233 Computational Techniques

Assessed Coursework #1

Handed out: 9th February 2007, Hand in: 23th February 2007, Nominal time: 1 hour

Problem 1

For each of the following statements, indicate whether it is true or false and briefly justify your answer. If a statement is false, a counterexample is sufficient as an explanation.

(a) If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ are linearly independent vectors then their dot product is equal to zero.
(b) The product $\boldsymbol{AB}$ of two matrices is defined whenever the number of rows of $\boldsymbol{A}$ is the same as the number of columns of $\boldsymbol{B}$.
(c) The norm of the sum of two vectors is at most the sum of their norms.
(d) The map $\boldsymbol{x} \mapsto \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$ is a vector norm.
(e) In $\mathbb{R}^m$, a set of less than m vectors is linearly independent.
(f) If $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, then $\text{range}(\boldsymbol{A}) \cap \text{null}(\boldsymbol{A}^T) = \{\boldsymbol{0}\}$.
(g) If, for given $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution, then $\boldsymbol{A}$ is singular.
(i) If $\boldsymbol{A}$ is a square matrix and $\boldsymbol{Ax} = \boldsymbol{b}$ has two different solutions, then $\boldsymbol{A}$ is singular.

Problem 2

For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix},$$

find

(a) the eigenvalues and
(b) a set of corresponding eigenvectors normalized such that their ℓ_2 norm is 1.
(c) Check that the eigenvectors form an orthonormal set.
(d) From *(a)* and *(c)*, find the spectral decomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{\Lambda}$ is diagonal.
(e) Check that your decomposition really gives the matrix $\boldsymbol{A}$.

Problem 3

With the notations in the notes, show that for any matrix $A \in \mathbb{R}^{m\times n}$ we have:

$$\|\boldsymbol{A}\|_1 = \max_{\boldsymbol{x}\neq 0} \frac{\|\boldsymbol{Ax}\|_1}{\|\boldsymbol{x}\|_1} = \max_{\|\boldsymbol{x}\|_1=1} \|\boldsymbol{Ax}\|_1.$$

Hint: 1) Assume that $\boldsymbol{A} = (a_{ji})$. Show that for all $\boldsymbol{x} \in \mathbb{R}^n$: $\|\boldsymbol{Ax}\|_1 = \sum_{j=1}^m |\sum_{i=1}^n a_{ji}x_i| \leq \sum_{j=1}^m \sum_{i=1}^n |a_{ji}||x_i| = \sum_{i=1}^n (\sum_{j=1}^m |a_{ji}|)|x_i|$.
2) Deduce that $\|\boldsymbol{Ax}\|_1 \leq \|\boldsymbol{A}\|_1 \|\boldsymbol{x}\|_1$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and that $\frac{\|\boldsymbol{Ax}\|_1}{\|\boldsymbol{x}\|_1} \leq \|\boldsymbol{A}\|_1$ for all nonzero vectors $\boldsymbol{x} \in \mathbb{R}^n$.
3) Find some nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $\frac{\|\boldsymbol{Ax}\|_1}{\|\boldsymbol{x}\|_1} = \|\boldsymbol{A}\|_1$ and deduce the first equality in the problem.
4) For the second equality, let $\boldsymbol{y} = \boldsymbol{x}/\|\boldsymbol{x}\|_1$ for $x \neq 0$.

Problems 1 and 2 carry 40% of the marks each, problem 3 carries 20%.