233 Computational Techniques

Problem Sheet for Tutorial 2

Problem 1
Which of the following pairs of vectors are orthogonal:

(a) $[1, 2]$ *and* $[-1, 1]$,

(b) $[2, 5, 1]$ *and* $[-3, 1, 1]$,

(c) $[3, 5, 3, -4]$ *and* $[4, -2, 2, 2]$.

Problem 2
For

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

decide which of the following products are defined, and compute them:
(a) $\mathbf{Au}$, (b) $\mathbf{Av}$, (c) $\mathbf{A}^T\mathbf{v}$, (d) $\mathbf{u}^T\mathbf{v}$, (e) $\mathbf{uv}^T$.

Problem 3
From the pair of vectors in problem 1(b), construct an orthonormal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *such that two of them are multiples of the given pair.*

Problem 4
Matrix multiplication is not commutative: *that is,* $\mathbf{AB} \neq \mathbf{BA}$ *in general. As an illustration, prove that a square* 2×2 *matrix* $\mathbf{A}$ *satisfying* $\mathbf{AX} = \mathbf{XA}$ *for every* 2×2 *matrix* $\mathbf{X}$ *must be a multiple of the unit matrix* $\mathbf{I}_2$*. In other words, prove the following:*

$$\mathbf{A} \in \mathbb{R}^{2\times 2} \text{ and } \mathbf{AX} = \mathbf{XA} \text{ for all } \mathbf{X} \in \mathbb{R}^{2\times 2} \iff \exists \lambda \in \mathbb{R} \text{ such that } \mathbf{A} = \lambda\,\mathbf{I}_2\,.$$

(This is true for square matrices of any size!) Hint*: Compare* $\mathbf{AX}$ *and* $\mathbf{XA}$ *for matrices* X *which have one entry equal to 1 and all others zero; for instance for*

$$\mathbf{E}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{E}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note: *The formulation was changed slightly in order to clarify the problem.*

Solution

Problem 1
Two vectors are orthogonal if their dot product is zero. The dot products are $1 \times (-1) + 2 \times 1 = 1$ for (a), $2 \times (-3) + 5 \times 1 + 1 \times 1 = 0$ for (b) and $3 \times 4 + 5 \times (-2) + 3 \times 2 + (-4) \times 2 = 0$ for (c); so the pairs (b) and (c) are orthogonal, the pair (a) is not.

Problem 2
(a)

$$\mathbf{Au} = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} .$$

(b) $\mathbf{Av}$ is not defined: the column dimension of $\mathbf{A}$ is 3, while the dimension of $\mathbf{v}$ is only 2.
(c)

$$\mathbf{A}^T\mathbf{v} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ 23 \end{bmatrix} .$$

(d) $\mathbf{u}^T\mathbf{v}$ is not defined: $\mathbf{u}$ and $\mathbf{v}$ do not have the same dimension.
(e)

$$\mathbf{uv}^T = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} [2, 3] = \begin{bmatrix} 2 & 3 \\ 4 & 6 \\ -2 & -3 \end{bmatrix}$$

is the *outer product* of $\mathbf{u}$ and $\mathbf{v}$. It can also be understood as a product of two matrices with dimensions 3×1 and 1×2 respectively.

Problem 3
$[2, 5, 1] =: \mathbf{u}_1$ and $[-3, 1, 1] =: \mathbf{u}_2$ are already orthogonal. So the easiest thing to do is to find a third vector $\mathbf{u}_3$ which is orthogonal to both of them, and then to *normalize* each of the three vectors, i.e. to divide each of them by its Euclidean norm, resulting in a vector of norm 1. (If $\mathbf{u} \neq 0$, then its norm is nonzero, and $\mathbf{v} := \mathbf{u}/\|\mathbf{u}\|_2$ has Euclidean norm $\|\mathbf{v}\|_2 = 1$.)
In three dimensions, the first step can be done by taking the *vector product*[1] of $\mathbf{u}_1$ and $\mathbf{u}_2$, since the vector product is always orthogonal to both vectors from which it is formed. So

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{bmatrix} 5 \times 1 - 1 \times 1 \\ 1 \times (-3) - 2 \times 1 \\ 2 \times 1 - 5 \times (-3)) \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 17 \end{bmatrix}$$

[1] The vector product of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is defined as the vector $[a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$.

Alternatively, let $\mathbf{u}_3^T = [a, b, c]$. Then the orthogonality conditions for the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are

$$0 = \mathbf{u}_3^T \mathbf{u}_1 = 2a + 5b + c \quad \text{and} \quad 0 = \mathbf{u}_3^T \mathbf{u}_2 = -3a + b + c\,.$$

We can rearrange the second equation as $c = 3a - b$ and use this to eliminate c from the first equation: $0 = 2a + 5b + 3a - b = 5a - 4b$, or $b = -(5/4)a$. We can now express c in terms of a alone as $c = 3a + (5/4)a = (17/4)a$. So we get $\mathbf{u}_3^T = [a, -(5/4)a, (17/4)a] = a[1, -5/4, 17/4]$ and we can check that this vector is really orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ for *any* choice of a. For instance, for $a = 4$, we obtain $\mathbf{u}_3^T = [4, -5, 17]$ as before.
The norms of the three vectors are

$$\begin{aligned} \|\mathbf{u}_1\| &= \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}\,, \quad \|\mathbf{u}_2\| = \sqrt{(-3)^2 + 1^2 + 1^2} = \sqrt{11}\,, \\ \|\mathbf{u}_3\| &= \sqrt{4^2 + (-5)^2 + 17^2} = \sqrt{330}\,, \end{aligned}$$

and so the resulting orthonormal set is

$$\mathbf{v}_1^T = [2, 5, 1]/\sqrt{30}\,, \quad \mathbf{v}_2^T = [-3, 1, 1]/\sqrt{11}\,, \quad \mathbf{v}_3^T = [4, -5, 17]/\sqrt{330}\,.$$

By the way, the $\mathbf{v}_i$ are only determined up to sign – orthogonality is a bilinear relation, and the negative of a vector has the same norm as the original vector. (So, for instance, if your third vector is $[-4, 5, -17]/\sqrt{330}$, that's also correct.)

Problem 4
Part "⇒": Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then for $\mathbf{X} = \mathbf{E}_{12}$,

$$\mathbf{A}\mathbf{E}_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}, \quad \mathbf{E}_{12}\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix},$$

and so $\mathbf{A}\mathbf{E}_{12} = \mathbf{E}_{12}\mathbf{A}$ if and only if $a = d$ and $c = 0$. Similarly for $\mathbf{X} = \mathbf{E}_{21}$:

$$\mathbf{A}\mathbf{E}_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}, \quad \mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix},$$

and so $\mathbf{A}\mathbf{E}_{21} = \mathbf{E}_{21}\mathbf{A}$ if and only if $a = d$ and $b = 0$. So from the hypothesis that $\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{A}$ for all $\mathbf{X}$, it follows that $a = d$ and $b = c = 0$, that is, $\mathbf{A}$ must be of the form

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \lambda \mathbf{I}_2 \quad \text{for } \lambda = a.$$

Part "⇐": The unit matrix satisfies $\mathbf{X}\mathbf{I}_m = \mathbf{I}_m\mathbf{X} = \mathbf{X}$ for every matrix $\mathbf{X}$ and in every dimension m; so if $\mathbf{A} = \lambda\mathbf{I}_2$ for $\lambda \in \mathbb{R}$, then $\mathbf{A}\mathbf{X} = (\lambda\mathbf{I}_2)\mathbf{X} = \lambda(\mathbf{I}_2\mathbf{X}) = \lambda\mathbf{X}$ and $\mathbf{X}\mathbf{A} = \mathbf{X}(\lambda\mathbf{I}_2) = (\mathbf{X}\lambda)\mathbf{I}_2 = \mathbf{X}\lambda = \lambda\mathbf{X}$.