Quantum Computing
Assessed Coursework 1

Matthieu Moutet - mm307 - MAC 2007

1 Exercise 1

$$
\begin{aligned}
|{+}{+}\rangle &= \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
&= \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \\
|{+}{-}\rangle &= \tfrac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right) \\
|{-}{+}\rangle &= \tfrac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right) \\
|{-}{-}\rangle &= \tfrac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)
\end{aligned}
$$

We know the behavior of the CNOT gate on the vectors of the standard computational basis. Therefore:

$$
\begin{aligned}
CNOT\,|{+}{+}\rangle &= \tfrac{1}{2}\left(|00\rangle + |01\rangle + |11\rangle + |10\rangle\right) &= |{+}{+}\rangle \\
CNOT\,|{+}{-}\rangle &= \tfrac{1}{2}\left(|00\rangle - |01\rangle + |11\rangle - |10\rangle\right) &= |{-}{-}\rangle \\
CNOT\,|{-}{+}\rangle &= \tfrac{1}{2}\left(|00\rangle + |01\rangle - |11\rangle - |10\rangle\right) &= |{-}{+}\rangle \\
CNOT\,|{-}{-}\rangle &= \tfrac{1}{2}\left(|00\rangle - |01\rangle - |11\rangle + |10\rangle\right) &= |{+}{-}\rangle
\end{aligned}
$$

Hence, the matrix representation of the CNOT in the basis $(|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle)$ is:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

2 Exercise 2

First recall that :

$$\begin{array}{ccccc} T|0\rangle = |0\rangle & T|1\rangle = e^{i\frac{\pi}{4}}|1\rangle & TT^\dagger|x\rangle = |x\rangle & H^2|x\rangle = |x\rangle & X^2|x\rangle = |x\rangle \\ T^\dagger|0\rangle = |0\rangle & T^\dagger|1\rangle = e^{-i\frac{\pi}{4}}|1\rangle & S^2|x\rangle = |x\rangle & S|0\rangle = |0\rangle & S|1\rangle = e^{i\frac{\pi}{2}}|1\rangle \end{array}$$

Therefore, we have the following input - output network:

$$\begin{array}{lcll} \underline{|00x\rangle} & \mapsto & T|0\rangle \otimes ST^\dagger T^\dagger|0\rangle \otimes H\left(TT^\dagger\right)\left(TT^\dagger\right)H|x\rangle & = |0\rangle \otimes |0\rangle \otimes H^2|x\rangle = \underline{|00x\rangle} \\ \underline{|01x\rangle} & \mapsto & |0\rangle \otimes ST^\dagger T^\dagger|1\rangle \otimes H\left(TT^\dagger\right)X\left(TT^\dagger\right)XH|x\rangle & = |0\rangle \otimes e^{i\frac{\pi}{2}}e^{-i\frac{\pi}{4}}e^{-i\frac{\pi}{4}}|1\rangle \otimes HX^2H|x\rangle \\ & & & = |0\rangle \otimes |1\rangle \otimes H^2|x\rangle \\ & & & = \underline{|01x\rangle} \\ \underline{|10x\rangle} & \mapsto & e^{i\frac{\pi}{4}}|1\rangle \otimes SXT^\dagger XT^\dagger|0\rangle \otimes HTX\left(T^\dagger T\right)XT^\dagger H|x\rangle & = e^{i\frac{\pi}{4}}|1\rangle \otimes SXT^\dagger|1\rangle \otimes HTX^2T^\dagger H|x\rangle \\ & & & = e^{i\frac{\pi}{4}}|1\rangle \otimes e^{-i\frac{\pi}{4}}|0\rangle \otimes H\left(TT^\dagger\right)H|x\rangle \\ & & & = |1\rangle \otimes |0\rangle \otimes H^2|x\rangle \\ & & & = \underline{|10x\rangle} \\ |11x\rangle & \mapsto & e^{i\frac{\pi}{4}}|1\rangle \otimes SXT^\dagger XT^\dagger|1\rangle \otimes HTXT^\dagger XTXT^\dagger XH|x\rangle & = e^{i\frac{\pi}{4}}|1\rangle \otimes SXT^\dagger\left(e^{-i\frac{\pi}{4}}|0\rangle\right) \otimes HTXT^\dagger XTXT^\dagger XH|x\rangle \\ & & & = |1\rangle \otimes e^{i\frac{\pi}{2}}|1\rangle \otimes HTXT^\dagger XTXT^\dagger XH|x\rangle \end{array}$$

Hence, we will focus on $|x\rangle \mapsto HTXT^\dagger XTXT^\dagger XH|x\rangle$:

$$\begin{array}{rcl|rcl} H|0\rangle & = & \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & H|0\rangle & = & \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ XH|0\rangle & = & \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) & & ... & \\ XT^\dagger XH|0\rangle & = & \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}|0\rangle + |1\rangle\right) & & ... & \\ XTXT^\dagger XH|0\rangle & = & \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}|1\rangle + e^{i\frac{\pi}{4}}|0\rangle\right) & & ... & \\ XT^\dagger XTXT^\dagger XH|0\rangle & = & \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{2}}|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right) & & ... & \\ TXT^\dagger XTXT^\dagger XH|0\rangle & = & \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{2}}(|0\rangle - |1\rangle) & & ... & \\ HTXT^\dagger XTXT^\dagger XH|0\rangle & = & e^{-i\frac{\pi}{2}}|1\rangle & HTXT^\dagger XTXT^\dagger XH|1\rangle & = & e^{-i\frac{\pi}{2}}|0\rangle \end{array}$$

$$\textit{Therefore,} \quad \underline{|110\rangle} \mapsto |1\rangle \otimes e^{i\frac{\pi}{2}}|1\rangle \otimes e^{-i\frac{\pi}{2}}|1\rangle = \underline{|111\rangle}$$

$$\textit{And,} \quad \underline{|111\rangle} \mapsto |1\rangle \otimes e^{i\frac{\pi}{2}}|1\rangle \otimes e^{-i\frac{\pi}{2}}|1\rangle = \underline{|110\rangle}$$

To sum up:

$$\begin{array}{c} |00x\rangle \mapsto |00x\rangle \\ |01x\rangle \mapsto |01x\rangle \\ |10x\rangle \mapsto |10x\rangle \\ |110\rangle \mapsto |111\rangle \\ |111\rangle \mapsto |110\rangle \end{array}$$

Thus, the network in the figure implements the Toffoli gate.