CT Notes in the lectures Weeks 2 & 3

Linear maps

$L : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

(i) $L(v+w) = L(v) + L(w)$ for $\forall v, w \in \mathbb{R}^n$

(ii) $L(rv) = rL(v)$ for $\forall v \in \mathbb{R}^n, \forall r \in \mathbb{R}$

For a fixed rectangular (orthonormal) coordinate system, we have a basis of

n unit vectors for $\mathbb{R}^n$ and

m unit vectors for $\mathbb{R}^m$.

With respect to such a basis a linear map is represented by a $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$

For a vector $v \in \mathbb{R}^m$, we get a linear map $v^T : \mathbb{R}^m \to \mathbb{R}$ defined by $w \mapsto v \cdot w$ (dot product)

Similarly for $A \in \mathbb{R}^{m \times n}$ we get a linear map represented by $A^T \in \mathbb{R}^{n \times m}$ (Transpose of A)

l_∞ norm:

For large p the l_p norm of a non zero vector $x \in \mathbb{R}^m$ can be written as

$$\|x\|_p = |x_{max}| \left(\sum_{i=1}^{m} \left|\frac{x_i}{x_{max}}\right|^p \right)^{\frac{1}{p}}$$

$$\longrightarrow |x_{max}| \quad \text{as } p \to \infty$$

where $|x_{max}| = \max\{|x_i| : i = 1, \ldots, m\}$.

Proof Note that $|x_{max}| \neq 0$ since $x \neq 0$.

We have $0 \leq \left|\frac{x_i}{x_{max}}\right| \leq 1$. Since $|x_i| = |x_{max}|$

for some i: $1 \leq \sum_{i=1}^{m} \left|\frac{x_i}{x_{max}}\right|^p \leq m$

and

$$1 \leq \left(\sum_{i=1}^{m} \left|\frac{x_i}{x_{max}}\right|^p \right)^{\frac{1}{p}} \leq m^{\frac{1}{p}}$$

But $m^{\frac{1}{p}} \to 1$ as $p \to \infty$. Thus $\|x\|_p \to |x_{max}|$ as $p \to \infty$ and we put $\|x\|_\infty = |x_{max}|$

Cauchy-Schwarz inequality

For $\forall$ $u, v \in \mathbb{R}^m$ we have

$$-\|u\|_2 \|v\|_2 \leq u^T v \leq \|u\|_2 \|v\|_2$$

Proof

Consider the vector $u + \lambda v$ where $\lambda \in \mathbb{R}$ is any real number.

Then $(u+\lambda v) \cdot (u + \lambda v) =$

$u \cdot u + 2\lambda u \cdot v + \lambda^2 v \cdot v \geqslant 0$

for all values of $\lambda \in \mathbb{R}$, since the dot product is always non-negative.

Now for fixed $u, v \in \mathbb{R}^m$, we have a quadratic (with a parabola as its graph)

$$y = (v \cdot v)\lambda^2 + 2\lambda (u \cdot v) + (u \cdot u)$$

For $y \geqslant 0$ irrespective of $\lambda \in \mathbb{R}$ we must have

$(u \cdot v)^2 - (u \cdot u)(v \cdot v) \leqslant 0$ i.e. $(u^T v)^2 \leq (u \cdot u)(v \cdot v)$

as required.

Orthogonality of range(A) and null(A^T).

For all $v \in$ range(A) and $w \in$ null(A^T) we have $v \cdot w = 0$ where $A \in \mathbb{R}^{m \times n}$.

Proof If $v \in$ range(A) Then $\exists\, x \in \mathbb{R}^n$ such That $Ax = v$. Then

$$v = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ are the columns of A.

Now $w \in$ Null(A^T) means $A^T w = 0$ i.e. w is orthogonal to all rows of A^T or colums of A.

So $w \cdot a_i = 0$ for $i = 1, \ldots, n$

Thus $v \cdot w = \sum_{i=1}^{n} x_i (a_i \cdot w) = 0$

$\mathrm{range}(A) \cap \mathrm{null}(A^T) = \{0\}$

i.e. The zero vector is the only vector in both range(A) & null(A^T).

<u>Proof</u>

Suppose $v \in \mathrm{range}(A) \cap \mathrm{null}(A^T)$

then by previous sheet we know That any vector in range(A) is orthogonal to any vector in null(A^T).

Therefore $v \cdot v = 0$, i.e. $v = 0$