233 Computational Techniques

Problem Sheet for Tutorial 5

Problem 1

The purpose of this exercise is to show you an application of eigenvalues and eigenvectors to a topic which, at first glance, might seem totally unrelated: the *Fibonacci series*. Recall (from the 1st year PPT classes) that the series is defined by $x_0 := 0$, $x_1 := 1$ and

$$x_{n+1} := x_n + x_{n-1} \tag{1}$$

for $n \geq 1$. This formula is *recursive*, that is, in order to find x_n for higher values of n, you have to know (or compute) the values for smaller n.

In many situations recursive formulae are not good enough, for instance if one wants to know how x_n grows with n. In this exercise you can find a formula for x_n which is *non-recursive* in the sense that it gives x_n as a *function of the index* n rather than as a function of previously computed values. Eigenvalues and -vectors are a good tool for this. Here is how to do it:

(a) Express (1) as a vector equation of the form

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \tag{2}$$

for some 2×2 matrix $\boldsymbol{A}$. This transforms the original series into a series of two-dimensional vectors.

(b) By recursive application of (2), express $[x_{n+1}, x_n]^T$ as a power of $\boldsymbol{A}$ times the "initial" vector (which one)?

(c) Now, find eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of $\boldsymbol{A}$. (Here the $\boldsymbol{u}_i$ need not be normalized.)

(d) Express the initial vector as a linear combination of the eigenvectors of $\boldsymbol{A}$.

(e) Use the results of *(b)–(d)* and the relation $\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$ to find the vector $[x_{n+1}, x_n]$—and hence x_n itself—as a function of n alone.

(f) Test your formula for $n = 0, \ldots, 4$.

Problem 2

Solve the following system of equations using Gauss-Jordan elimination. Identify basic variables. Express all solutions in terms of non-basic variables. Determine the space of solutions and verify the result.

$$\begin{array}{rcrcrcrcrcr}
2x_1 & + & x_2 & - & x_3 & + & 2x_4 & - & x_5 & = & -2 \\
4x_1 & + & 2x_2 & & & + & 3x_4 & - & 2x_5 & = & 2 \\
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3
\end{array}$$

Problem 3
(a) Find the *Cholesky factorization* of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & -5 \\ -1 & -5 & 6 \end{bmatrix}.$$

(b) Then solve $\boldsymbol{Ax} = \boldsymbol{b}$ for $\boldsymbol{b} = [1, -3, 6]^T$ by forward and backward substitution, using the triangular shape of the factorization matrices. (See the end of section 3.5.1 in the lecture notes.)