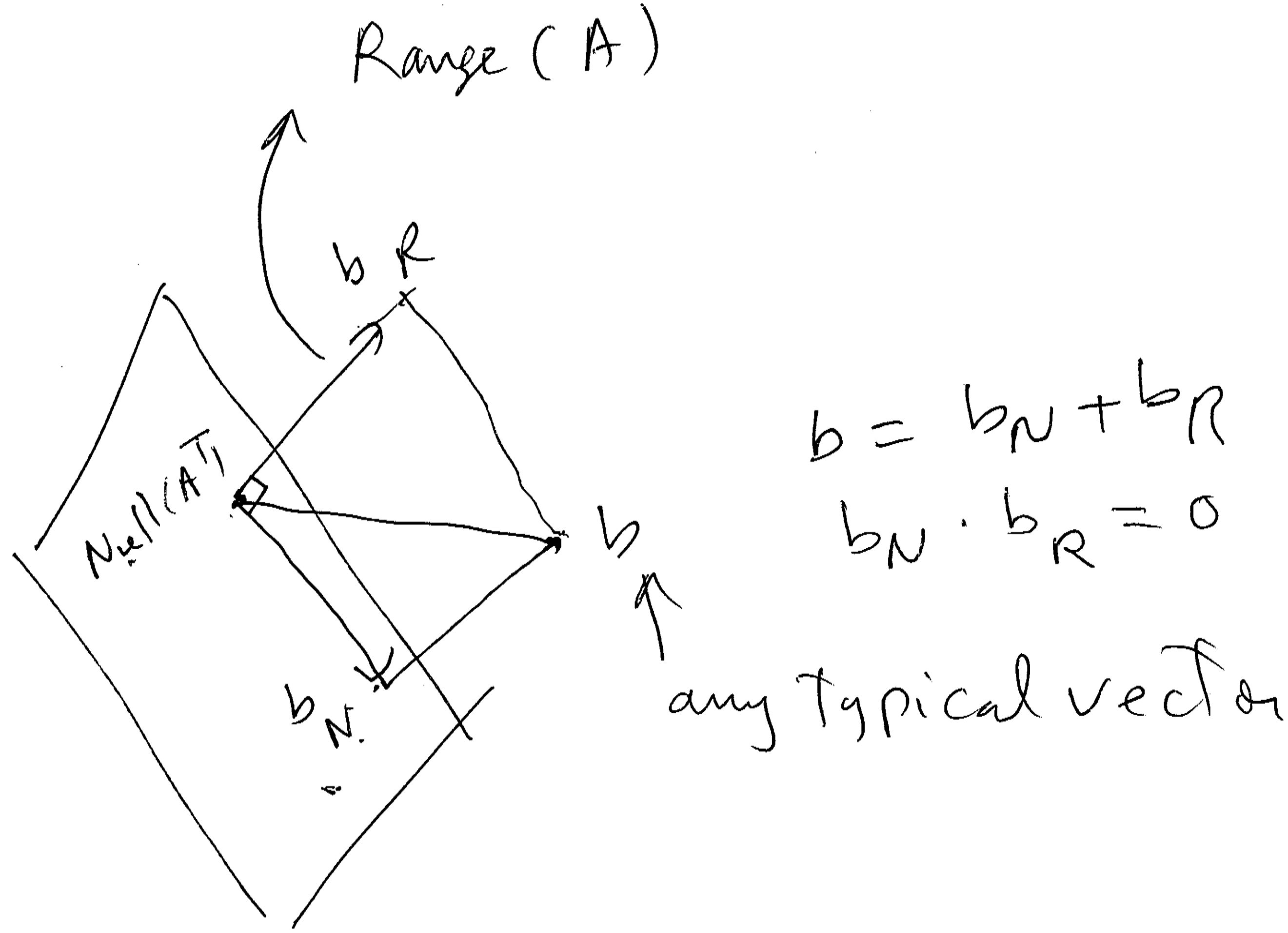
Prove $\left\{ \begin{array}{l} A \,\&\, B \text{ nonsingular} \Rightarrow \\ AB \text{ nonsingular} \end{array} \right.$

A nonsingular $\Leftrightarrow \{Az = 0 \text{ iff } z = 0\}$

Show $(AB)z = 0 \Leftrightarrow z = 0$

$ABz = 0 \Rightarrow A(Bz) = 0 \underset{\text{since } A \text{ is nonsingular}}{\Rightarrow} Bz = 0 \underset{\text{since } B \text{ is nonsingular}}{\Rightarrow}$

$z = 0$



$$b = b_N + b_R$$

$$b_N \cdot b_R = 0$$