233 Computational Techniques

Problem Sheet for Tutorial 4

Problem 1
Which of the following sets of vectors are linearly independent:
(a) $[1, 5]$, $[2, 3]$;
(b) $[2, 1, -3]$, $[-1, 1, -6]$, $[1, 1, -4]$;
(c) $[1, 0, 3]$, $[-1, 1, 2]$, $[2, 0, -5]$?

Problem 2
For

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} ,$$

find
(a) *the nullspace of* $\mathbf{A}$,
(b) *the nullspace of* $\mathbf{A}^T$,
(c) *the range of* $\mathbf{A}$,
(d) *the range of* $\mathbf{A}^T$.
(e) *Check that null*$\mathbf{A}^T$ *is orthogonal to range*$\mathbf{A}$, *and that null*$\mathbf{A}$ *is orthogonal to range*$\mathbf{A}^T$.
(f) *For* $\mathbf{x} = [1, 1, 1]^T$, *find the two vectors* $\mathbf{x}_R \in$ *range*$\mathbf{A}^T$ *and* $\mathbf{x}_N \in$ *null*$\mathbf{A}$ *which satisfy* $\mathbf{x} = \mathbf{x}_R + \mathbf{x}_N$. *Check that* $\mathbf{x}_R$ *and* $\mathbf{x}_N$ *are orthogonal!*

Problem 3
Prove:
(a) *If* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is invertible, then its right and left inverses are equal; that is, if* $\mathbf{AB} = \mathbf{I}$ *and* $\mathbf{CA} = \mathbf{I}$ *then* $\mathbf{B} = \mathbf{C}$. *(*Hint: *Compute* $\mathbf{CAB}$ *in two different ways.)*
(b) *If* $\mathbf{A}$ *has an inverse, then the columns of* $\mathbf{A}$ *are linearly independent.*
(c) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are both nonsingular, then* $\mathbf{AB}$ *is nonsingular, and* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
(d) *Suppose* $\alpha \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *such that* $\alpha\mathbf{u}^T\mathbf{v} \neq 1$. *Then* $\mathbf{E} = \mathbf{I}_n - \alpha\mathbf{u}\mathbf{v}^T$ *is nonsingular, and its inverse is* $\mathbf{I}_n - \beta\mathbf{u}\mathbf{v}^T$, *where*

$$\beta = \frac{\alpha}{\alpha\mathbf{u}^T\mathbf{v} - 1} .$$

Solution

Problem 1

The sets *(a)* and *(c)* are independent, *(b)* is not. There are two ways of doing this: by working from the definition of linear dependence, and by using determinants.
From the definition (*(a)* only): A set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of vectors is linearly dependent if
(i) there are real numbers $x_1, \ldots, x_n$, not all zero, such that $x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$,
or equivalently if
(ii) there is a real vector $\mathbf{x} = [x_1, \ldots, x_n]^T \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$, where $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ is the matrix which has $\mathbf{a}_1, \ldots, \mathbf{a}_n$ as its columns.
If the only numbers x_i satisfying (i) are $x_1 = \ldots = x_n = 0$ (the only vector $\mathbf{x}$ satisfying (ii) is the zero vector), then the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly independent.
So let x_1, x_2 be such that

$$x_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{1}$$

From the first component one finds $x_1 = -2x_2$, and the second component gives $0 = 5x_1 + 3x_2 = -7x_2$ (substituting for x_1 according to the first equation). From the second equation, it follows that $x_2 = 0$, and from the first that $x_1 = 0$ as well. So the only pair x_1, x_2 satisfying (1) is $x_1 = x_2 = 0$. According to (i), the vectors $[1, 5]^T$ and $[2, 3]^T$ are linearly independent.
Using determinants: Recall the following facts about determinants:
• A square matrix $\mathbf{A}$ is singular if and only if its determinant is zero.
• The determinant of a 1×1 matrix (i.e. a real number) is the number itself.
• The determinant of a 2×2 matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc .$$

• The determinant of a 3×3 matrix can be computed in various ways; one of them is

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} .$$

(Note the pattern: • The signs are alternating. • Each term in the sum is the product of an element $*$ of the first row and the determinant of the 2×2 matrix which remains if the row and column of $*$ are deleted.)
The first fact leads to the following criterion for linear dependence:
(iii) n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in n-dimensional space are linearly dependent if and only if the matrix $[\mathbf{a}_1, \ldots, \mathbf{a}_n]$ has zero determinant.
(*Proof:* $\det[\mathbf{a}_1, \ldots, \mathbf{a}_n] = 0 \iff [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ singular $\iff$ *(ii)* above.)
Application:
(a) The determinant of $\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$ is $1 \times 3 - 2 \times 5 = -7 \neq 0$, so $[1, 5]$ and $[2, 3]$ are linearly

independent. (So are $[1,2]$ and $[5,3]$!)
(b) The same in 3 dimensions:

$$\begin{aligned}\det\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ -3 & -6 & -4 \end{bmatrix} &= 2\cdot\det\begin{bmatrix} 1 & 1 \\ -6 & -4 \end{bmatrix} - (-1)\cdot\det\begin{bmatrix} 1 & 1 \\ -3 & -4 \end{bmatrix} + 1\cdot\det\begin{bmatrix} 1 & 1 \\ -3 & -6 \end{bmatrix} \\ &= 2\times 2 - (-1)\times(-1) + 1\times(-3) = 0\ ,\end{aligned}$$

so $[2,1,-3]$, $[-1,1,-6]$ and $[1,1,-4]$ are linearly dependent.
(c) Here the determinant of the resulting matrix is -11, so the vectors $[1,0,3]$, $[-1,1,2]$ and $[2,0,-5]$ are linearly independent.

Problem 2
(a) The null space of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. If $\mathbf{x}$ is such a vector, then

$$\mathbf{Ax} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_3 \\ 3x_1 - x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ ,$$

and so $x_3 = -2x_1$ and $x_2 = 3x_1 + 2x_3 = -x_1$. In vector notation, $\mathbf{x}^T = [x_1, -x_1, -2x_1] = x_1[1,-1,-2]$, that is, the null space of $\mathbf{A}$ is the set of all scalar multiples of the vector $[1,-1,-2]^T$. A concise notation for this statement is

$$\mathrm{null}\mathbf{A} = \mathbb{R}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\ .$$

(b) If $\mathbf{y} \in \mathrm{null}\mathbf{A}^T$, then

$$\mathbf{0} = \mathbf{A}^T\mathbf{y} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 + 3y_2 \\ -y_2 \\ y_1 + 2y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\ ,$$

and so $y_2 = y_1 = 0$. It follows that $\mathrm{null}\mathbf{A}^T = \{\mathbf{0}\}$. Another way to see this is to notice that the two columns in $\mathbf{A}^T$ are linearly independent; therefore $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ forces $\mathbf{y} = \mathbf{0}$. A third way is to do *(c)* first and to use the first of the dimension formulae

$$\begin{aligned}\dim(\mathrm{range}\mathbf{A}) + \dim(\mathrm{null}\mathbf{A}^T) &= \#\text{ of rows of }\mathbf{A}\ ,\\ \dim(\mathrm{range}\mathbf{A}^T) + \dim(\mathrm{null}\mathbf{A}) &= \#\text{ of columns of }\mathbf{A}.\end{aligned}$$

For the range space of $\mathbf{A}$ is two-dimensional, and so the null space of $\mathbf{A}^T$ is zero-dimensional. The only zero-dimensional vector space is $\{\mathbf{0}\}$.
(c) The range of $\mathbf{A}$ consists of all vectors in $\mathbb{R}^2$ which can be written in the form $\mathbf{Ax}$ for some $\mathbf{x} \in \mathbb{R}^3$. As

$$\mathbf{Ax} = x_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 2 \end{bmatrix}\ ,$$

the range space of $\mathbf{A}$ is spanned by the three vectors $[2,3]^T$, $[0,-1]^T$, and $[1,2]^T$. Now it is easy to see that any two of these vectors are linearly independent; so their span is the entire $\mathbb{R}^2$. In other words, range$\mathbf{A} = \mathbb{R}^2$, and dim(range$\mathbf{A}$) = 2.
(d) The range of $\mathbf{A}^T$ is

$$\text{range}\mathbf{A}^T = \{\mathbf{A}^T\mathbf{y} : \mathbf{y} \in \mathbb{R}^2\} = \left\{ y_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad : \quad y_1, y_2 \in \mathbb{R} \right\} .$$

Its dimension is indeed two, since the two generating vectors are linearly independent.
(e) null$\mathbf{A}^T = \{\mathbf{0}\}$, and the zero vector is orthogonal to any vector. For null$\mathbf{A}$ and range$\mathbf{A}^T$, it suffices to check that the vectors which generate them are orthogonal. Indeed,

$$[1,-1,-2] \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0 \qquad \text{and} \qquad [1,-1,-2] \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 0 .$$

(f) One way to do this is to compute the coefficients λ_i in

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\lambda_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}_{\mathbf{x}_N} + \underbrace{\lambda_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}_{\mathbf{x}_R} ;$$

$\mathbf{x}_N$ and $\mathbf{x}_R$ are then given by the bracketed terms.
$\mathbf{x}_N = -(1/3)[1,-1,-2] = [-1/3, 1/3, 2/3]$ and $\mathbf{x}_R = [1,1,1]-[-1/3,1/3,2/3] = [4/3,2/3,1/3]$.
These two vectors are indeed orthogonal: $\mathbf{x}_N^T\mathbf{x}_R = -(1/3)\times 4/3 + 1/3\times 2/3 + 2/3\times 1/3 = 0$.

Problem 3
(a) $\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}$.
(b) If $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ and $\mathbf{x} = [x_1, \ldots, x_n]^T$, then $\mathbf{Ax} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$. Now suppose that the right hand side is zero. Then $\mathbf{Ax} = \mathbf{0}$, and by pre-multiplying with the inverse $\mathbf{A}^{-1}$, it follows that $\mathbf{x} = \mathbf{0}$, and so the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent.
(c) To prove that $\mathbf{AB}$ is nonsingular, we have to show $\mathbf{ABz} = \mathbf{0} \Rightarrow \mathbf{z} = \mathbf{0}$. So suppose that $\mathbf{ABz} = \mathbf{0}$. Then, as $\mathbf{A}$ is nonsingular, it follows that $\mathbf{Bz} = \mathbf{0}$, and since $\mathbf{B}$ is also nonsingular, that $\mathbf{z} = \mathbf{0}$. For the inverse of $\mathbf{AB}$, it suffices to check that

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I} ,$$

so $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is indeed the inverse of $\mathbf{AB}$.
(d) To prove that $\mathbf{E}$ is nonsingular, we have to show that $\mathbf{Ez} = \mathbf{0} \Rightarrow \mathbf{z} = \mathbf{0}$. So suppose that $\mathbf{Ez} = \mathbf{0}$. As $\mathbf{E} = \mathbf{I} - \alpha\mathbf{uv}^T$, this is equivalent to

$$\mathbf{z} = \alpha(\mathbf{v}^T\mathbf{z})\mathbf{u} . \tag{2}$$

We compute $\mathbf{v}^T\mathbf{z}$ by taking the scalar product of the last equation with $\mathbf{v}^T$; the result is $\mathbf{v}^T\mathbf{z} = \alpha(\mathbf{v}^T\mathbf{z})(\mathbf{v}^T\mathbf{u})$, or $\mathbf{v}^T\mathbf{z}(1-\alpha\mathbf{u}^T\mathbf{v}) = 0$. But it was assumed that $\alpha\mathbf{u}^T\mathbf{v} \neq 1$; so it follows that $\mathbf{v}^T\mathbf{z} = 0$. By (2), this also implies $\mathbf{z} = \mathbf{0}$.
Again, it remains to check that $\mathbf{I} - \beta\mathbf{u}\mathbf{v}^T$ is really the inverse of $\mathbf{E}$:

$$\begin{aligned}(\mathbf{I} - \alpha\mathbf{u}\mathbf{v}^T)(\mathbf{I} - \beta\mathbf{u}\mathbf{v}^T) &= \mathbf{I} - \beta\mathbf{u}\mathbf{v}^T - \alpha\mathbf{u}\mathbf{v}^T + \alpha\beta(\mathbf{v}^T\mathbf{u})\mathbf{u}\mathbf{v}^T \\ &= \mathbf{I} + \{\alpha\beta(\mathbf{u}^T\mathbf{v}) - \alpha - \beta\}\mathbf{u}\mathbf{v}^T \ ;\end{aligned}$$

here the term in the curly bracket is

$$\beta(\alpha\mathbf{u}^T\mathbf{v} - 1) - \alpha = \alpha - \alpha = 0 \ .$$