233 Computational Techniques

Problem Sheet for Tutorial 5

Problem 1

The purpose of this exercise is to show you an application of eigenvalues and eigenvectors to a topic which, at first glance, might seem totally unrelated: the *Fibonacci series*. Recall (from the 1st year PPT classes) that the series is defined by $x_0 := 0$, $x_1 := 1$ and

$$x_{n+1} := x_n + x_{n-1} \tag{1}$$

for $n \geq 1$. This formula is *recursive*, that is, in order to find x_n for higher values of n, you have to know (or compute) the values for smaller n.

In many situations recursive formulae are not good enough, for instance if one wants to know how x_n grows with n. In this exercise you can find a formula for x_n which is *non-recursive* in the sense that it gives x_n as a *function of the index* n rather than as a function of previously computed values. Eigenvalues and -vectors are a good tool for this. Here is how to do it:

(a) Express (1) as a vector equation of the form

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \tag{2}$$

for some 2×2 matrix $\boldsymbol{A}$. This transforms the original series into a series of two-dimensional vectors.

(b) By recursive application of (2), express $[x_{n+1}, x_n]^T$ as a power of $\boldsymbol{A}$ times the "initial" vector (which one)?

(c) Now, find eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of $\boldsymbol{A}$. (Here the $\boldsymbol{u}_i$ need not be normalized.)

(d) Express the initial vector as a linear combination of the eigenvectors of $\boldsymbol{A}$.

(e) Use the results of *(b)*–*(d)* and the relation $\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$ to find the vector $[x_{n+1}, x_n]$—and hence x_n itself—as a function of n alone.

(f) Test your formula for $n = 0, \ldots, 4$.

Problem 2

Solve the following system of equations using Gauss-Jordan elimination. Identify basic variables. Express all solutions in terms of non-basic variables. Determine the space of solutions and verify the result.

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & - & x_5 & = & -2 \\ 4x_1 & + & 2x_2 & & & + & 3x_4 & - & 2x_5 & = & 2 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \end{array}$$

Problem 3

(a) Find the *Cholesky factorization* of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & -5 \\ -1 & -5 & 6 \end{bmatrix}.$$

(b) Then solve $\boldsymbol{Ax} = \boldsymbol{b}$ for $\boldsymbol{b} = [1, -3, 6]^T$ by forward and backward substitution, using the triangular shape of the factorization matrices. (See the end of section 3.5.1 in the lecture notes.)

Solution

Problem 1
(a) The matrix in (2) is

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} .$$

(b) The initial vector is $[x_1, x_0]^T = [1, 0]^T$, and

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \boldsymbol{A}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

(c) The characteristic polynomial of $\boldsymbol{A}$ is

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}_2) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1 ,$$

with zeros $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$; these are the eigenvalues of $\boldsymbol{A}$. A corresponding choice of eigenvectors is

$$\boldsymbol{u}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} , \qquad \boldsymbol{u}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} .$$

They are not normalized – this is not necessary here as we do not need the explicit orthogonal matrix from the spectral decomposition.
(d)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\boldsymbol{u}_1 - \boldsymbol{u}_2}{\sqrt{5}} .$$

(e) Multiplying both sides of the last equation by $\boldsymbol{A}^n$ gives

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \frac{\boldsymbol{A}^n \boldsymbol{u}_1 - \boldsymbol{A}^n \boldsymbol{u}_2}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left\{ \lambda_1^n \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \lambda_2^n \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \right\} .$$

Here the second component gives

$$x_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} = \frac{\{(1+\sqrt{5})/2\}^n - \{(1-\sqrt{5})/2\}^n}{\sqrt{5}} . \qquad (3)$$

(f) Thus:

$$\begin{aligned}
x_0 &= 0\ , \\
x_1 &= \frac{(1+\sqrt{5})/2-(1-\sqrt{5})/2}{\sqrt{5}} = 1\ , \\
x_2 &= \frac{(1+2\sqrt{5}+5)/4-(1-2\sqrt{5}+5)/4}{\sqrt{5}} = 1\ , \\
x_3 &= \frac{(1+3\sqrt{5}+15+5\sqrt{5})/8-(1-3\sqrt{5}+15-5\sqrt{5})/8}{\sqrt{5}} = 2\ , \\
x_4 &= \frac{(1+4\sqrt{5}+30+20\sqrt{5}+25)/16-(1-4\sqrt{5}+30-20\sqrt{5}+25)/16}{\sqrt{5}} = 3
\end{aligned}$$

in agreement with (1).
Obviously the recursive formula is better for small values of n as it avoids the "detour" into the real numbers. However for large n, (3) with real arithmetic can be much faster than (1) or (2) with integer arithmetic.

Problem 2
In tableau notation, a possible sequence of steps is the following (with the third equation as the first row; pivot elements underlined):

$$\left[\begin{array}{ccccc|c} \underline{1} & 1 & 1 & 1 & 1 & 3 \\ 2 & 1 & -1 & 2 & -1 & -2 \\ 4 & 2 & 0 & 3 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & \underline{-1} & -3 & 0 & -3 & -8 \\ 0 & -2 & -4 & -1 & -6 & -10 \end{array}\right] \longrightarrow$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & -2 & 5 \\ 0 & 1 & 3 & 0 & 3 & 8 \\ 0 & 0 & \underline{2} & -1 & 0 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3/2 & 3 & -1 \\ 0 & 0 & 1 & -1/2 & 0 & 3 \end{array}\right] .$$

The first three columns of the final tableau contain the three unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. (A different choice of pivot elements would have produced them in other columns.) So x_1, x_2 and x_3 are the basic variables, and we can read off

$$x_1 = 1 + 2x_5\ , \qquad x_2 = -1 - \frac{3}{2}x_4 - 3x_5\ , \qquad x_3 = 3 + \frac{1}{2}x_4\ ,$$

where the non-basic variables x_4 and x_5 can take arbitrary values. Hence the set of solutions is

$$\left\{\boldsymbol{x} = [1,-1,3,0,0]^T + x_4[0,-3/2,1/2,1,0]^T + x_5[2,-3,0,0,1]^T\ : x_4, x_5 \in \mathbb{R}\right\}\ .$$

(Note that this is *not* a vector space! For instance, the zero vector $\boldsymbol{x} = \mathbf{0}$ is not a solution. In general, for a given matrix $\boldsymbol{A}$ and a given vector $\boldsymbol{b}$ (with dimension equal to the row dimension of $\boldsymbol{A}$) the set $\{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ is a vector space if and only if $\boldsymbol{b} = \mathbf{0}$.)

Problem 3

(a) The aim is to split $\boldsymbol{A}$ in the following way:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & -5 \\ -1 & -5 & 6 \end{bmatrix} = \boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T = \begin{bmatrix} \ell_{11} & & \\ \ell_{21} & \ell_{22} & \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ & \ell_{22} & \ell_{32} \\ & & \ell_{33} \end{bmatrix} .$$

With the usual sign convention $\ell_{ii} > 0$ for the diagonal elements of $\boldsymbol{L}$, one obtains

- from the first column of $\boldsymbol{A}$:

$$1 = \ell_{11}^2 \Rightarrow \ell_{11} = 1 , \qquad 1 = \ell_{21}\ell_{11} = \ell_{21} , \qquad -1 = \ell_{31}\ell_{11} = \ell_{31} ,$$

- from the second column of $\boldsymbol{A}$, starting with a_{22} (as $\boldsymbol{A}$ is symmetric, the equation for a_{21} is the same as the one for a_{12}, which we have just solved)

$$5 = \ell_{21}^2 + \ell_{22}^2 = 1 + \ell_{22}^2 \Rightarrow \ell_{22} = 2 , \qquad -5 = \ell_{31}\ell_{21} + \ell_{32}\ell_{22} = -1 + 2\ell_{32} \Rightarrow \ell_{32} = -2 ,$$

- and from the third column (only the last element can give anything new),

$$6 = \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 = 1 + 4 + \ell_{33}^2 \Rightarrow \ell_{33} = 1 .$$

So

- the factorization was successful (which implies that $\boldsymbol{A}$ is positive definite), and
- the Cholesky factor of $\boldsymbol{A}$ is

$$\boldsymbol{L} = \begin{bmatrix} 1 & & \\ 1 & 2 & \\ -1 & -2 & 1 \end{bmatrix} . \tag{4}$$

(b) The idea is to solve $\boldsymbol{L}\boldsymbol{L}^T\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ in two steps by defining $\boldsymbol{y} := \boldsymbol{L}^T\boldsymbol{x}$ and solving (1) $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ by forward substitution and (2) $\boldsymbol{L}^T\boldsymbol{x} = \boldsymbol{y}$ by backward substitution. So:

$$\begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix} = \boldsymbol{b} = \boldsymbol{L}\boldsymbol{y} = \begin{bmatrix} 1 & & \\ 1 & 2 & \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_1 + 2y_2 \\ -y_1 - 2y_2 + y_3 \end{bmatrix} \Rightarrow \boldsymbol{y} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \boldsymbol{y} = \boldsymbol{L}^T\boldsymbol{x} = \begin{bmatrix} 1 & 1 & -1 \\ & 2 & -2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_2 - 2x_3 \\ x_3 \end{bmatrix} \Rightarrow \boldsymbol{x} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} .$$