# 233 Computational Techniques
## Assessed Coursework #1

**Problem 1**

For each of the following statements, indicate whether it is true or false and briefly justify your answer. If a statement is false, a counterexample is sufficient as an explanation.

*(a)* If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ are linearly independent vectors then their dot product is equal to zero.
No. Eg: $a = (1, 0)^T$ and $b = (1, 1)^T$.

*(b)* The product $\boldsymbol{AB}$ of two matrices is defined whenever the number of rows of $\boldsymbol{A}$ is the same as the number of columns of $\boldsymbol{B}$.
No. The number of columns of $\boldsymbol{A}$ must be the same as the number of rows of $\boldsymbol{B}$.

*(c)* The norm of the sum of two vectors is at most the sum of their norms.
Correct. It is the triangular inequality.

*(d)* The map $\boldsymbol{x} \mapsto \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$ is a vector norm.
It is the $\ell_2$ norm.

*(e)* In $\mathbb{R}^m$, a set of less than $m$ vectors is linearly independent.
No. Eg. $a = (0, 0)^T$ in $\mathbb{R}^2$.

*(f)* If $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, then $\text{range}(\boldsymbol{A}) \cap \text{null}(\boldsymbol{A}^T) = \{\boldsymbol{0}\}$.
Yes. Since if $u \in \text{range}(\boldsymbol{A}) \cap \text{null}(\boldsymbol{A}^T)$, then $u \cdot u = 0$, which implies $\sum_{i=1}^m u_i^2 = 0$ i.e. $u_i = 0$ for $i = 1, \cdots, m$.

*(g)* If, for given $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution, then $\boldsymbol{A}$ is singular.
Yes, since it means that $b \notin \text{range}(\boldsymbol{A})$, i.e. $\boldsymbol{A}$ is rank deficient.

*(i)* If $\boldsymbol{A}$ is a square matrix and $\boldsymbol{Ax} = \boldsymbol{b}$ has two different solutions, then $\boldsymbol{A}$ is singular.
Yes. Since if $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Ay} = \boldsymbol{b}$ for $\boldsymbol{x} \neq \boldsymbol{y}$, then $\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{y}) = 0$ for $\boldsymbol{x} - \boldsymbol{y} \neq 0$.

**Problem 2**

For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix},$$

find

*(a)* the eigenvalues
$1, -4, 6$.
and

*(b)* a set of corresponding eigenvectors normalized such that their $\ell_2$ norm is 1.
$a_1 = \frac{1}{5}(0, -3, 4)^T$, $a_2 = \frac{1}{\sqrt{50}}(5, -4, -3)^T$ and $a_3 = \frac{1}{\sqrt{50}}(5.4, 3)^T$.

*(c)* Check that the eigenvectors form an orthonormal set.

*(d)* From *(a)* and *(c)*, find the spectral decomposition $\boldsymbol{A} = \boldsymbol{Q\Lambda Q}^T$ where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{\Lambda}$ is diagonal.
$Q = [a_1, a_2, a_3]$ and $\Lambda = \text{diag}(1, -4, 6)$. *(e)* Check that your decomposition really gives the matrix $\boldsymbol{A}$.

**Problem 3**

With the notations in the notes, show that for any matrix $A \in \mathbb{R}^{m\times n}$ we have:

$$\|\boldsymbol{A}\|_1 = \max_{\boldsymbol{x}\neq 0} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_1}{\|\boldsymbol{x}\|_1} = \max_{\|\boldsymbol{x}\|_1=1} \|\boldsymbol{A}\boldsymbol{x}\|_1.$$

1) We have for all $x \in \mathbb{R}^n$: $\|Ax\|_1 = \sum_{j=1}^m |\sum_{i=1}^n a_{ji}x_i| \leq \sum_{j=1}^m \sum_{i=1}^n |a_{ji}||x_i| = \sum_{i=1}^n (\sum_{j=1}^m |a_{ji}|)|x_i|$.
2) Since $\|A\|_1 = \max_{1\leq j\leq m} \sum_{j=1}^m |a_{ji}|$, the sum between brackets in the last term in (1) is bounded by $\|A\|_1$ for each $i = 1, \cdots, n$, so we get: for all $x \in \mathbb{R}^n$: $\|Ax\|_1 \leq \|A\|_1 \sum_{i=1}^n |x_i| = \|A\|_1\|x\|_1$.
3) Suppose $k$ with $1 \leq k \leq n$ is such that $\|A\|_1 = \max_{1\leq i\leq n} \sum_{j=1}^m |a_{ji}| = \sum_{j=1}^m |a_{jk}|$.Then put $x = (x_1, x_2, \cdots, x_n)^T$ with $x_k = 1$ and $x_i = 0$ for $i \neq k$. Thus, the LHS can actually be equal to the RHS and the first equality is proved.
4) For the second equality, we let $y = x/\|x\|_1$ for $x \neq 0$.

*Problems 1 and 2 carry 40% of the marks each, problem 3 carries 20%.*