233 Computational Techniques

Problem Sheet for Tutorial 3

Problem 1
In 2 dimensions, the ℓ_p norm of a vector $\boldsymbol{x} = (x_1, x_2)$ is given by

$$\|\boldsymbol{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p} \quad \text{for} \quad 1 \le p < \infty \,, \qquad \|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|\} \,.$$

(a) Sketch the surfaces of constant ℓ_p norm of 1,

$$C_p := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\|_p = 1\}$$

for $p = 1, 2, \infty$ in a rectangular coordinate system.
(b) Hence explain graphically the inequality $\|x\|_1 \ge \|x\|_2 \ge \|x\|_\infty$.

Problem 2
Using the definition of the angle between two vectors, prove the *cosine theorem* of trigonometry:

$$\|\boldsymbol{u} - \boldsymbol{v}\|_2^2 = \|\boldsymbol{u}\|_2^2 + \|\boldsymbol{v}\|_2^2 - 2\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2 \cos\phi \qquad (1)$$

for all $\boldsymbol{u}, \boldsymbol{v} \ne \boldsymbol{0}$, where ϕ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Which theorem is the special case $\phi = \pi/2$?
Hint given in the tutorial: Use $\boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|_2^2$ (both sides are the sum of squares of the components of $\boldsymbol{x}$.)

Problem 3
From (1) and the fact that the sum of angles in a triangle is equal to π, deduce

$$\textit{(a)} \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \,, \qquad \textit{(b)} \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2} \,.$$

Hint: Take $\boldsymbol{u}$ and $\boldsymbol{v}$ in (1) as defining two sides of a triangle enclosing the desired angle. What is the third side? For *(a)*, take an equilateral triangle, for *(b)* one with one right angle and the other two angles $\pi/4$.

Problem 4
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -9 & 2 & 3 \\ -4 & 8 & 6 \\ 1 & 5 & 7 \end{bmatrix}.$$

Determine $\|\boldsymbol{A}\|_1$, $\|\boldsymbol{A}\|_\infty$ and $\|\boldsymbol{B}\|_1$, $\|\boldsymbol{B}\|_\infty$.