233 Computational Techniques

Problem Sheet for Tutorial 3

Problem 1
In 2 dimensions, the ℓ_p norm of a vector $\boldsymbol{x} = (x_1, x_2)$ is given by

$$\|\boldsymbol{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p} \quad \text{for} \quad 1 \le p < \infty \ , \qquad \|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|\} \ .$$

(a) Sketch the surfaces of constant ℓ_p norm of 1,

$$C_p := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\|_p = 1\}$$

for $p = 1, 2, \infty$ in a rectangular coordinate system.
(b) Hence explain graphically the inequality $\|x\|_1 \ge \|x\|_2 \ge \|x\|_\infty$.

Problem 2
Using the definition of the angle between two vectors, prove the *cosine theorem* of trigonometry:

$$\|\boldsymbol{u} - \boldsymbol{v}\|_2^2 = \|\boldsymbol{u}\|_2^2 + \|\boldsymbol{v}\|_2^2 - 2\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2 \cos\phi \qquad (1)$$

for all $\boldsymbol{u}, \boldsymbol{v} \neq \boldsymbol{0}$, where ϕ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Which theorem is the special case $\phi = \pi/2$?
Hint given in the tutorial: Use $\boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|_2^2$ (both sides are the sum of squares of the components of $\boldsymbol{x}$.)

Problem 3
From (1) and the fact that the sum of angles in a triangle is equal to π, deduce

$$(a) \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \ , \qquad (b) \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2} \ .$$

Hint: Take $\boldsymbol{u}$ and $\boldsymbol{v}$ in (1) as defining two sides of a triangle enclosing the desired angle. What is the third side? For *(a)*, take an equilateral triangle, for *(b)* one with one right angle and the other two angles $\pi/4$.

Problem 4
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -9 & 2 & 3 \\ -4 & 8 & 6 \\ 1 & 5 & 7 \end{bmatrix} .$$

Determine $\|\boldsymbol{A}\|_1, \ \|\boldsymbol{A}\|_\infty$ and $\|\boldsymbol{B}\|_1, \ \|\boldsymbol{B}\|_\infty$.

Solution

Problem 1
(a)

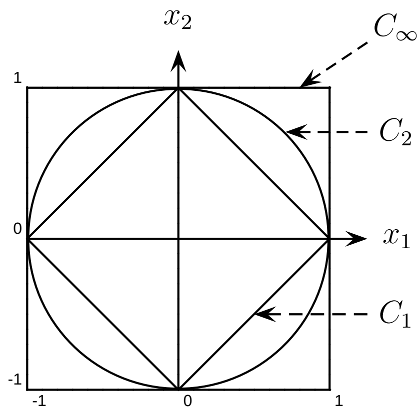

Figure 1:

Explanation: $\|\boldsymbol{x}\|_2 = \sqrt{x_1^2 + x_2^2}$ is the Euclidean distance of $\boldsymbol{x}$ from the origin; so C_2 is the set of all points which have Euclidean distance 1 from the origin, which is the unit circle.
Next, note that the ℓ_p-norm of a vector $\boldsymbol{x} = [x_1, x_2]$ depends only on the *absolute* value of x_1 and x_2, not on their signs. This means that once we know the part of the surface C_p which lies, say, in the *positive quadrant* $x_1, x_2 \geq 0$, then we can obtain the entire surface by simply taking this part together with its four reflections at the axes $x_1 = 0$ and $x_2 = 0$.
For C_1, the part lying in the positive quadrant is the set of all $\boldsymbol{x} = [x_1, x_2]$ such that $x_1, x_2 \geq 0$ and $x_1 + x_2 = 1$, or $x_2 = 1 - x_1$. The graph of this function is a straight line through the points $[0, 1]$ and $[1, 0]$; so the part of C_1 in the positive quadrant is the line segment between $[0, 1]$ and $[1, 0]$, including the endpoints. Hence C_1 is the "diamond"-shaped curve shown in the diagram.
For C_∞, the part lying in the positive quadrant is the set of all $\boldsymbol{x} = [x_1, x_2]$ such that $x_1, x_2 \geq 0$ and $\max\{x_1, x_2\} = 1$. So if $x_1 \leq x_2$, then $x_2 = 1$, and $0 \leq x_1 \leq 1$; in this case, $\boldsymbol{x}$ lies on the line segment between $[0, 1]$ and $[1, 1]$, including both endpoints. However if $x_1 > x_2$, then $x_1 = 1$ and $0 \leq x_2 < 1$, and $\boldsymbol{x}$ lies on the line segment between $[1, 0]$ and $[1, 1]$ (including the former, but not the latter). This gives the positive quadrant of the square shown in the diagram.
(b) First suppose $\|\boldsymbol{x}\|_2 = 1$, i.e. that $\boldsymbol{x}$ lies on C_2, which is the unit circle. If $\boldsymbol{x}$ is one of the corners of C_1 (the "diamond"), it also lies on C_∞ (the square) and hence all three norms of $\boldsymbol{x}$ are equal. Otherwise $\boldsymbol{x}$ lies on the outside of C_1 and on the inside of C_∞; this means that the ℓ_1 norm of $\boldsymbol{x}$ is greater than 1, and that the ℓ_∞ norm of $\boldsymbol{x}$ is smaller than one. So the inequality is true in this case!

Note that the inequality is also true if $\|\boldsymbol{x}\|_2 = 0$; for in this case, $\boldsymbol{x}$ is the zero vector, and so the other two norms are also zero.
Now the remaining case: $\|\boldsymbol{x}\|_2 \notin \{0, 1\}$. But we know already that the inequality is true for vectors with ℓ_2 norm equal to 1. So we divide $\boldsymbol{x}$ by its ℓ_2 norm and get $\hat{\boldsymbol{x}} = \lambda \boldsymbol{x}$ with $\lambda = 1/\|\boldsymbol{x}\|_2 \neq 0$. It follows from the general properties of a vector norm that $\hat{\boldsymbol{x}}$ has ℓ_2 norm $\|\hat{\boldsymbol{x}}\|_2 = |\lambda| \, \|\boldsymbol{x}\|_2 = 1$. So we have the inequality $\|\hat{\boldsymbol{x}}\|_1 \geq \|\hat{\boldsymbol{x}}\|_2 \geq \|\hat{\boldsymbol{x}}\|_\infty$, or

$$|\lambda| \|\boldsymbol{x}\|_1 \geq \|\lambda\| \|\boldsymbol{x}\|_2 \geq \|\lambda\| \|\boldsymbol{x}\|_\infty ,$$

and, dividing by $|\lambda|$ we obtain the general result.

Problem 2
The left hand side of (1) is

$$(\boldsymbol{u} - \boldsymbol{v})^T (\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{u}^T \boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{v} - \boldsymbol{v}^T \boldsymbol{u} + \boldsymbol{v}^T \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{u} + \boldsymbol{v}^T \boldsymbol{v} - 2\boldsymbol{u}^T \boldsymbol{v} .$$

Here $\boldsymbol{u}^T \boldsymbol{u} = \|\boldsymbol{u}\|_2^2$ and $\boldsymbol{v}^T \boldsymbol{v} = \|\boldsymbol{v}\|_2^2$. The remaining term is

$$-2\boldsymbol{u}^T \boldsymbol{v} = -2\|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_2 \frac{\boldsymbol{u}^T \boldsymbol{v}}{\|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_2} = -2\|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_2 \cos\phi ,$$

by the definition of the angle between two vectors which was given in the lecture. ■
The cosine theorem allows to compute the angles of a triangle provided the lengths of the sides are known. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two sides of the triangle, then their lengths are $\|\boldsymbol{u}\|_2$ and $\|\boldsymbol{v}\|_2$, and the length of the third side is $\|\boldsymbol{u} - \boldsymbol{v}\|_2$ (see Fig. 1). If $\phi = \pi/2$, then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is a right angle, and the cosine theorem then says that the squared lengths of the sides adjoining the right angle add up to the squared length of the side opposite. This is *Pythagoras' theorem.*

Problem 4
Reminder:

$$\|\boldsymbol{A}\|_1 = \max_j \|\boldsymbol{a}_j\|_1 \quad \text{the maximum absolute column sum,}$$

$$\|\boldsymbol{A}\|_\infty = \max_i \|\boldsymbol{a}^i\|_1 \quad \text{the maximum absolute row sum.}$$

Therefore, for $\boldsymbol{A}$: $\|\boldsymbol{A}\|_1 = \max\{3+1,\ 2,\ 4+3\} = \max\{4, 2, 7\} = 7$
and $\|\boldsymbol{A}\|_\infty = \max\{3+4,\ 1+2+3\} = \max\{7, 6\} = 7$.
Accidentally, the two norms are equal.

For $\boldsymbol{B}$: $\|\boldsymbol{B}\|_1 = \max\{14, 15, 16\} = 16$ and $\|\boldsymbol{B}\|_\infty = \max\{14, 18, 13\} = 18$.