A Computational Model for Metric Spaces

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Abstract

For every metric space X, we define a continuous poset $\mathbf{B}X$ such that X is homeomorphic to the set of maximal elements of $\mathbf{B}X$ with the relative Scott topology. The poset $\mathbf{B}X$ is a dcpo iff X is complete, and ω-continuous iff X is separable. The computational model $\mathbf{B}X$ is used to give domain-theoretic proofs of Banach's fixed point theorem and of two classical results of Hutchinson: on a complete metric space, every hyperbolic iterated function system has a unique non-empty compact attractor, and every iterated function system with probabilities has a unique invariant measure with bounded support. We also show that the probabilistic power domain of $\mathbf{B}X$ provides an ω-continuous computational model for measure theory on a separable complete metric space X.

1 Introduction

In this paper, we establish new connections between the theory of metric spaces and domain theory, the two basic mathematical structures in computer science. For every metric space X, we define a continuous poset (not necessarily a dcpo) $\mathbf{B}X$ of *formal balls* as a computational model for X (Section 2). The space X with the metric topology is homeomorphic to the subspace of maximal elements of $\mathbf{B}X$ with the relative Scott topology. The order-theoretic properties of $\mathbf{B}X$ are closely related with the metric properties of X. Ascending sequences in $\mathbf{B}X$ correspond to Cauchy sequences in X, and least upper bounds in $\mathbf{B}X$ to limits in X. Hence, the poset $\mathbf{B}X$ is directed complete if and only if the metric space X is complete. This gives a simple connection between completeness of a partial order and that of a metric space. Furthermore, bases of the continuous poset $\mathbf{B}X$ correspond to dense subsets of X.

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Hence, the poset $\mathbf{B}X$ is ω-continuous (has a countable basis) if and only if X is separable (has a countable dense subset). All these results offer pleasing connections between classical notions of metric space theory and analogous notions of domain theory.

These results extend those in [4] which were for a locally compact Hausdorff space. They present a considerably simpler framework than that of an ω-algebraic cpo with distance and weight used by Weihrauch and Schreiber in [17] to embed a complete separable metric space into a domain. Our construction is also much simpler than Lawson's recent construction of an MP hull for a Polish space [15]. Furthermore, our results lead to a simple computational model for Hilbert and Banach spaces. In fact, for a normed vector space X, $\mathbf{B}X$ is isomorphic to the poset of closed balls ordered by reverse inclusion. It follows that for a separable Banach space and in particular for a Hilbert space X, $\mathbf{B}X$ is an ω-continuous dcpo.

In Section 3, we give a domain-theoretic proof of the Banach fixed point theorem for a contracting function on a complete metric space X from the dcpo fixed point theorem applied to a suitable pointed subdcpo of $\mathbf{B}X$. The technique of this proof is then used to give a domain-theoretic proof of a classical result of Hutchinson [9]: on a complete metric space, every hyperbolic iterated function system (IFS) has a unique non-empty compact attractor (Section 4). In our proof, we work in the Plotkin power domain of $\mathbf{B}X$, and completely avoid the Hausdorff metric on compact subsets of X.

Our framework also provides a simple domain-theoretic model for measure theory on Polish spaces, the basic structures used in probability theory and stochastic processes. In Section 5, we show that the probabilistic power domain of $\mathbf{B}X$ provides an ω-continuous computational model for measure theory on a separable complete metric space X. In particular, we establish that every normalised measure on a complete separable metric space or indeed on any Polish space can be obtained as the least upper bound of an increasing chain of simple valuations on its space of formal balls. This result can be used to do generalised Riemann integration on complete separable metric spaces or on Polish spaces.

The computational model for measure theory is used to give a domain-theoretic proof of another classical result of Hutchinson [9] without the need for the Hutchinson metric: We use the space of formal balls to show that an iterated function system with probabilities on a complete metric space has a unique invariant measure with bounded support (Section 6). As shown in [5] for locally compact Hausdorff spaces, our domain-theoretic framework can be used to derive algorithms for generating invariant measures and for computing expected values of functions with respect to invariant measures.

Some Notation

Let $f : X \to Y$ be a function. For $A \subseteq X$, we write f^+A for $\{fa \mid a \in A\}$, and for $B \subseteq Y$, we write f^-B for $\{x \in X \mid fx \in B\}$. The set $\mathbf{R}_+$ is the set of real numbers r with $0 \le r < \infty$. For convenience, we sometimes denote the restriction or extension of a map by the same symbol as the map itself.

Let (X, d) be a metric space. For x, x' in X, $d(x, x') \in \mathbf{R}_+$ gives the distance between x and x'. For x in X and r in $\mathbf{R}_+$, let $O(x, r) = \{x' \in X \mid d(x, x') < r\}$ and $C(x, r) = \{x' \in X \mid d(x, x') \le r\}$ be, respectively, the open and closed ball of radius r centred at x. The sets $O(x, r)$ form a basis of a topology on X, the *metric topology*. In this topology, the sets $O(x, r)$ are open and the sets $C(x, r)$ are closed.

2 The Poset $\mathbf{B}X$ of Formal Balls

For every metric space X, we define a continuous poset $\mathbf{B}X$ whose subspace of maximal elements is homeomorphic to X. This poset is directed complete iff the metric space is complete.

2.1 Closed Balls and Formal Balls

In [4], for every locally compact Hausdorff space X, a continuous bounded complete dcpo $\mathbf{U}X$ is constructed which consists of the non-empty compact subsets of X, ordered by inverse inclusion. The space X is shown to be homeomorphic to the space of maximal points of $\mathbf{U}X$. These results do not directly extend to all metric spaces since the poset $\mathbf{U}X$ is not continuous in general.

If (X, d) is a metric space, one may consider the poset of closed balls $C(x, r)$ of X ordered by reverse inclusion '$\supseteq$'. The points x of X can be identified with the maximal closed balls $C(x, 0)$. Thus we have found a poset whose set of maximal elements is in one-to-one correspondence with X. However, even if X is complete, the poset of closed balls may be not complete. A standard example is the space $X = \{x_n \mid n \in \mathbf{N}\}$ with $d(x_n, x_m) = 1 + \frac{1}{n+m}$ for distinct n and m. This space is clearly complete. Let $C_n = C(x_n, 1 + \frac{1}{2n})$. Then $C_n = \{x_i \mid i \ge n\}$. Hence, $C_1 \supseteq C_2 \supseteq \cdots$ with empty intersection.

¿From the triangle inequality it follows that $C(x, r) \supseteq C(y, s)$ holds if $d(x, y) \le r - s$; the opposite implication is not always true, e.g., for the balls C_n of the previous paragraph. This observation is taken as the basis of the following

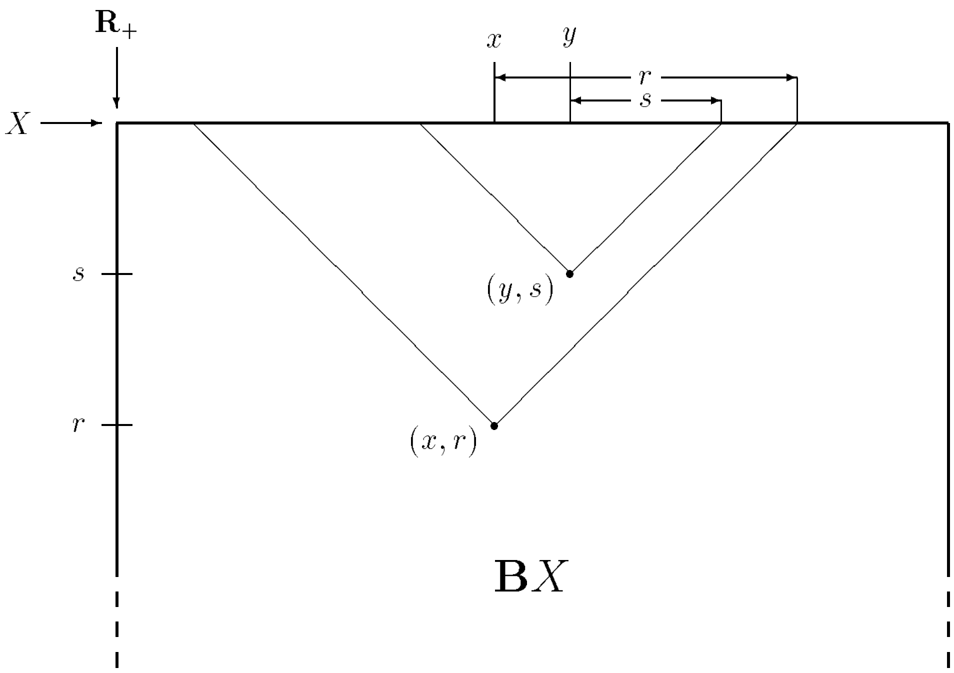

Fig. 1. Poset $\mathbf{B}X$ with two formal balls

definition:

A *formal ball* in a metric space X is a pair (x, r) with x in X and r in $\mathbf{R}_+$ (i.e., $0 \leq r < \infty$). The set of formal balls in X is called $\mathbf{B}X$. Formal balls are ordered by

$$(x, r) \sqsubseteq (y, s) \iff d(x, y) \leq r - s.$$

Note that $(x, r) \sqsubseteq (y, s)$ implies $r \geq s$. For, $d(x, y) \leq r - s$ implies $0 \leq r - s$, whence $s \leq r$.

Proposition 1 *For every metric space X, $\mathbf{B}X$ is a poset.*

Proof. Reflexivity: $(x, r) \sqsubseteq (x, r)$ since $d(x, x) = 0 = r - r$.

Transitivity: $(x, r) \sqsubseteq (y, s) \sqsubseteq (z, t)$ implies $d(x, z) \leq d(x, y) + d(y, z) \leq r - s + s - t = r - t$.

Antisymmetry: $(x, r) \sqsubseteq (y, s) \sqsubseteq (x, r)$ implies $r \geq s \geq r$. This gives $r = s$ and thus, $d(x, y) \leq 0$, whence $x = y$. $\square$

The structure of poset $\mathbf{B}X$ is visualised in Figure 1. It shows two formal balls $(x, r) \sqsubseteq (y, s)$, and indicates their upper cones $\uparrow(x, r)$ and $\uparrow(y, s)$. The maximal elements of $\mathbf{B}X$ are the formal balls $(x, 0)$, which obviously are in one-to-one correspondence with the points of X. The corresponding closed balls are obtained as intersection of the upper cone with this one-to-one image of X: $C(x, r) = \{z \in X \mid (z, 0) \in \uparrow(x, r)\}$.

Formal balls were first introduced by Weihrauch and Schreiber [17]. However, they do not consider the poset of formal balls itself, but complete it to a bounded complete ω-algebraic cpo. They also embed the metric space in the cpo, but not simply as the set of maximal points. In contrast, they have to impose additional non-order structures on their cpo in order to characterise the image of the embedding. Compared with their work, our approach is much simpler and more direct.

2.2 Ascending Sequences in $\mathbf{B}X$

In a metric space X, it is not necessary to study limits of general nets; it suffices to consider sequences. The poset $\mathbf{B}X$ has an analogous property: when considering least upper bounds of directed sets, it suffices to look at ascending sequences.

Theorem 2 *For every directed subset D of $\mathbf{B}X$, there is an ascending sequence $(x_1, r_1) \sqsubseteq (x_2, r_2) \sqsubseteq \cdots$ of elements of D which has the same upper bounds as D.*

Proof. Let $s = \inf\{r \mid (x, r) \in D\}$. For every n in $\mathbf{N}$, there is (y_n, s_n) in D such that $s_n \leq s + 1/n$. Let $(x_1, r_1) = (y_1, s_1)$, and for every $n > 1$, let (x_n, r_n) be an upper bound of (x_{n-1}, r_{n-1}) and (y_n, s_n) in D, which exists since D is directed.

We have to show that every upper bound of the sequence is an upper bound of all of D. Thus, let (z, t) be an upper bound of all (x_n, r_n), and let (a, u) be an element of D. Since D is directed, there are upper bounds (b_n, v_n) of (a, u) and (x_n, r_n) in D. Then for all n in $\mathbf{N}$,

$$\begin{aligned} d(a, z) &\leq d(a, b_n) + d(b_n, x_n) + d(x_n, z) \\ &\leq (u - v_n) + (r_n - v_n) + (r_n - t) \\ &\leq u - t + 2(r_n - s) \\ &\leq u - t + 2/n \end{aligned}$$

Hence, $d(a, z) \leq u - t$, i.e., $(a, u) \sqsubseteq (z, t)$ as required. $\square$

In the sequel, we compare ascending sequences in $\mathbf{B}X$ with sequences in X.

Proposition 3 *If $(x_1, r_1) \sqsubseteq (x_2, r_2) \sqsubseteq \cdots$ is an ascending sequence in $\mathbf{B}X$, then the sequence $(r_n)_{n\in\mathbf{N}}$ is descending and convergent, and the sequence $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in X.*

Proof. The relation $(x_n, r_n) \sqsubseteq (x_{n+1}, r_{n+1})$ implies $r_n \geq r_{n+1}$. Hence, $(r_n)_{n\in\mathbf{N}}$ is descending. Since $r_n \geq 0$ for all n, the sequence $(r_n)_{n\in\mathbf{N}}$ is convergent and in particularly Cauchy. Thus, for every $\epsilon > 0$, there is N in $\mathbf{N}$ such that for all $m, n \geq N$, $|r_n - r_m| < \epsilon$ holds. For $n \geq m \geq N$, $(x_m, r_m) \sqsubseteq (x_n, r_n)$ holds, whence $d(x_m, x_n) \leq r_m - r_n < \epsilon$. Thus, $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence. $\square$

A converse of Prop. 3 can also be proved.

Lemma 4 *Every Cauchy sequence $(x_n)_{n\in\mathbf{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbf{N}}$ such that $(x_{n_k}, 2^{-k})$ is ascending in $\mathbf{B}X$.*

Proof. Let $n_0 = 0$. For every k in $\mathbf{N}$, there is $n_k > n_{k-1}$ in $\mathbf{N}$ such that $d(x_i, x_j) \leq 2^{-(k+1)}$ for all $i, j \geq n_k$. Hence in particular, $d(x_{n_k}, x_{n_{k+1}}) \leq 2^{-(k+1)} = 2^{-k} - 2^{-(k+1)}$, i.e., $(x_{n_k}, 2^{-k}) \sqsubseteq (x_{n_{k+1}}, 2^{-(k+1)})$. $\square$

The following theorem exhibits a close relationship between least upper bounds in $\mathbf{B}X$ and limits in X.

Theorem 5 *For an ascending sequence $(x_n, r_n)_{n\in\mathbf{N}}$ in $\mathbf{B}X$ and an element (y, s) of $\mathbf{B}X$, the following are equivalent:*

(i) (y, s) *is the least upper bound of* $(x_n, r_n)_{n\in\mathbf{N}}$*;*
(ii) (y, s) *is an upper bound of* $(x_n, r_n)_{n\in\mathbf{N}}$*, and* $\lim_{n\to\infty} r_n = s$*;*
(iii) $\lim_{n\to\infty} x_n = y$ *and* $\lim_{n\to\infty} r_n = s$*.*

Proof.

(i) $\Rightarrow$ *(ii)* : Since $(x_n, r_n)_{n\in\mathbf{N}}$ is ascending with upper bound (y, s), $(r_n)_{n\in\mathbf{N}}$ is descending with lower bound s. Thus, we only have to show that for all $\epsilon > 0$, there is n in $\mathbf{N}$ such that $r_n < s + \epsilon$. Assume the contrary, i.e., there is $\epsilon > 0$ such that $r_n \geq s + \epsilon$ for all n in N. Since $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence (Prop. 3), there is N in $\mathbf{N}$ such that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N$. Hence for all $n \geq N$, $d(x_n, x_N) < \epsilon/2 \leq r_n - (s + \epsilon/2)$, or $(x_n, r_n) \sqsubseteq (x_N, s + \epsilon/2)$. Since the sequence is ascending, $(x_N, s+\epsilon/2)$ is an upper bound of the whole sequence. Thus $(y, s) \sqsubseteq (x_N, s + \epsilon/2)$, whence $s \geq s + \epsilon/2$, a contradiction.

$(ii) \Rightarrow (iii):$ For every n in $\mathbf{N}$, $(x_n, r_n) \sqsubseteq (y, s)$ holds, or $d(x_n, y) \leq r_n - s$. Since $\lim_{n\to\infty} r_n = s$ holds, $\lim_{n\to\infty} x_n = y$ follows.

$(iii) \Rightarrow (i):$ First, we show that (y, s) is an upper bound. For every $m \geq n$, $(x_n, r_n) \sqsubseteq (x_m, r_m)$ holds, whence $d(x_n, x_m) \leq r_n - r_m$. Letting m tend to infinity, we obtain $d(x_n, y) \leq r_n - s$, whence $(x_n, r_n) \sqsubseteq (y, s)$.

If (z, t) is an arbitrary upper bound of the sequence, then $(x_n, r_n) \sqsubseteq (z, t)$ holds for all n in $\mathbf{N}$, whence $d(x_n, z) \leq r_n - t$. Letting n tend to infinity, $d(y, z) \leq s - t$ follows, i.e., $(y, s) \sqsubseteq (z, t)$. □

¿From Theorem 5, we may conclude a direct connection between metric completeness and dcpo completeness. This reveals a deep relationship between these two kinds of completeness.

Theorem 6 *For a metric space X, the following are equivalent:*

(i) X is a complete metric space, i.e., every Cauchy sequence has a limit.
(ii) In $\mathbf{B}X$, every ascending sequence has a least upper bound.
(iii) $\mathbf{B}X$ is a dcpo, i.e., every directed set has a least upper bound.

Proof. The equivalence of (ii) and (iii) follows from Theorem 2.

Let X be complete. If $(x_n, r_n)_{n\in\mathbf{N}}$ is an ascending sequence in $\mathbf{B}X$, then by Prop. 3, $(r_n)_{n\in\mathbf{N}}$ converges to some s in $\mathbf{R}_+$, and $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in X. By completeness, $(x_n)_{n\in\mathbf{N}}$ converges to some y in X. By Theorem 5, (y, s) is the least upper bound of $(x_n, r_n)_{n\in\mathbf{N}}$.

Assume (ii) holds, and let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in X. By Lemma 4, it has a subsequence $(x_{n_k})_{k\in\mathbf{N}}$ such that $(x_{n_k}, 2^{-k})$ is ascending. By (ii), this sequence has a least upper bound (y, s). By Theorem 5, $\lim_{k\to\infty} x_{n_k} = y$. Since the whole sequence $(x_n)_{n\in\mathbf{N}}$ is Cauchy and a subsequence converges to y, the whole sequence converges to y. □

2.3 Continuity of $\mathbf{B}X$

In this subsection, we show that for every metric space X, the poset $\mathbf{B}X$ is *continuous*. Usually, this notion and related ones are defined for dcpo's only, but they can easily be generalised to arbitrary posets; one only has to replace quantifications such as 'for every directed set' by 'for every directed set which has a least upper bound'. This is done in Section 1.4 of [18], where continuous posets are defined and it is shown that their elementary properties are in complete analogy with those of continuous dcpo's. Thus for instance, the way-below relation in a continuous poset P is interpolative, and the sets $\Uparrow b$ with b in P form a basis of the Scott topology.

First, we characterise the way-below relation on $\mathbf{B}X$.

Proposition 7 *$(x, r) \ll (y, s)$ iff $d(x, y) < r - s$*

Proof. First assume $(x, r) \ll (y, s)$. Consider the sequence $a_n = (y, s + 1/n)$ which is obviously ascending. By Theorem 5, (y, s) is its least upper bound. ¿From the hypothesis, there is some n such that $(x, r) \sqsubseteq (y, s + 1/n)$, whence $d(x, y) \leq r - s - 1/n < r - s$.

For the opposite direction, assume $d(x, y) < r - s$. Hence, there is $\epsilon > 0$ such that $d(x, y) < r - s - \epsilon$. Let S be a directed subset of $\mathbf{B}X$ such that $\bigsqcup S = (z, t)$ exists and $(y, s) \sqsubseteq (z, t)$ holds. By Theorem 2, there is an ascending sequence $(z_n, t_n)_{n \in \mathbf{N}}$ in S with least upper bound (z, t). By Theorem 5, $\lim_{n \to \infty} z_n = z$ and $\lim_{n \to \infty} t_n = t$ holds. Hence, there is some n in $\mathbf{N}$ such that $d(z_n, z) < \epsilon/2$ and $t_n < t + \epsilon/2$. The estimation

$$\begin{aligned} d(x, z_n) &\leq d(x, y) + d(y, z) + d(z, z_n) \\ &< (r - s - \epsilon) + (s - t) + \epsilon/2 \\ &= r - (t + \epsilon/2) < r - t_n \end{aligned}$$

shows $(x, r) \sqsubseteq (z_n, t_n)$. $\square$

We now come to the continuity theorem. We prove that $\mathbf{B}X$ is continuous, and also show how a basis of $\mathbf{B}X$ can be obtained from dense subsets of X and $\mathbf{R}_+$. Recall that a subset B of a poset P is a basis of P iff for all p in P, the set $\Downarrow_B p = \{b \in B \mid b \ll p\}$ is directed with least upper bound p. In case of $P = \mathbf{B}X$, Theorem 2 implies that B is a basis iff for all (x, r) in P, the set $\Downarrow_B (x, r)$ contains an ascending sequence with least upper bound (x, r).

Theorem 8 *Let X be a metric space with a dense subset A, and let Q be a dense subset of $\mathbf{R}_+ = [0, \infty)$. Then $A \times Q$ is a basis of $\mathbf{B}X$.*

Proof. Consider (x, r) in $\mathbf{B}X$. For every n in $\mathbf{N}$, there is a_n in A such that $d(a_n, x) < 4^{-n}$, and q_n in Q such that $r + 2 \cdot 4^{-n} < q_n < r + 3 \cdot 4^{-n}$. Then $d(a_n, x) < 4^{-n} < 2 \cdot 4^{-n} < q_n - r$, whence $(a_n, q_n) \ll (x, r)$. For $n > 1$,

$$\begin{aligned} d(a_{n-1}, a_n) &\leq d(a_{n-1}, x) + d(a_n, x) \\ &< 4^{-(n-1)} + 4^{-n} = 5 \cdot 4^{-n} \end{aligned}$$

$$= (r + 2 \cdot 4^{-(n-1)}) - (r + 3 \cdot 4^{-n})$$
$$< q_{n-1} - q_n$$

whence $(a_{n-1}, q_{n-1}) \sqsubseteq (a_n, q_n)$. Clearly, $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} q_n = r$, whence (x, r) is the least upper bound of $(a_n, q_n)_{n\in\mathbf{N}}$ by Theorem 5. $\square$

It is also possible to show an implication in the opposite direction:

Proposition 9 *If B is a basis of $\mathbf{B}X$, then $\{a \in X \mid (a, r) \in B\}$ is dense in X.*

Proof. For every x in X, there is an ascending sequence $(a_n, r_n)_{n\in\mathbf{N}}$ in B with least upper bound $(x, 0)$. By Theorem 5, $(a_n)_{n\in\mathbf{N}}$ converges to x. $\square$

Corollary 10 *For every metric space X, $\mathbf{B}X$ is a continuous poset. It is ω-continuous (i.e., has a countable basis) iff X is separable (i.e., has a countable dense subset).*

Proof. A poset is continuous iff it has a basis. Theorem 8 shows that $\mathbf{B}X = X \times \mathbf{R}_+$ is a basis. The equivalence follows from Theorem 8 with Q being the positive rationals, and Prop. 9 for the opposite direction. $\square$

2.4 Embedding X into $\mathbf{B}X$

In this subsection, we study the function $\mathsf{i} : X \to \mathbf{B}X$ with $\mathsf{i}x = (x, 0)$. Clearly, this function is a one-to-one correspondence between X and the subset $X \times \{0\}$ of $\mathbf{B}X$, which we call X^+.

Proposition 11 *The elements of X^+ are precisely the maximal elements of $\mathbf{B}X$.*

Unless otherwise stated we consider $\mathbf{B}X$ equipped with its Scott topology. When we consider measures on X, we need the following result:

Proposition 12 *X^+ is a G_δ-subset of $\mathbf{B}X$.*

Proof. For every n in $\mathbf{N}$, the set $O_n = \bigcup_{x\in X} \Uparrow(x, 1/n)$ is Scott open in $\mathbf{B}X$. The equality $O_n = X \times [0, 1/n)$ is easily verified. Therefore, the intersection $\bigcap_{n\in\mathbf{N}} O_n$ is X^+. $\square$

For (x, r) in $\mathbf{B}X$, $\mathsf{i}^{-}(\Uparrow(x, r)) = O(x, r)$ holds by Prop. 7. The sets $\Uparrow(x, r)$ form a basis of the Scott topology on $\mathbf{B}X$, while the open balls $O(x, r)$ are a basis of the metric topology on X. Hence, i is a topological embedding, which makes X homeomorphic to the subspace X^{+} of $\mathbf{B}X$ in the relative Scott topology.

The complements of the sets $\uparrow(x, r)$ together with the sets $\Uparrow(x, r)$ form a subbasis of the Lawson topology on $\mathbf{B}X$. By definition of '$\sqsubseteq$' in $\mathbf{B}X$, $\mathsf{i}^{-}(\uparrow(x, r))$ is $C(x, r)$. Thus, i is also a topological embedding w.r.t. the Lawson topology. Summarising, we obtain:

Theorem 13 *On the subset $X^{+} = \mathrm{Max}(\mathbf{B}X)$, the relative Scott and the relative Lawson topology agree. The function $\mathsf{i} : X \to \mathbf{B}X$ with $\mathsf{i}x = (x, 0)$ is a homeomorphism from X with the metric topology onto X^{+} with the relative Scott or Lawson topology.*

If X is a separable complete metric space, then $\mathbf{B}X$ is an ω-continuous dcpo by Cor. 10. Thus, $\mathbf{B}X$ can be given an effective structure along the lines of [2]. This structure can be used to derive an effective structure for X via the homeomorphism between X and X^{+}. This provides a computational framework for X. Recall that a *Polish space* is a topological space which is metrisable with a separable complete metric. Thus, we obtain a computational model for a Polish space by choosing a separable complete metric for it.

Some authors have studied the subspace of maximal points of a domain with the relative Scott topology (MP space). Kamimura and Tang [12] considered MP spaces of bounded complete ω-continuous domains; in the algebraic case, these spaces were shown to be Polish spaces.

This result has now been elegantly generalised by Lawson [15]. He shows that for all ω-continuous domains where the relative Scott and relative Lawson topologies agree on the maximal points, the MP spaces are Polish spaces. He also provides a construction for the opposite direction which shows that every Polish space can be obtained as an MP space of such an ω-continuous domain. The construction presented in the paper at hand proves the same in a much simpler fashion. Furthermore, as we will see in the following, we obtain new results for Banach spaces and a functorial construction; we are also able to prove various fixed point theorems.

2.5 Normed Vector Spaces

In Subsection 2.1, we mentioned that $C(x, r) \supseteq C(y, s)$ does not imply $d(x, y) \leq r - s$ in general. Here, we prove that the equivalence does hold in non-trivial normed vector spaces. Normed vector spaces $(X, \|\,.\,\|)$ (over $\mathbf{R}$) form a special class of metric spaces where the distance is given by $d(x, y) = \|x - y\|$. A normed vector space is non-trivial if it contains elements different from 0.

Theorem 14 *In a non-trivial normed vector space, $C(x,r) \supseteq C(y,s)$ holds iff $d(x,y) \leq r - s$.*

Proof. Let $t = d(x,y) + s$. In the case $x = y$, let u be an element of X different from x, which exists by non-triviality. In the case $x \neq y$, let $u = y$. In both cases, let $z = x + \frac{t}{d(x,u)}(u - x)$. Then $d(x,z) = t$ and $d(y,z) = s$. Hence, z is in $C(y,s) \subseteq C(x,r)$. Thus, $d(x,y) + s = t = d(x,z) \leq r$, whence $d(x,y) \leq r - s$. $\square$

Corollary 15 *For a non-trivial normed vector space, the poset of formal balls is order isomorphic with the poset of closed balls ordered by reverse inclusion.*

A *Banach space* is a normed vector space which is complete in the metric induced by the norm. A *Hilbert space* is a Banach space whose norm is derived from an inner product.

Corollary 16 *For a separable non-trivial Banach space, in particular for a separable non-trivial Hilbert space, the poset of closed balls ordered by reverse inclusion is an ω-continuous dcpo.*

2.6 Completion

Starting from an arbitrary metric space (X, d), we may first construct its *metric completion* $(\overline{X}, \overline{d})$, and then the poset $\mathbf{B}\overline{X}$, which is a continuous dcpo by Theorem 6. On the other hand, we may construct $\mathbf{B}X$, which is a continuous poset, consider $(\mathbf{B}X, \ll)$ as an abstract basis and construct its *rounded ideal completion* $\mathcal{I}(\mathbf{B}X)$ [1], which is a continuous dcpo. We claim that the two continuous dcpo's $\mathbf{B}(\overline{X})$ and $\mathcal{I}(\mathbf{B}X)$ are isomorphic.

We may consider X as a subset of $\overline{X}$ such that $\overline{d}$ restricted to X is d. Since X is dense in $\overline{X}$, the set $\mathbf{B}X = X \times \mathbf{R}_+$ is a basis of $\mathbf{B}\overline{X}$ by Theorem 8. By Prop. 7, the way-below relation of $\mathbf{B}\overline{X}$ restricts to that of $\mathbf{B}X$. By the theory of continuous domains, $\mathbf{B}\overline{X}$ is isomorphic to the rounded ideal completion of its basis $\mathbf{B}X$.

With these results, the metric completion can be constructed domain-theoretically. Starting from an arbitrary metric space X, construct the continuous poset $\mathbf{B}X$ and use rounded ideal completion to obtain $D = \mathcal{I}(\mathbf{B}X)$. Since we know $D \cong \mathbf{B}\overline{X}$, Theorem 13 tells us that $\overline{X}$ is (homeomorphic to) $\mathrm{Max}\, D$, the space of maximal elements of D in the relative Scott topology. Thus, we obtain $\overline{X}$ as a topological space. In a future paper, the metric of X will be extended to a partial metric on $\mathbf{B}X$ and D, whose restriction to $\operatorname{Max} D$ will recover the metric on $\overline{X}$.

3 The Functor B and the Fixed Point Theorem

In this section, we make $\mathbf{B}$ into a functor from a category of metric spaces to the category of continuous posets and continuous functions. This functor is then used to show how Banach's fixed point theorem for complete metric spaces can be derived from the dcpo fixed point theorem.

3.1 The Functor B

Let $f : X \to Y$ be a function between two metric spaces. A *Lipschitz constant* for f is a number c in $\mathbf{R}_+$ such that for all x, x' in X, $d(fx, fx') \leq c \cdot d(x, x')$ holds. If f has a Lipschitz constant then it is continuous. However, not every continuous map has a Lipschitz constant.

The collection of pairs (f, c) of a function and a Lipschitz constant forms a category with $\mathsf{id}_X = (\mathsf{id}_X, 1)$ and $(g, c') \circ (f, c) = (g \circ f, c' \cdot c)$. The functor $\mathbf{B}$ is defined on this category by $\mathbf{B}(f, c)(x, r) = (fx, c \cdot r)$ for every (x, r) in $\mathbf{B}X$. The functorial properties of $\mathbf{B}$ are obvious. We also want to show that $g = \mathbf{B}(f, c) : \mathbf{B}X \to \mathbf{B}Y$ is Scott continuous.

The function g is monotonic since

$$\begin{aligned}(x, r) \sqsubseteq (x', s) &\Rightarrow d(x, x') \leq r - s \\ &\Rightarrow d(fx, fx') \leq c \cdot d(x, x') \leq cr - cs \\ &\Rightarrow (fx, cr) \sqsubseteq (fx', cs) \\ &\Rightarrow g(x, r) \sqsubseteq g(x', s)\,.\end{aligned}$$

For Scott continuity, it suffices to consider ascending sequences. If $(x_n, r_n)_{n\in\mathbf{N}}$ is ascending with least upper bound (x, r), then by Theorem 5 (ii), (x, r) is an upper bound of $(x_n, r_n)_{n\in\mathbf{N}}$ and $\lim_{n\to\infty} r_n = r$. Hence $\lim_{n\to\infty} cr_n = cr$, and by monotonicity of g, $g(x, r) = (fx, cr)$ is an upper bound of $(g(x_n, r_n))_{n\in\mathbf{N}} = (fx_n, cr_n)_{n\in\mathbf{N}}$. By Theorem 5 again, $g(x, r)$ is the least upper bound of $(g(x_n, r_n))_{n\in\mathbf{N}}$.

Let us see how g cooperates with i. For every x in X, $g(\mathsf{i}x) = \mathbf{B}(f, c)(x, 0) = (fx, 0) = \mathsf{i}(fx)$ holds, whence $g \circ \mathsf{i} = \mathsf{i} \circ f$. If we identify X and X^+, this means that g extends f.

Theorem 17 *With the definition* $\mathbf{B}(f,c)(x,r) = (fx, c \cdot r)$*,* $\mathbf{B}$ *becomes a functor from the category of maps with Lipschitz constants between metric spaces to the category of Scott continuous functions between continuous posets. This functor has the property* $\mathbf{B}(f,c) \circ \mathsf{i} = \mathsf{i} \circ f$ *for all maps* f *with Lipschitz constant* c*.*

3.2 The Fixed Point Theorem for Contracting Functions

As announced earlier, we now give a domain-theoretic proof of the Banach fixed point theorem.

Theorem 18 *A contracting map on a complete metric space has a unique fixed point which is the limit of the orbit of any point in the space.*

Proof. Let (X,d) be a complete metric space and $f : X \to X$ a contracting map. This means f has a Lipschitz constant $c < 1$. The function $g = \mathbf{B}(f,c) : \mathbf{B}X \to \mathbf{B}X$ is Scott continuous.

Let x be an arbitrary point of X, and let $R_x = d(x, fx)/(1-c)$. For every $r \geq R_x$, we obtain $d(x, fx) \leq (1-c)r = r - cr$, whence $(x,r) \sqsubseteq (fx, cr) = g(x,r)$. Using monotonicity, this relation implies that g maps $\uparrow(x,r)$ into itself.

Since $\uparrow(x,r)$ is a dcpo with least element (x,r), the dcpo fixed point theorem implies that $(f^n x, c^n r)_{n \in \mathbf{N}}$ is an ascending sequence whose least upper bound (y,s) is the least fixed point of g on $\uparrow(x,r)$. By Theorem 5, $s = \lim_{n\to\infty} c^n r = 0$ and $y = \lim_{n\to\infty} f^n x$ holds. Since $(y,s) = (y,0)$ is maximal in $\mathbf{B}X$ by Prop. 11, it is not only the least, but also the unique fixed point of g on $\uparrow(x,r)$.

Clearly, $(y,0) = (\lim_{n\to\infty} f^n x, 0)$ is independent from $r \geq R_x$. We show that it is the unique fixed point of g not only in $\uparrow(x,r)$, but in the whole of $\mathbf{B}X$. Let (z,t) be an arbitrary fixed point of g. Let $r \geq \max(R_x d(x,z) + t)$. Then (z,t) is in $\uparrow(x,r)$ and thus must equal $(y,0)$.

So far, we have shown that $(y,0)$ is the unique fixed point of g in $\mathbf{B}X$. Hence, y is independent from x, i.e., $y = \lim_{n\to\infty} f^n x$ for all x in X. Since $z \in X$ is a fixed point of $f : X \to X$ iff $(z,0)$ is a fixed point of $g : \mathbf{B}X \to \mathbf{B}X$, it follows that y is the unique fixed point of f. $\square$

4 Iterated Function Systems

Banach's fixed point theorem can be extended to systems $(X; f_1, \ldots, f_N)$ of several contracting maps $f_i : X \to X$. Such systems are called *(hyperbolic) iterated function systems (IFS).*

In the classical work of Hutchinson [9], one considers the space $\mathbf{H}X$ of non-empty compact subsets of X with the *Hausdorff metric* d_H. One shows that $(\mathbf{H}X, d_H)$ is a complete metric space and that the map $f : \mathbf{H}X \to \mathbf{H}X$ defined by $f(K) = \bigcup_{1 \le i \le N} f_i^+(K)$ is contracting. It will therefore have a unique fixed point in $\mathbf{H}X$, the so-called *attractor* of the IFS.

With the techniques used in the proof of Theorem 18, a domain-theoretic proof of this result can be given which completely avoids the Hausdorff metric. This extends the corresponding result for a compact metric space as in [8,5]. The proof is performed in the Plotkin power domain [1] of the domain $\mathbf{B}X$. We first recall some basic definitions (see [1]).

Let D be a continuous dcpo. For subsets A and B of D, we define

- $A \prec_L B$ iff for all a in A, there is b in B such that $a \ll b$;
- $A \prec_U B$ iff for all b in B, there is a in A such that $a \ll b$;
- $A \prec B$ iff $A \prec_L B$ and $A \prec_U B$.

Let $\mathcal{P}_{\mathit{fin}} D$ be the set of non-empty finite subsets of D. The structure $(\mathcal{P}_{\mathit{fin}} D, \prec)$ is an abstract basis, whose rounded ideal completion is $\mathbf{C}D$, the *Plotkin power domain* of D. The points of $\mathbf{C}D$ are the rounded ideals of $(\mathcal{P}_{\mathit{fin}} D, \prec)$, i.e., non-empty subsets $\mathcal{I}$ of $\mathcal{P}_{\mathit{fin}} D$ with

(i) if $F \in \mathcal{I}$ and $G \prec F$, then $G \in \mathcal{I}$;
(ii) for every F, G in $\mathcal{I}$, there is H in $\mathcal{I}$ such that $F, G \prec H$.

Property (ii) implies that for every F in $\mathcal{I}$, there is H in $\mathcal{I}$ such that $F \prec H$. The order of $\mathbf{C}D$ is set inclusion '$\subseteq$' among the rounded ideals.

Let $\mathcal{P}_{\mathit{com}} D$ be the set of all non-empty compact subsets of D. (These sets are not required to be saturated.) For every K in $\mathcal{P}_{\mathit{com}} D$, there is a corresponding element $K^* = \{F \in \mathcal{P}_{\mathit{fin}} D \mid F \prec K\}$ of $\mathbf{C}D$. Note that $K_1^* = K_2^*$ does not imply $K_1 = K_2$. For finite F and G, $F^* \sqsubseteq G^*$ holds iff $F \sqsubseteq_L G$ and $F \sqsubseteq_U G$, where '$\sqsubseteq_L$' and '$\sqsubseteq_U$' are derived from '$\sqsubseteq$' on D in the same manner as '$\prec_L$' and '$\prec_U$' are derived from '$\ll$'.

The function $+ : \mathbf{C}D \times \mathbf{C}D \to \mathbf{C}D$ with $\mathcal{I} + \mathcal{J} = \{H \in \mathcal{P}_{\mathit{fin}} D \mid \exists F \in \mathcal{I}, G \in \mathcal{J} : H \prec F \cup G\}$ is continuous, commutative, associative, and idempotent. For K, L in $\mathcal{P}_{\mathit{com}} D$, $K^* + L^* = (K \cup L)^*$ holds. For continuous $h : D \to D$, there is a continuous function $\mathbf{C}h : \mathbf{C}D \to \mathbf{C}D$ with $\mathbf{C}h(\mathcal{I}) = \{G \in \mathcal{P}_{fin} D \mid \exists F \in \mathcal{I} : G \prec h^+ F\}$. For K in $\mathcal{P}_{com} D$, $\mathbf{C}h(K^*) = (h^+ K)^*$ holds.

We apply these general results to $D = \mathbf{B}X$ for a complete metric space X. By Cor. 10 and Theorem 6, D is a continuous dcpo. In this special case, we can say more about the structure of $\mathbf{C}D$.

For F in $\mathcal{P}_{fin} D$, we define $\mathsf{r}F$ to be the greatest radius of formal balls in F, i.e., $\mathsf{r}F = \max\{r \mid (x,r) \in F\}$. Note that $F \prec G$ implies $\mathsf{r}F > \mathsf{r}G$. For $\mathcal{I}$ in $\mathbf{C}D$, we define $\bar{\mathsf{r}}\mathcal{I} = \inf\{\mathsf{r}F \mid F \in \mathcal{I}\}$. For F in $\mathcal{P}_{fin} D$, $\bar{\mathsf{r}}F^*$ equals $\mathsf{r}F$. Later, we shall see that the elements $\mathcal{I}$ of $\mathbf{C}D$ with $\bar{\mathsf{r}}\mathcal{I} = 0$ are maximal in D and can be identified with the non-empty compact subsets of X. For this, we need a lemma.

Lemma 19 *If an element $\mathcal{I}$ of $\mathbf{C}D$ contains a sequence $(F_n)_{n \in \mathbf{N}}$ with $\lim_{n \to \infty} \mathsf{r}F_n = 0$, then for every G in $\mathcal{I}$, $G \prec F_n$ holds for some n in $\mathbf{N}$.*

Proof. For G in $\mathcal{I}$, there is H in $\mathcal{I}$ such that $G \prec H$. Let $\delta = \min\{r - s - d(x,y) \mid (x,r) \in G (y,s) \in H (x,r) \ll (y,s)\}$, which is greater than 0. Thus, there is n in $\mathbf{N}$ such that $\mathsf{r}F_n < \delta/2$. Since both H and F_n are in $\mathcal{I}$, there is M in $\mathcal{I}$ such that $H, F_n \prec M$. We claim $G \prec F_n$.

To prove $G \prec_L F_n$, let (x,r) be in G. Because of $G \prec H \prec M \succ F_n$, there are (y,s) in H, (z,t) in M, and (a,u) in F_n such that $(x,r) \ll (y,s) \ll (z,t) \gg (a,u)$. The computation

$$\begin{aligned} d(x,a) &\le d(x,y) + d(y,z) + d(z,a) \\ &< (r - s - \delta) + (s - t) + (u - t) \\ &\le r + u - \delta < r - u \end{aligned}$$

shows $(x,r) \ll (a,u)$. The last '$<$' follows from $u \le \mathsf{r}F_n < \delta/2$.

To prove $G \prec_U F_n$, let (a,u) be in F_n. Because of $G \prec H \prec M \succ F_n$, there are (z,t) in M, (y,s) in H, and (x,r) in G such that $(x,r) \ll (y,s) \ll (z,t) \gg (a,u)$. As above, $(x,r) \ll (a,u)$ follows. $\square$

Theorem 20 *Every $\mathcal{I}$ in $\mathbf{C}D$ with $\bar{\mathsf{r}}\mathcal{I} = 0$ is maximal in $\mathbf{C}D$.*

Proof. Assume $\mathcal{I} \subseteq \mathcal{J}$ for some $\mathcal{J}$ in $\mathbf{C}D$. We have to show $\mathcal{J} \subseteq \mathcal{I}$. For every n in $\mathbf{N}$, $\mathcal{I}$ contains some F_n with $\mathsf{r}F_n < 1/n$. Because of $\mathcal{I} \subseteq \mathcal{J}$, all F_n are in $\mathcal{J}$. By Lemma 19 applied to $\mathcal{J}$, every G in $\mathcal{J}$ satisfies $G \prec F_n \in \mathcal{I}$ for some n. By property (i) of rounded ideals, G is in $\mathcal{I}$. $\square$

Next, we show how to characterise the least upper bound of an ascending sequence with r-values tending to zero.

Proposition 21 *For a sequence $(F_n)_{n\in\mathbf{N}}$ in $\mathcal{P}_{\mathit{fin}} D$ with $F_n^* \sqsubseteq F_{n+1}^*$ for all n and $\lim_{n\to\infty} \mathsf{r} F_n = 0$, the least upper bound $\bigcup_{n\in\mathbf{N}} F_n^*$ is maximal in $\mathbf{C}D$ and equals A^*, where A is the filtered intersection $\bigcap_{n\in\mathbf{N}} \uparrow F_n$. This set A is a non-empty compact subset of* $\operatorname{Max} D = X^+$.

Proof. Let $\mathcal{I} = \bigcup_{n\in\mathbf{N}} F_n^*$. For every n in $\mathbf{N}$, $\bar{\mathsf{r}}\,\mathcal{I} \leq \bar{\mathsf{r}} F_n^* = \mathsf{r} F_n$ holds, whence $\bar{\mathsf{r}}\,\mathcal{I} = 0$. By Theorem 20, $\mathcal{I}$ is maximal in $\mathbf{C}D$.

Let $A = \bigcap_{n\in\mathbf{N}} \uparrow F_n$. Since $F_n^* \sqsubseteq F_{n+1}^*$ implies $\uparrow F_n \supseteq \uparrow F_{n+1}$, this is a filtered intersection of non-empty compact saturated subsets of D. Since D is sober as a continuous dcpo, A is non-empty and compact in D. Since $\lim_{n\to\infty} \mathsf{r} F_n = 0$, A is a subset of X^+.

We claim $\mathcal{I} = A^*$. Because of maximality of $\mathcal{I}$, it suffices to show '$\subseteq$'. If G is in $\mathcal{I}$, $G \prec F_m$ holds for some m in $\mathbf{N}$. Relation $G \prec_U A$ follows from $A \sqsupseteq_U F_m \succ_U G$. For $G \prec_L A$, let c be in G. Then $c \ll b$ for some b in F_m. Using $F_n \sqsubseteq_L F_{n+1}$ for $n \geq m$ and $F_n \sqsupseteq_U F_{n-1}$ for $1 < n \leq m$, we obtain a sequence $b_1 \sqsubseteq b_2 \sqsubseteq \cdots$ with b_n in F_n and $b_m = b$. Let $a = \bigsqcup_{n\in\mathbf{N}} b_n$. Then a is in A, and $c \ll b_m \sqsubseteq a$ holds. This proves $G \prec A$, i.e., $G \in A^*$. $\square$

Let K be an element of $\mathcal{P}_{\mathit{com}} X$, i.e., a non-empty compact subset of X. Then $K^+ = \mathsf{i}^+ K = \{(x,0) \mid x \in K\}$ is a non-empty compact subset of $D = \mathbf{B}X$. Hence, $\bar{\mathsf{i}} K = (K^+)^*$ is an element of $\mathbf{C}D$. This defines a function $\bar{\mathsf{i}} : \mathcal{P}_{\mathit{com}} X \to \mathbf{C}D$.

Theorem 22 *The function $\bar{\mathsf{i}} : \mathcal{P}_{\mathit{com}} X \to \mathbf{C}D$ is a one-to-one correspondence between $\mathcal{P}_{\mathit{com}} X$ and the elements $\mathcal{I}$ of $\mathbf{C}D$ with $\bar{\mathsf{r}}\,\mathcal{I} = 0$.*

Proof. First we prove $\bar{\mathsf{r}}\,\mathcal{I} = 0$ for $\mathcal{I} = \bar{\mathsf{i}} K$ where K in $\mathcal{P}_{\mathit{com}} X$. Let $K^+ = \{(x,0) \mid x \in K\}$, and fix $r > 0$. Then $K^+ \subseteq \bigcup_{x\in K} \Uparrow(x,r)$ holds. Since K^+ is compact in D, there is a finite subset S of K with $K^+ \subseteq \bigcup_{x\in S} \Uparrow(x,r)$. Thus $F \prec_U K^+$ where $F = \{(x,r) \mid x \in S\}$. Relation $F \prec_L K^+$ holds since $(x,r) \ll (x,0)$. Hence F is in $\bar{\mathsf{i}} K$. Clearly, $\mathsf{r} F$ is r. Thus $\bar{\mathsf{r}}\,\mathcal{I} \leq r$ for every $r > 0$, whence $\bar{\mathsf{r}}\,\mathcal{I} = 0$.

Second, we show injectivity of $\bar{\mathsf{i}}$. Let $\bar{\mathsf{i}} K = \bar{\mathsf{i}} L$, and assume there is x in K which is not in L. For every y in L, let $s_y = \frac{1}{2} d(x,y) > 0$. By arguments as above, $\bar{\mathsf{i}} L$ contains some finite subset G of $\{(y, s_y) \mid y \in L\}$. ¿From $\bar{\mathsf{i}} K = \bar{\mathsf{i}} L$, $G \prec K^+$ follows. In particular, there is some (y, s_y) in G such that $(x,0) \gg (y, s_y)$, or $d(x,y) < s_y$, a contradiction.

Finally, for $\mathcal{I}$ in $\mathbf{C}D$ with $\bar{\mathsf{r}}\mathcal{I} = 0$, we find a non-empty compact subset K of X with $\mathcal{I} = \bar{\mathsf{i}}K$. By property (ii) of rounded ideals and $\bar{\mathsf{r}}\mathcal{I} = 0$, there is a sequence $F_1 \prec F_2 \prec \cdots$ in $\mathcal{I}$ with $\mathsf{r}F_n < 1/n$. Hence, $F_n^* \sqsubseteq F_{n+1}^*$ follows. By Lemma 19, $\mathcal{I} = \bigcup_{n\in\mathbf{N}} F_n^*$ holds, and by Prop. 21, $\bigcup_{n\in\mathbf{N}} F_n^* = A^*$ for some non-empty compact subset A of X^+. Since X and X^+ are homeomorphic by i, $K = \mathsf{i}^- A$ is a non-empty compact subset of X. ¿From $\mathcal{I} = A^*$, $\mathcal{I} = \bar{\mathsf{i}}K$ follows. □

After these preliminaries, we now come to the theorem about hyperbolic IFS.

Theorem 23 *An hyperbolic IFS on a complete metric space has a unique non-empty and compact attractor.*

Proof. Given $f_1, \ldots, f_N : X \to X$ where each f_i has a Lipschitz constant $c_i < 1$, we have to show that $f : \mathcal{P}_{com}X \to \mathcal{P}_{com}X$ with $f(K) = \bigcup_{1\leq i\leq N} f_i^+ K$ has a unique fixed point.

The functions $g_i = \mathbf{B}(f_i, c_i) : \mathbf{B}X \to \mathbf{B}X$ are continuous, whence $g : \mathbf{CB}X \to \mathbf{CB}X$ with $g(\mathcal{I}) = \sum_{1\leq i\leq N} \mathbf{C}g_i(\mathcal{I})$ is continuous. For compact $K \subseteq \mathbf{B}X$, $g(K^*) = \{(f_i x, c_i r) \mid 1 \leq i \leq N (x, r) \in K\}^*$ holds. Hence, $g \circ \bar{\mathsf{i}} = \bar{\mathsf{i}} \circ f$ follows. ¿From this and injectivity of $\bar{\mathsf{i}}$, we conclude that K is a fixed point of f iff $\bar{\mathsf{i}}K$ is a fixed point of g.

The following is similar to the proof of Theorem 18. Fix any x in X, let $R = \max_{1\leq i\leq N} d(x, f_i x)/(1 - c_i)$, and consider an arbitrary $r \geq R$. For $a = (x, r)$, $g_i a \sqsupseteq a$ holds for all i, whence $g\{a\}^* \sqsupseteq \{a\}^*$. Therefore, g maps the subdomain $\uparrow\{a\}^*$ to itself and thus has a least fixed point $\mathcal{I}_a = \bigsqcup_{n\in\mathbf{N}} g^n\{a\}^*$ on it. For every n in $\mathbf{N}$, $\bar{\mathsf{r}}\mathcal{I}_a \leq \bar{\mathsf{r}}(g^n\{a\}^*) \leq c^n \cdot r$ holds where $c = \max_{1\leq i\leq N} c_i < 1$. Hence, $\bar{\mathsf{r}}\mathcal{I}_a$ is 0. By Theorem 20, $\mathcal{I}_a$ is maximal in $\mathbf{C}D$, and thus the unique fixed point of g in $\uparrow\{a\}^*$.

If $r' \geq r$, then $\uparrow\{(x, r')\}^* \supseteq \uparrow\{(x, r)\}^*$. By the uniqueness statement just proved, $\mathcal{I}_{(x,r)}$ and $\mathcal{I}_{(x,r')}$ are identical. Let us write $\mathcal{I}_x$ for it. By Theorem 22, there is some K_x in $\mathcal{P}_{com}X$ such that $\mathcal{I}_x = \bar{\mathsf{i}}K_x$. By the arguments at the beginning of this proof, K_x is a fixed point of f.

Let L be an arbitrary fixed point of f. By compactness of L, there is some $r \geq R$ such that $L \subseteq O(x, r)$. Then $(x, r) \ll (y, 0)$ for all y in L, whence $\{(x, r)\} \prec \{(y, 0) \mid y \in L\}$, which implies $\bar{\mathsf{i}}L \in \uparrow\{(x, r)\}^*$. Since $\bar{\mathsf{i}}L$ is a fixed point of g, $\bar{\mathsf{i}}L = \mathcal{I}_x = \bar{\mathsf{i}}K_x$ follows, whence by Theorem 22, $L = K_x$. □

The unique non-empty compact attractor K of the IFS can be obtained as a filtered intersection of finite unions of closed balls. Fix x in X and $r \geq R$ as in the proof above. ¿From the proof, we know $\bar{\mathsf{i}}K = \bigsqcup_{n \in \mathbf{N}} g^n \{(x, r)\}^*$. For every n in $\mathbf{N}$, $g^n\{(x,r)\}^* = F_n^*$ holds where $F_n = \{(f_{i_1} \ldots f_{i_n} x, c_{i_1} \cdots c_{i_n} \cdot r) \mid i_1, \ldots, i_n \in \Sigma\}$ with $\Sigma = \{1, \ldots, N\}$. By Prop. 21, $\bar{\mathsf{i}}K = A^*$ follows where A is the filtered intersection $\bigcap_{n \in \mathbf{N}} \uparrow F_n$. Hence, $K = \mathsf{i}^- A = \bigcap_{n \in \mathbf{N}} \mathsf{i}^-(\uparrow F_n)$. Since $\mathsf{i}^-(\uparrow(y, s)) = C(y, s)$, K is the filtered intersection $\bigcap_{n \in \mathbf{N}} \bigcup_{i_1, \ldots, i_N \in \Sigma} C(f_{i_1} \ldots f_{i_n} x, c_{i_1} \cdots c_{i_n} \cdot r)$.

5 A Computational Framework for Measure Theory

In this section, we embed the space $\mathbf{M}^1 X$ of normalised Borel measures on a metric space X into the probabilistic powerdomain $\mathbf{PB}X$ of valuations on $\mathbf{B}X$. Thus, $\mathbf{PB}X$ becomes a computational model for $\mathbf{M}^1 X$, similar as $\mathbf{B}X$ itself is a model for X.

5.1 Measures and Valuations

A *(Borel) measure* on a topological space X is a function μ from the Borel sets of X to $\mathbf{R}_+$ which is *strict* ($\mu\emptyset = 0$), *modular* ($\mu(A \cup B) + \mu(A \cap B) = \mu A + \mu B$ for Borel sets A and B), and *countably continuous* ($\mu(\bigcup_{n \in \mathbf{N}} B_n) = \sup_{n \in \mathbf{N}} \mu B_n$ for Borel sets $B_1 \subseteq B_2 \subseteq \cdots$).

A measure μ on X is *normalised* if $\mu X = 1$, and *continuous* if $\mu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mu U_i$ for every directed family $(U_i)_{i \in I}$ of opens. All measures on a separable metric space are continuous.

The *support* of a measure μ is the closed set of all x such that $\mu O > 0$ for all opens O containing x. We call a measure μ *well-supported* if $\mu S' = 0$ holds for the complement S' of the support of μ. Obviously, every continuous measure is well-supported. By Theorem 2.2.16 in [7], all measures on metric spaces have separable support. Hence, well-supported measures on metric spaces are continuous. Therefore, the notions 'well-supported' and 'continuous' coincide on metric spaces.

It is consistent with set theory to assume that all measures are well-supported (see [7]). Apparently, this assumption was implicitly adopted by Hutchinson [9].

The computational framework for measure theory will be defined in terms of valuations on $\mathbf{B}X$. A *valuation* on a topological space Y is a function from the opens of Y to $\mathbf{R}_+$ which is strict, modular, and Scott continuous. The *probabilistic power domain* $\mathbf{P}Y$ [10,11] of Y consists of all valuations ν on Y with $\nu(Y) \leq 1$, ordered by their values on Scott open sets.

For a in Y, the *point valuation* $\delta(a)$ is defined by $\delta(a)(O) = 1$ if a in O, and $= 0$ otherwise. The function $\delta(.)$ is an order embedding, i.e., $\delta(a) \sqsubseteq \delta(b)$ holds iff $a \sqsubseteq b$. For continuous $h : Y \to Y'$, the function $\mathbf{P}h : \mathbf{P}Y \to \mathbf{P}Y'$ with $\mathbf{P}h(\nu)(O) = \nu(h^{-}O)$ is well-defined and continuous. Obviously, $\mathbf{P}h(\delta(a)) = \delta(ha)$ holds.

A *simple valuation* is a finite linear combination $r_1 \cdot \delta(a_1) + \cdots + r_n \cdot \delta(a_n)$ of point valuations. If Y is an (ω-)continuous dcpo with its Scott topology, then $\mathbf{P}Y$ is (ω-)continuous again [10,11] with a basis of simple valuations. In fact, simple valuations with rational coefficients over elements of a basis of Y are sufficient.

¿From the definitions, it is obvious that every continuous measure yields a valuation when restricted to the opens. Different measures yield different valuations since Borel measures are uniquely determined by their values on open sets. On a metric space, every valuation μ extends uniquely to a continuous measure by defining $\mu B = \inf\{\mu O \mid O \text{ open } \supseteq B\}$ for Borel sets B. Hence, continuous measures and valuations can be identified on metric spaces. A corresponding result for continuous dcpo's is not known. For ω-continuous dcpo's however, it is known that every valuation extends uniquely to a Borel measure [13,16].

5.2 Valuations on $\mathbf{B}X$

For a complete metric space X, let $\mathbf{M}^1X$ be the set of normalised well-supported measures on X. We embed $\mathbf{M}^1X$ into the probabilistic power domain $\mathbf{PB}X$ of $\mathbf{B}X$, which forms a continuous dcpo. Let SSX be the set of all ν in $\mathbf{PB}X$ with $\nu O = 1$ for all open supersets O of X^+. We shall prove that SSX is a subset of the maximal elements of $\mathbf{PB}X$, and define a one-to-one correspondence $\bar{\mathsf{i}} : \mathbf{M}^1X \to SSX$. For both statements, we need an auxiliary lemma.

Lemma 24 *Let ν be in $\mathbf{PB}X$, U an open of $\mathbf{B}X$, and c in $\mathbf{R}_+$ such that $\nu U > c$. Then there are opens V and W of $\mathbf{B}X$ such that $V \subseteq U$, $\nu V > c$, $V \cap W = \emptyset$, and $U \cup W \supseteq X^+$.*

Proof. For n in $\mathbf{N}$, let $V_n = \{(x,r) \in \mathbf{B}X \mid (x, r + 1/n) \in U\}$. These sets are open as inverse images of U under the continuous functions $(x,r) \mapsto (x, r + 1/n)$. Clearly, $V_1 \subseteq V_2 \subseteq \cdots$ holds. Since $(x,r) = \bigsqcup_{n \in \mathbf{N}}(x, r+1/n)$, we obtain $U = \bigcup_{n \in \mathbf{N}} V_n$. By the countable continuity property, $\nu U = \sup_{n \in \mathbf{N}} \nu V_n > c$ follows, whence there is some n such that $\nu V_n > c$.

Let $V = V_n$ and $W = \bigcup\{\Uparrow(x, 1/n) \mid (x,0) \notin U\}$. For every x in X, $(x,0) \notin U$ implies $(x,0) \in W$. Hence, X^+ is a subset of $U \cup W$. If (x,r) is in $V \cap W$, then so is $(x,0)$. Thus $(x,1/n)$ is in U, and there is $(y,0) \notin U$ such that $(x,0) \gg (y,1/n)$. The latter relation implies $(y,0) \gg (x,1/n)$. This is impossible since $(x,1/n)$ is in U, but $(y,0)$ is not. Therefore, $V \cap W$ is empty. $\square$

The lemma can be used to prove the maximality claim.

Theorem 25 *For every complete metric space X, the elements of SSX are maximal in $\mathbf{PB}X$.*

Proof. Let ν be in SSX, and let $\nu \sqsubseteq \nu'$ for some ν' in $\mathbf{PB}X$. Assume $\nu' U > \nu U$ for some open U. By Lemma 24, there are opens $V \subseteq U$ and W with $\nu' V > \nu U$, $V \cap W = \emptyset$, and $U \cup W \supseteq X^+$.

$$\begin{aligned}
\nu'(V \cup W) &= \nu' V + \nu' W && \text{since } V \cap W = \emptyset \\
&> \nu U + \nu W && \text{since } \nu' V > \nu U \text{ and } \nu' \sqsupseteq \nu \\
&\geq \nu(U \cup W) && \text{by modularity} \\
&= 1 && \text{since } X^+ \subseteq U \cup W \text{ and } \nu \text{ in } SSX
\end{aligned}$$

The result $\nu'(V \cup W) > 1$ is impossible. $\square$

Next, we show by a similar argument that for ν in SSX, the value νO only depends on $\mathsf{i}^- O$, where $\mathsf{i} : X \to \mathbf{B}X$ with $\mathsf{i}x = (x,0)$ is the embedding of X into $\mathbf{B}X$.

Proposition 26 *Let ν be in SSX. If U and U' are two opens of $\mathbf{B}X$ with $\mathsf{i}^- U \subseteq \mathsf{i}^- U'$, then $\nu U \leq \nu U'$.*

Proof. Assume $\nu U > \nu U'$, and apply Lemma 24. It gives opens $V \subseteq U$ and W with $\nu V > \nu U'$, $V \cap W = \emptyset$, and $U \cup W \supseteq X^+$.

$$\begin{aligned}
\nu(V \cup W) &= \nu V + \nu W && \text{since } V \cap W = \emptyset \\
&> \nu U' + \nu W && \text{since } \nu V > \nu U' \\
&\geq \nu(U' \cup W) && \text{by modularity} \\
&= 1 && \text{since } X \subseteq \mathsf{i}^- U \cup \mathsf{i}^- W \subseteq \mathsf{i}^- U' \cup \mathsf{i}^- W
\end{aligned}$$

However, $\nu(V \cup W) > 1$ is impossible. $\square$

Now we come to the correspondence between $\mathbf{M}^1X$ and SSX.

Theorem 27 *For every complete metric space X, the function $\bar{\mathsf{i}} : \mathbf{M}^1X \to SSX$ with $\bar{\mathsf{i}}\mu(V) = \mu(\mathsf{i}^- V)$ for opens V of $\mathbf{B}X$ is a bijection.*

Proof. Since μ in $\mathbf{M}^1X$ restricts to a continuous valuation on X, $\bar{\mathsf{i}}\mu$ is a continuous valuation on $\mathbf{B}X$. If $X^+ \subseteq V$, then $\mathsf{i}^- V = X$ and so $\bar{\mathsf{i}}\mu(V) = \mu X = 1$. Thus, $\bar{\mathsf{i}}\mu$ is in SSX.

For the opposite direction, let ν be in SSX. Since $\mathsf{i} : X \to \mathbf{B}X$ is a topological embedding, every open U of X can be written as $U = \mathsf{i}^- V$ for some open V of $\mathbf{B}X$. Let U^* be the union of these V. The assignment $U \mapsto U^*$ is monotonic, and satisfies $\mathsf{i}^- U^* = U$ for opens U of X and $V \subseteq (\mathsf{i}^- V)^*$ for opens V of $\mathbf{B}X$. We can use it to define $\nu' U = \nu(U^*)$ for opens U of X. The function ν' is a valuation on X with $\nu' X = 1$. To prove this note that $\mathsf{i}^- V = \mathsf{i}^- W$ implies $\nu V = \nu W$ by Prop. 26. Hence, $\nu((U_1 \cup U_2)^*) = \nu(U_1^* \cup U_2^*)$ holds, and likewise for '$\cap$'. Thus $\nu'(U_1 \cup U_2) + \nu'(U_1 \cap U_2) = \nu(U_1^* \cup U_2^*) + \nu(U_1^* \cap U_2^*) = \nu' U_1 + \nu' U_2$. Continuity is shown analogously.

Consider $\mu \mapsto \bar{\mathsf{i}}\mu \mapsto (\bar{\mathsf{i}}\mu)'$. For opens U of X, $(\bar{\mathsf{i}}\mu)'(U) = \mu(\mathsf{i}^- U^*) = \mu U$ holds. Finally, consider $\nu \mapsto \nu' \mapsto \bar{\mathsf{i}}\nu'$. For opens V of $\mathbf{B}X$, $\bar{\mathsf{i}}\nu'(V) = \nu((\mathsf{i}^- V)^*) \geq \nu V$ holds. By Theorem 25, $\bar{\mathsf{i}}\nu' = \nu$ follows. $\square$

5.3 The Separable Case

We can obtain much stronger results when we restrict ourselves to *separable* complete metric spaces X. In this case, $\mathbf{B}X$ is an ω-continuous dcpo, and $\mathbf{PB}X$ is ω-continuous as well [10,11]. Thus it admits an effective treatment along the lines of [2]. This generalises the results of [4,3], which were restricted to locally compact spaces.

By Theorem 27, the set $\mathbf{M}^1X$ of normalised measures of X corresponds to the subset SSX of $\mathbf{PB}X$. By [6], this correspondence is a homeomorphism if $\mathbf{M}^1X$ is equipped with the weak topology and SSX with the relative Scott topology.

By ω-continuity, every valuation ν on $\mathbf{B}X$ extends uniquely to a Borel measure [13,16]; for convenience this unique extension will be denoted by ν as well. Since X^+ is a G_δ-subset of $\mathbf{B}X$ (Prop. 12), the defining condition of SSX can be simplified to $\nu(X^+) = 1$.

By Theorem 25, every element of SSX is maximal in $\mathbf{PB}X$. In the separable case, the opposite statement also holds. Lawson [14] has proved this independently from us even for the more general situation where $\mathbf{B}X$ is replaced by any ω-continuous dcpo D such that the relative Scott and Lawson topology agree on $\operatorname{Max} D$ and X is homeomorphic to $\operatorname{Max} D$. While Lawson's proof uses the von Neumann Selection Lemma, we present an elementary proof for the case of $D = \mathbf{B}X$.

Theorem 28 *If X is a separable complete metric space, then every maximal element of $\mathbf{PB}X$ is in SSX.*

Proof. Let $D = \mathbf{B}X$, and let μ be maximal in $\mathbf{P}D$. Let W be an open of D with $X^+ \subseteq W$. We show $\mu W = 1$.

In the first part of the proof, we consider μ as a valuation. Let S be a countable dense set of X. Fix any x in S, and let U be the complement of $\downarrow(x, 0)$ in D. Let $\mu' O = \mu(O \cap U)$ for open sets O of D, and let $\nu = \mu' + (1 - \mu U) \cdot \delta(x, 0)$. Obviously, μ' and ν are valuations on X. Since $\nu D = \mu U + 1 - \mu U = 1$, ν is in $\mathbf{P}D$. If $(x, 0)$ is not in O, then $O \subseteq U$, and thus $\nu O = \mu(O \cap U) = \mu O$. If $(x, 0)$ is in O, then $\nu O = \mu(O \cap U) + 1 - \mu U \geq \mu(O \cap U) + \mu(O \cup U) - \mu U = \mu O$. Hence, $\mu \sqsubseteq \nu$ in $\mathbf{P}D$, whence $\mu = \nu$ follows by maximality of μ. Applying this to D and W, we obtain $\mu D = \nu D = 1$, and since $(x, 0) \in X^+ \subseteq W$, $\mu W = \mu(W \cap U) + 1 - \mu U$, whence $\mu(W \cup U) = 1$ by modularity.

¿From now on, we consider μ as a Borel measure. ¿From $\mu D = \mu(W \cup U) = 1$, we conclude $\mu(C \cap \downarrow(x, 0)) = 0$, where C is the complement of W. By additivity of μ and countability of S, $\mu(C \cap \bigcup_{x \in S} \downarrow(x, 0))$ is 0. If (y, s) is in C, then $s > 0$ because C is disjoint from X^+. Since S is dense in X, $O(y, s)$ contains some x in S, and $(y, s) \in \downarrow(x, 0)$ follows. Therefore, C is a subset of $\bigcup_{x \in S} \downarrow(x, 0)$. Thus we obtain $\mu C = 0$, whence by complementation $\mu W = 1$. $\square$

Corollary 29 *For any complete separable metric space X, the space $\mathbf{M}^1 X$ of normalised measures on X is homeomorphic with the space of maximal elements of the ω-continuous dcpo $\mathbf{PB}X$. These maximal elements are characterised by $\mu(X^+) = 1$. Every normalised measure on X can be obtained via this homeomorphism as the least upper bound of an increasing chain of simple valuations on $\mathbf{B}X$.*

This domain-theoretic framework for measure theory can be used to do generalised Riemann integration theory on complete separable metric spaces or indeed on Polish spaces as in the case of compact metric spaces in [3].

6 Iterated Function Systems with Probabilities

An *IFS with probabilities* is an IFS $(X; f_1, \ldots, f_N)$ together with a vector $(p_1, \ldots, p_N)$ of real numbers $p_i > 0$ with $\sum_{i=1}^{N} p_i = 1$. For every $f_i : X \to X$, there is a function $\mathbf{M}f_i : \mathbf{M}^1 X \to \mathbf{M}^1 X$ with $\mathbf{M}f_i\,(\mu)(B) = \mu\,(f_i^{-} B)$ for Borel sets B. Using these functions, one can define $f = \sum_{i=1}^{N} p_i {\cdot} \mathbf{M}f_i : \mathbf{M}^1 X \to \mathbf{M}^1 X$. In [9], it is shown that f has a unique (well-supported) fixed point with bounded support, the so-called *invariant measure* of the IFS. Hutchinson proved his result by defining a complete metric on the set of normalised (well-supported) measures with bounded support with respect to which f is contracting. He also shows that the invariant measure has compact support.

In [4], a domain-theoretic proof of this result is established for compact metric spaces. Here, we will use the domain-theoretic framework of the previous section to extend the above result to any complete (not necessarily separable) metric space. Thus we obtain Hutchinson's theorem in its full generality.

Theorem 30 *A hyperbolic IFS with probabilities on a complete metric space has an invariant measure which is unique among the normalised well-supported measures with bounded support. The invariant measure has compact support.*

Proof. Given the probabilistic IFS $(X; f_1, \ldots, f_N; p_1, \ldots, p_N)$, let $\Sigma = \{1, \ldots, N\}$ and $f = \sum_{i \in \Sigma} p_i \cdot \mathbf{M}f_i : \mathbf{M}^1 X \to \mathbf{M}^1 X$. Let $c_i < 1$ be Lipschitz constants for the functions f_i. The functions $g_i = \mathbf{B}(f_i, c_i) : \mathbf{B}X \to \mathbf{B}X$ are continuous. They induce continuous functions $\mathbf{P}g_i : \mathbf{PB}X \to \mathbf{PB}X$ with $\mathbf{P}g_i(\nu)(O) = \nu(g_i^{-} O)$. Hence $g = \sum_{i \in \Sigma} p_i \cdot \mathbf{P}g_i : \mathbf{PB}X \to \mathbf{PB}X$ is continuous. The equality $g \circ \bar{\mathsf{i}} = \bar{\mathsf{i}} \circ f$ is easily verified. With injectivity of $\bar{\mathsf{i}}$, it follows that μ is a fixed point of f iff $\bar{\mathsf{i}}\mu$ is a fixed point of g.

The following is similar to the proof of Theorem 18. Fix any x in X, let $R = \max_{i \in \Sigma} d(x, f_i x)/(1 - c_i)$, and consider an arbitrary $r \geq R$. For $a = (x, r)$, $g_i a \sqsupseteq a$ holds for all i, whence $g(\delta(a)) = \sum_{i \in \Sigma} p_i {\cdot} \delta(g_i a) \sqsupseteq \sum_{i \in \Sigma} p_i {\cdot} \delta(a) = \delta(a)$. Therefore, g maps the subdomain $\uparrow\!\delta(a)$ to itself and thus has a least fixed point $\nu_a = \bigsqcup_{n \in \mathbf{N}} \nu_{n,a}$ on it where $\nu_{n,a} = g^n(\delta(a))$.

We introduce some abbreviating notations. For $\sigma = (i_1, \ldots, i_n)$ in Σ^n, let $\sigma i = (i_1, \ldots, i_n, i)$, $p_\sigma = p_{i_1} \cdot \ldots \cdot p_{i_n}$ and $g_\sigma = g_{i_1} \circ \ldots \circ g_{i_n}$. The notations c_σ and f_σ are defined analogously.

For every n in $\mathbf{N}$, $\nu_{n,a} = \sum_{\sigma \in \Sigma^n} p_\sigma \cdot \delta(g_\sigma a)$ holds. This can be shown by induction using linearity of g. Let $F_n = \{g_\sigma a \mid \sigma \in \Sigma^n\}$.

Since $g_i a \sqsupseteq a$ for all i in Σ, and all the g_σ are monotonic, $g_\sigma a \sqsubseteq g_{\sigma i}\, a$ follows. Consider the tree with node set $\Sigma^* = \bigcup_{n \in \mathbf{N}} \Sigma^n$ and edges $\sigma \to \sigma i$. This is an infinite tree with constant degree N at every node. If node σ is labeled by $g_\sigma a$, then infinite paths in the tree correspond to ascending sequences in $\mathbf{B}X$. Let K be the set of least upper bounds of all these sequences. If $K \subseteq O$ for some open set O, then $F_n \subseteq O$ for some n by König's Lemma. This property suffices to show compactness of K. Since $F_n \subseteq O$ implies $\nu_{n,a} O = 1$, we conclude:

(i) If $K \subseteq O$ open, then $\nu_a O = 1$.

If some x in K is contained in an open set O, then some F_n meets O. Since $p_i > 0$ for all i in Σ, $\nu_{n,a} O > 0$ follows. Thus, we obtain:

(ii) If K meets an open set O, then $\nu_a O > 0$.

Since $g_\sigma a = (f_\sigma x, c_\sigma \cdot r)$ and $c_i < 1$, the set K is a subset of X^+. Therefore, property (i) implies $\nu_a \in SSX$. By Theorem 25, ν_a is maximal in $\mathbf{PB}X$ and thus the unique fixed point of g in $\uparrow\delta(a)$. Since $r' > r$ implies $\uparrow\delta(x, r') \supseteq \uparrow\delta(x, r)$, $\nu_{(x,r)}$ does not depend on r. Hence, we may write ν_x for it.

By Theorem 27, there is a normalised well-supported measure μ_x on X with $\nu_x = \bar{\mathsf{i}}\mu_x$. Let $S = \mathsf{i}^- K$. Since K is a compact subset of X^+, S is a compact subset of X. Properties (i) and (ii) imply analogous properties for S: if $S \subseteq U$ open, then $\mu_x U = 1$, and if S meets an open set U, then $\mu_x U > 0$. These two properties imply that S is the support of μ_x.

So far, we have shown the existence of invariant normalised measures μ_x for every x in X. These measures are continuous with compact support. For uniqueness, let μ be an arbitrary invariant normalised well-supported measure with bounded support. Then there is some $r \geq R$ such that the support is contained in $O(x, r)$. Since μ is well-supported, $\mu(O(x, r)) = 1$ holds. Let $\nu = \bar{\mathsf{i}}\mu$. This is a fixed point of g. If (x, r) is in some open V of $\mathbf{B}X$, then $\Uparrow(x, r) \subseteq V$, whence $O(x, r) = \mathsf{i}^-(\Uparrow(x, r)) \subseteq \mathsf{i}^- V$, whence $\nu V = \mu(\mathsf{i}^- V) = 1$. This implies $\delta(x, r) \sqsubseteq \nu$. Since ν_x is the unique fixed point of g on $\uparrow\delta(x, r)$, $\nu = \nu_x$ follows whence $\mu = \mu_x$ by injectivity of $\bar{\mathsf{i}}$. $\square$

7 Future Work

To obtain a full domain-theoretic treatment of metric completion, the metric of X will be extended to a *partial metric* on $\mathbf{B}X$. This partial metric should be easily extendible to the rounded ideal completion of $\mathbf{B}X$, to obtain the metric completion $\overline{X}$.

A major area for further work is in developing the domain-theoretic measure and integration theory on Polish spaces. Given a Borel measure on a complete separable metric space X, we need to explicitly construct an increasing chain of simple valuations on $\mathbf{B}X$ whose least upper bound is the given measure. One should then be able to use generalised Riemann integration theory to compute the expected value of well-behaved continuous functions with respect to the Borel measure to any degree of accuracy. This can provide new computational techniques in probability theory and stochastic processes.

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