Computable Banach Spaces via Domain Theory [1]

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Abstract

This paper extends the domain-theoretic approach to computable analysis to complete metric spaces and Banach spaces. We employ the domain of formal balls to define a computability theory for complete metric spaces. For Banach spaces, the domain specialises to the domain of closed balls, ordered by reversed inclusion. We characterise computable linear operators as those which map computable sequences to computable sequences and are effectively bounded. We show that the domain-theoretic computability theory is equivalent to the well-established approach by Pour-El & Richards.

1 Introduction

This paper is part of a programme to introduce the theory of continuous domains as a new approach to computable analysis, mediating between the traditional approaches that are either machine oriented as e.g. Type 2 Theory of Effectivity (TTE) [12,20,22], the so-called Russian [4] and Polish [9,14] approaches and the Blum-Shub-Smale model [3], or which are analysis-oriented as the work of Hauck [11] and Pour-El & Richards [17]. For a more detailed discussion see the introduction of [7].

Continuous domains were introduced independently by Dana Scott [19] and Yuri Ershov [8]. Their traditional use is in programming semantics, but recently they were successfully employed as computational models for spaces

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in classical mathematics. The survey paper [5] gives an overview over this subject.

A continuous domain is a partially ordered set equipped with the notions of *completeness* and *approximation*. The completeness axiom requires existence of least upper bounds for all directed subsets, approximation means that all elements arise as directed suprema of their essential parts or approximants. (All definitions are formally given below.) The approximation axiom provides the link to the machine-based level of recursion theory or Turing machines: We will enumerate a convenient set of approximants and let the machine operate on this set.

In our earlier paper [7], we started a systematic exploration of the use of domains for computable analysis by considering computability on the real line, the complex plane, and $\mathbb{R}^n$. It was shown that the resulting notion of computability is equivalent to the Polish approach and hence to the above mentioned TTE-approach and the work of Pour-El & Richards.

The present paper extends the domain-theoretic framework to deal with metric spaces and Banach spaces. The domain employed for this purpose is the *domain of formal balls,* defined in [6], which, for Banach spaces, coincides with the collection of all closed balls ordered by reversed set inclusion. Hence every point is represented as intersection of a shrinking sequence of balls, a straightforward generalisation of the well-established representation of real numbers by shrinking sequences of intervals.

Our main result is that the domain-theoretic notion of computability structure for a Banach space coincides with the notion of computability structure in terms of computable sequences suggested by Pour-El & Richards [17]. This latter notion is used by theoretical physicists such as in the work of Penrose [16]. The domain-theoretic setting gives rise in a natural way to a notion of *computable operator.* Such a definition does not appear in the work of Pour-El & Richards but is a natural and useful notion for classification of operators.

2 Preliminaries and Notations

2.1 Recursion theory

We will use the relevant notions from recursion theory as in the fairly standard language of Rogers [18]. The set $\{0, 1, 2, \ldots\}$ of natural numbers is denoted by $\mathbb{N}$. The nth partial recursive function is denoted by ϕ_n, the nth recursively enumerable (r.e.) set by W_n, so that $W_n = \mathsf{dom}(\phi_n)$. We will make use of a standard pairing function $\langle \cdot, \cdot \rangle \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which could be defined as $\langle n, m \rangle = \frac{1}{2}(n^2 + 2nm + m^2 + 3n + m)$. The projections are denoted by π_1, π_2; they satisfy $\langle \pi_1(n), \pi_2(n) \rangle = n$ as well as $\pi_1(\langle n, m \rangle) = n$ and $\pi_2(\langle n, m \rangle) = m$ for all $n, m \in \mathbb{N}$. As usual, a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ is said to be r.e. if the set $\{\langle n, m \rangle \mid (n, m) \in R\}$ is r.e. We will conveniently say that R is r.e. in n, m. Similarly for relations of higher arity.

Many of our results have the form *"There is f with property A iff there is g with property B"*. Sometimes, we add the phrase *"This equivalence is effective."* This means that there are partial recursive function $\psi_1, \psi_2 \colon \mathbb{N} \to \mathbb{N}$ such that if ϕ_n has property A then $\psi_1(n)$ is defined and $\phi_{\psi_1(n)}$ has property B and, conversely, if ϕ_m has property B then $\psi_2(m)$ is defined and $\phi_{\psi_2(m)}$ has property A. Similarly for r.e. sets in place of functions. This is referred to as *uniformity* by Rogers in *loc.cit.*

We will employ the dovetailing principle in the form of the following construction: Every r.e. set $A \subseteq \mathbb{N}$ can be written as the union $A = \bigcup_{n \in \mathbb{N}} A_n$ of an increasing chain $A_0 \subseteq A_1 \subseteq A_2 \cdots$ such that the relation $i \in A_n$ is recursive in i, n. To see that this is true, we take a Turing machine based approach to recursion theory. If M is a machine which produces a list of the elements of A, then define A_n to contain those elements produced after n steps of computation by M. A consequence of effectiveness of this construction is the Selection Theorem [18, Theorem 5-XVIII]. It says that there is a partial recursive function $\psi \colon \mathbb{N} \to \mathbb{N}$ such that $\psi(n)$ is defined iff the set W_n is not empty in which case $\psi(n) \in W_n$.

2.2 Continuous domains

In the earlier paper [7], we gave a short introduction to continuous domains as computational models. Here we just cite the relevant definitions. See [1] for a detailed treatment of domain theory.

A *directed complete partial order* (dcpo) is a partially ordered set closed under suprema (we also say least upper bound or lub) of directed subsets. A dcpo is *pointed,* if it contains a least element $\bot$. We write $x = \bigvee^{\uparrow} A$ to denote that the set $A \subseteq D$ is directed and has lub x. If $(D, \sqsubseteq)$ is a dcpo and $x, y \in D$ then we say that x *approximates* y (x is *way-below* y, $x \ll y$) if for every directed subset $A \subseteq D$ with $y \sqsubseteq \bigvee^{\uparrow} A$ there is some $a \in A$ with $x \sqsubseteq a$.

A *continuous domain* is a dcpo $(D, \sqsubseteq)$ such that for every element $x \in D$ the set $\mathord{\downarrow\!\!\downarrow} x = \{y \in D \mid y \ll x\}$ is directed and has x as its supremum: $\bigvee^{\uparrow} \mathord{\downarrow\!\!\downarrow} x = x$. We often refer to a continuous domain simply as a *domain.*

The *Scott-topology* of a dcpo $(D, \sqsubseteq)$ consists of all subsets O of D which are upwards closed ($x \in O,\ x \sqsubseteq y \implies y \in O$) and inaccessible by directed suprema, i.e. if $\bigvee^{\uparrow} A \in O$ then $O \cap A \neq \emptyset$. If D is a continuous domain, then the sets $\twoheaduparrow x = \{y \in D \mid x \ll y\}$ for $x \in D$ are Scott-open and the collection of these sets forms a base for the Scott-topology. A function $f\colon (D, \sqsubseteq) \to (E, \sqsubseteq)$ between dcpo's is continuous with respect to the Scott-topologies on D and E if and only if it is monotone and preserves suprema of directed subsets: $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ and $f(\bigvee^{\uparrow} A) = \bigvee^{\uparrow} f(A)$. A function f between pointed domains is called *strict* if $f(\bot) = \bot$. The collection of all Scott-continuous functions from D to E is denoted by $[D \to E]$. It is endowed with the *pointwise order,* i.e., $f \sqsubseteq g$ iff $f(x) \sqsubseteq g(x)$ for all $x \in D$, which makes $[D \to E]$ a dcpo.

A subset B of a domain $(D, \sqsubseteq)$ is a *basis* if every element of D is the directed supremum of all basis elements approximating it: $x = \bigvee^{\uparrow}(\twoheaddownarrow x \cap B)$. A domain is *$\omega$-continuous* if it has a countable basis.

If D and D' are domains then so is their direct product, $D \times D'$, partially ordered coordinatewise. The order of approximation on $D \times D$ is obtained coordinatewise and the Scott-topology coincides with the product topology of the Scott-topologies on the factors. A function $f\colon D \times D' \to E$ is Scott-continuous if and only if it is Scott-continuous in both variables separately.

The *interval domain* $\mathcal{I}$ is the collection of all compact intervals on $\mathbb{R}$, endowed with a least element:

$$\mathcal{I} = \{[a, b] \subseteq \mathbb{R} \mid a, b \in \mathbb{R},\ a \leq b\} \cup \{\bot\}$$

The order is reversed subset inclusion, i.e. $\bot \sqsubseteq I$ for all $I \in \mathcal{I}$ and $[a, b] \sqsubseteq [c, d]$ iff $a \leq c$ and $b \geq d$ in the usual ordering of real numbers. One can think of the least element $\bot$ as the set $\mathbb{R}$. Directed suprema are filtered intersections of intervals. The order of approximation is given by $\bot \ll I$ for all $I \in \mathcal{I}$ and $[a, b] \ll [c, d]$ iff $a < c$ and $b > d$. The maximal elements are the intervals $[a, a]$, i.e. the singleton sets. The real line is homeomorphic to the set of maximal elements of $\mathcal{I}$ endowed with the relative Scott-topology via $x \mapsto \{x\}$. Every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ has a Scott-continuous extension $g\colon \mathcal{I} \to \mathcal{I}$ in the sense that $g(\{x\}) = \{f(x)\}$ holds for all $x \in \mathbb{R}$. There is a largest strict extension, given by $g(I) = \{f(x) \mid x \in I\}$ for $I \neq \bot$.

For an interval $I = [a, b]$ we define its *width* $|I| = b - a$, for $I = \bot = \mathbb{R}$ we set $|I| = \infty$.

2.3 Effectively given domains

We collect the necessary definitions and results on effective domain theory as presented in Section 3 of [7] along the lines of [23].

We say that an ω-continuous pointed domain $(D, \sqsubseteq)$ with countable basis $D_0 = \{b_0, b_1, b_2, \cdots\}$ is *effectively given* with respect to b if the relation $b_n \ll b_m$ is r.e. in n, m. An element $x \in D$ is *computable,* if the set $\{n \in \mathbb{N} \mid b_n \ll x\}$ is r.e. We will assume without loss of generality that $b_0 = \perp$. An element $x \in D$ is computable iff it is the lub of an effective chain in D_0, i.e. iff there is $f: \mathbb{N} \to \mathbb{N}$ total recursive such that $b_{f(0)} \sqsubseteq b_{f(1)} \sqsubseteq b_{f(2)} \sqsubseteq \cdots$ and $x = \bigvee^{\uparrow}{}_{n \in \mathbb{N}} b_{f(n)}$. This equivalence is effective.

The computability structure on two effectively given domains induces an effective structure on their product as follows: If $(D, \sqsubseteq)$ and $(D', \sqsubseteq)$ are effectively given via $(b_n)_{n \in \mathbb{N}}$ and $(b'_n)_{n \in \mathbb{N}}$, respectively, then the product $D \times D'$ has the effective basis $(b^*_n)_{n \in \mathbb{N}}$ with $b^*_{\langle n,m \rangle} = (b_n, b'_m)$.

An enumeration ξ_D of the set D_c of all computable elements is defined by constructing a total recursive function $\eta_D: \mathbb{N} \to \mathbb{N}$ with the property that for each natural number n the function $\phi_{\eta_D(n)}$ is total. Its range describes a chain $b_{\phi_{\eta_D(n)}(0)} \sqsubseteq b_{\phi_{\eta_D(n)}(1)} \sqsubseteq \cdots$ satisfying the condition that if ϕ_n did already describe a chain, then the lubs of the two chains coincide. The nth computable element $\xi_D(n)$ is defined to be the lub $\bigvee^{\uparrow}{}_{i \in \mathbb{N}} b_{\phi_{\eta_D(n)}(i)}$. Every element b_n from D_0 is computable, the index of b_n as a computable element of D is a recursive function of n. Least upper bounds of effective chains of computable elements are computable with effectively obtainable index.

Suppose that the domains $(D, \sqsubseteq)$ and $(D', \sqsubseteq)$ are effectively given via b and b', respectively. A continuous function $f: D \to D'$ is *computable* if the relation $b'_m \ll f(b_n)$ is r.e. in n, m. An alternative characterization of computability for a continuous function $f: D \to D'$ is the existence of a total recursive function $\psi: \mathbb{N} \to \mathbb{N}$ such that $f \circ \xi_D = \xi_{D'} \circ \psi$. A third characterization is in terms of computable sequences. A sequence $(x_n)_{n \in \mathbb{N}}$ in D is *computable,* if there is a total recursive function $h: \mathbb{N} \to \mathbb{N}$ such that $x_n = \xi_D(h(n))$. A continuous function is computable iff it maps computable sequences to computable sequences.

If $f: D \to D'$ is computable then the relation $b'_m \ll f(\xi_D(n))$ is r.e. in n, m. This means in particular that the relation $b_n \ll \xi_D(m)$ is r.e. in n and m.

2.4 Computability on the real line

Let us briefly state the relevant definition and results from [7]. We denote by $q_0, q_1, q_2, \ldots$ a standard numeration of the rationals, e.g. $q_{\langle n, \langle m,k \rangle \rangle} = \frac{n-m}{k+1}$. The arithmetic operations $+, -, \cdot, /$, when defined, as well as the comparisons $<, \leq, =$ and the absolute value function $|\cdot|$ on rationals are recursive in the indices.

The interval domain $\mathcal{I}$ is ω-continuous. An example for a countable basis is the collection $\mathcal{I}_0$ of all intervals with rational endpoints together with the least element $\perp$. We set $I_0 = \perp$ and

$$I_{\langle n,m\rangle+1} = [q_n - |q_m|, q_n + |q_m|]$$

to enumerate this basis and consider the interval domain as effectively given with respect to I. Note that $I_n \ll I_m$ iff

$$n = 0 \text{ or } \Big(m, n > 0 \text{ and } |q_{\pi_1(n-1)} - q_{\pi_1(m-1)}| + |q_{\pi_2(m-1)}| < |q_{\pi_2(n-1)}|\Big).$$

The resulting enumeration of the computable elements of $\mathcal{I}$ is denoted by $\xi_{\mathcal{I}}: \mathbb{N} \to \mathcal{I}$.

We say that a real number x is *computable,* if the set $\{x\}$ is a computable element of the interval domain, and a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is *computable,* if the sequence $(\{x_n\})_{n\in\mathbb{N}}$ is computable in $\mathcal{I}$. In other words, if there exists $f: \mathbb{N} \to \mathbb{N}$ total recursive such that $\{x_n\} = \xi_{\mathcal{I}}(f(n))$. The sequence is said to *converge effectively* to $x \in \mathbb{R}$, if there is $g: \mathbb{N} \to \mathbb{N}$ recursive (the *modulus of convergence*) such that $k \geq g(n)$ implies $|x_k - x| \leq 2^{-n}$. If $(x_n)_{n\in\mathbb{N}}$ is an effectively convergent computable sequence, then its limit x is computable. Moreover, an index for $\{x\}$ can be obtained effectively from the indices for the sequence and the modulus of convergence.

Our notions of computable number and sequence coincide with the classical notions, see e.g. [17]. Hence a real number x is computable iff there is a recursive function $h: \mathbb{N} \to \mathbb{N}$ such that $|q_{h(n)} - x| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$, and a sequence $(x_n)_{n\in\mathbb{N}}$ is computable iff there is a recursive function $r: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $|q_{r(n,k)} - x_n| \leq 2^{-k}$ for all $n, k \in \mathbb{N}$.

A function $f: \mathbb{R} \to \mathbb{R}$ is *computable* iff there is an extension $g: \mathcal{I} \to \mathcal{I}$ (i.e. $g(\{x\}) = \{f(x)\}$ for all $x \in \mathbb{R}$) which is computable in the sense of Section 2.3. It follows that every computable function $\mathbb{R} \to \mathbb{R}$ is continuous. A continuous function is computable if and only if it maps computable sequences of intervals to computable sequences of intervals.

Our approach is equivalent to the classical definition of computable real function. Using the re-formulation of [10] which Pour-El & Richards employ as a starting point for a detailed treatment of computable analysis [17] this means that a function $f: \mathbb{R} \to \mathbb{R}$ is computable iff (1) it maps computable sequences to computable sequences and (2) it is effectively uniformly continuous on intervals $[-n, n]$, i.e. there is a recursive function $h: \mathbb{N}^2 \to \mathbb{N}$ such that $|x - y| \leq \frac{1}{2^{h(n,k)}}$ & $x, y \in [-n, n]$ implies $|f(x) - f(y)| \leq \frac{1}{2^k}$ for all $n, k \in \mathbb{N}$.

3 Computability on metric spaces

3.1 The domain of formal balls

Edalat and Heckman investigate in [6] the *poset of formal balls* which endows metric spaces with a computability structure similar to the interval domain for the reals. In the sequel, we will briefly give the relevant definitions.

Suppose (X, d) is a complete metric space. Let $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ denote the set of nonnegative real numbers. We use the notation

$$C(x, r) = \{y \in X \mid d(x, y) \leq r\}$$

for the closed ball of radius r around x. The *domain of formal balls* is the set

$$B(X) = \{\bot\} \cup (X \times \mathbb{R}_0^+)$$

endowed with the order defined by $\bot \sqsubseteq A$ for all $A \in B(X)$ and

$$\langle x, r\rangle \sqsubseteq \langle y, s\rangle \iff d(x, y) + s \leq r.$$

Formal balls, as their name suggests, are thought of as balls in the metric space: The pair $\langle x, r\rangle$ corresponds to the closed ball $C(x, r)$. The set $C(X)$ of all closed balls of X, partially ordered by reversed inclusion, is not a dcpo but the map $e\colon \langle x, r\rangle \mapsto C(x, r)\colon B(X) \to C(X)$ is monotone and continuous in the sense that $e(\bigvee^{\uparrow}{}_{i\in I}\, a_i) = \bigvee^{\uparrow}{}_{i\in I}\, e(a_i)$ for all directed families $(a_i)_{i\in I}$ of elements of $B(X)$. This means in particular that the supremum $\bigvee^{\uparrow}{}_{i\in I}\, e(a_i)$ exists. The poset $(B(X), \sqsubseteq)$ is a continuous domain. The way-below relation turns out to be $\bot \ll A$ for all $A \in B(X)$ and

$$\langle x, r\rangle \ll \langle y, s\rangle \iff d(x, y) + s < r.$$

An element is maximal iff it is of the form $\langle x, 0\rangle$ for some $x \in X$. The space X in the metric topology is via $x \mapsto \langle x, 0\rangle$ homeomorphic to the set of maximal elements with the relative Scott-topology. If $X_0 \subseteq X$ is a dense subset of X then the set

$$\{\langle x, r\rangle \mid x \in X_0,\ r \in \mathbb{Q},\ r \geq 0\}$$

is a basis for $(B(X), \sqsubseteq)$. An important property of the domain of formal balls is that every directed subset contains an ascending sequence with the same lub [6, Theorem 2.2]. Hence openness and Scott-continuity can be verified by just looking at ascending sequences. The lub of such a sequence $\langle x_0, r_0\rangle \sqsubseteq \langle x_1, r_1\rangle \sqsubseteq \langle x_2, r_2\rangle \sqsubseteq \cdots$ is easily calculated: The element $\langle x, r\rangle$ is the lub iff $lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} r_n = r$ [6, Theorem 2.5].

Suppose that $f: X \to Y$ is a Lipschitz-continuous function between the metric spaces (X, d) and (Y, d) with Lipschitz-constant M, i.e. that $d(f(x), f(x')) \leq Md(x, x')$ for all $x, x' \in X$. Then there is a Scott-continuous extension $g: B(X) \to B(Y)$ to the domains of formal balls. It is defined by $g(\perp) = \perp$ and $g(\langle x, r \rangle) = \langle f(x), Mr \rangle$ [6, Theorem 3.1].

3.2 Effective metric spaces

If (X, d) is a separable metric space, then the domain of formal balls is ω-continuous and we are in the position to endow it with an effective structure.

Definition 1 *An* effectively given metric space *is a complete metric space with a countable dense subset $\{x_0, x_1, x_2, \ldots\}$ such that the domain of formal balls is effectively given with respect to the basis $\{B_0, B_1, \ldots\}$ with $B_0 = \perp$ and*

$$B_{\langle n,m \rangle + 1} = \langle x_n, |q_m| \rangle.$$

Theorem 2 *A dense sequence $x_0, x_1, x_2, \ldots$ in a complete metric space gives rise to an effective representation if and only if the relation*

$$d(x_n, x_m) < q_k$$

is r.e. in n, m, k.

PROOF. The domain of formal balls is effectively given with respect to the basis $\{B_0, B_1, \ldots\}$ as defined above iff the order of approximation is r.e. This is the case iff the relation $d(x_n, x_{n'}) + |q_{m'}| < |q_m|$ is r.e. As addition of rational numbers and the absolute value function are recursive in the indices, this proves the theorem. $\square$

In [21] Weihrauch defines a computable metric space to be a metric space with a dense sequence $(a_n)_{n \in \mathbb{N}}$ such that the relation $d(a_n, a_m) < q_k$ is r.e. in n, m, k. So Theorem 2 means in particular that a complete metric space is effectively given in our sense if and only if it is a computable metric space in the sense of *loc.cit.* The stronger assumption of both $q_k < d(a_n, a_m)$ and $d(a_n, a_m) < q_k$ being r.e. is part of Moschovakis' definition of a recursive metric space [15], the other part being countability of X. This stronger condition is implied by the requirement of $\ll$ being recursive rather than merely r.e. It appears in the work of Blanck [2] where metric spaces are modelled using algebraic domains.

Note that our model for metric spaces is quite similar to our model for the reals. The dense sequence $(x_n)_{n \in \mathbb{N}}$ takes over the role of the rationals $(q_n)_{n \in \mathbb{N}}$.

Writing the intervals in the form $[x-r, x+r]$ makes this similarity even more apparent. In fact, the interval domain is isomorphic to the domain of formal balls of the real line. The isomorphism is given by

$$[a,b] \mapsto \langle \tfrac{a+b}{2}, \tfrac{b-a}{2} \rangle : \mathcal{I} \to B(\mathbb{R})$$

with inverse

$$\langle x, r \rangle \mapsto [x-r, x+r] : B(\mathbb{R}) \to \mathcal{I}.$$

So it is not surprising that many of the definitions and results from [7] easily transfer to the more general setting of complete metric spaces.

Definition 3 *Suppose (X,d) is an effective metric space, where the resulting enumeration of computable elements of $B(X)$ is denoted by ξ_X.*

1) An element $x \in X$ is computable *if $\xi_X(n) = \langle x, 0 \rangle$ for some $n \in \mathbb{N}$.*

2) A sequence $(y_n)_{n\in\mathbb{N}}$ in X is computable *if there is a total recursive function $f: \mathbb{N} \to \mathbb{N}$ such that $\langle y_n, 0 \rangle = \xi_X(f(n))$.*

Theorem 4 *An element $x \in X$ is computable iff it is an effective limit of a sequence $(x_{n_i})_{i\in\mathbb{N}}$, i.e. iff there is $f: \mathbb{N} \to \mathbb{N}$ recursive such that $d(x, x_{f(n)}) \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. This equivalence is effective.*

PROOF. The proof is similar to the proofs of Theorems 15 and 18 of [7].

(only if) Suppose that x is computable and $n \in \mathbb{N}$ with $\xi_X(n) = \langle x, 0 \rangle$. The relation R with

$$\langle m, k, i \rangle \in R \iff B_i \ll \xi_X(m) \ \&\ |q_{\pi_2(i-1)}| \leq \frac{1}{2^k} \tag{1}$$

is r.e. in m, k, i. Moreover, it is true that if $\xi_X(m)$ is maximal and $k \in \mathbb{N}$ then there is $i \in \mathbb{N}$ with $\langle m, k, i \rangle \in R$. By the Selection Theorem, there exists a partial recursive function $g: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that

$$\langle m, k, g(m,k) \rangle \in R \tag{2}$$

holds for all k whenever $\xi_X(m)$ is maximal. Define $j: \mathbb{N}^2 \to \mathbb{N}$ by $j(m,k) = \pi_1(g(m,k) - 1)$. Then $x_{j(m,k)}$ is the centre of the formal ball $B_{g(m,k)}$. Now $(x_{j(n,k)})_{k\in\mathbb{N}}$ is an effectively converging sequence with limit x_n: By (2) and (1) we have in particular $B_{g(n,k)} \ll \langle x, 0 \rangle$. Thus $d(x_{j(n,k)}, x) + 0 < |q_{\pi_2(g(n,k)-1)}|$. Therefore, by (1), we have $d(x, x_{j(n,k)}) \leq 2^{-k}$ as required.

(if) Assume $f: \mathbb{N} \to \mathbb{N}$ is such that $d(x_{f(n)}, x) \leq 2^{-n}$ for all $n \in \mathbb{N}$. Define $h: \mathbb{N} \to \mathbb{N}$ total recursive such that

$$B_{h(n)} = \langle x_{f(n+2)}, \frac{1}{2^n} \rangle.$$

Then $B_{h(n)} \sqsubseteq \langle x, 0 \rangle$ and the sequence $(B_{h(n)})_{n \in \mathbb{N}}$ is increasing as $d(x_{f(n+2)}, x_{f(n+3)}) + \frac{1}{2^{n+1}} \leq d(x_{f(n+2)}, x) + d(x, x_{f(n+3)}) + \frac{1}{2^{n+1}} \leq \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \frac{1}{2^{n+1}} \leq \frac{1}{2^n}$. Since $\lim_{n \to \infty} \frac{1}{2^n} = 0$, the lub of this chain is maximal, hence this lub must be $\langle x, 0 \rangle$. Therefore, an index for the function h is an index for $\langle x, 0 \rangle$. $\square$

As in the case of the real line, effectivity allows us to characterise the computable sequences.

Corollary 5 *A sequence $(y_n)_{n \in \mathbb{N}}$ is computable if and only if there is a total recursive function $r: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $d(x_{r(n,k)}, y_n) \leq 2^{-k}$ for all $n, k \in \mathbb{N}$. This equivalence is effective.*

PROOF. The proof is similar to that of Theorem 20 of [7].

(if) Assume that $(y_n)_{n \in \mathbb{N}}$ is a sequence in X and that $r: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is recursive such that $d(x_{r(n,k)}, y_n) \leq 2^{-k}$ for all $n, k \in \mathbb{N}$. This means that $(x_{r(n,k)})_{k \in \mathbb{N}}$ is a computable sequence effectively convergent to y_n. So we can apply Theorem 4 to effectively get an index for y_n from r and n. Hence the sequence $(y_n)_{n \in \mathbb{N}}$ is computable.

(only if) Now assume that there is $f: \mathbb{N} \to \mathbb{N}$ with $\xi_X(f(n)) = \langle y_n, 0 \rangle$. By Theorem 4 there is a recursive function $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the sequence $(x_{h(f(n),k)})_{k \in \mathbb{N}}$ is effectively converging to y_n. Thus we have a double sequence as required. $\square$

3.3 Computable functions

Definition 6 *A function $f: X \to Y$ between effectively given metric spaces is* computable *iff there is an extension $g: B(X) \to B(Y)$ (i.e. $g(\langle x, 0 \rangle) = \langle f(x), 0 \rangle$ for all $x \in X$) which is computable in the sense of Section 2.3.*

Theorem 7 *A function with a given Lipschitz constant between effectively given metric spaces is computable if and only if it maps computable sequences of points to computable sequences and it is effectively Lipschitz-continuous in the sense that a computable Lipschitz-constant can be constructed.*

PROOF. Necessity of this condition is immediate. For sufficiency, assume that $f: X \to Y$ is Lipschitz-continuous and maps computable sequences to computable sequences. Denote the dense sequences specifying the effective structure of X and Y by $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively. As f maps computable sequences to computable sequences, there is $h: \mathbb{N} \to \mathbb{N}$ recursive such that $\langle f(x_n), 0\rangle = \xi_Y(h(n))$ for all $n \in \mathbb{N}$. Assume $M \in \mathbb{R}$ is a computable Lipschitz constant for f. Then the extension $g: B(X) \to B(Y)$ of f is defined by $g(\perp) = \perp$ and $g(\langle x, r\rangle) = \langle f(x), Mr\rangle$. It remains to show computability of this function. Observe that

$$\begin{aligned}
\langle y_i, |q_j|\rangle \ll g(\langle x_n, |q_m|\rangle) &\iff d(y_i, f(x_n)) + M|q_m| < |q_j| \\
&\iff d(y_i, f(x_n)) < |q_j| - M|q_m| \\
&\iff d(y_i, f(x_n)) < \max(0, |q_j| - M|q_m|) \\
&\iff \langle y_i, \max(0, |q_j| - M|q_m|)\rangle \ll \langle f(x_n), 0\rangle \\
&\iff \langle y_i, \max(0, |q_j| - M|q_m|)\rangle \ll \xi_Y(h(n)).
\end{aligned}$$

As h is recursive and as the arithmetic operations on the rationals are computable, this relation is r.e. in the indices n, m, i, j. $\square$

4 Computability on Banach spaces

A *Banach space* is a complete normed vector space. For simplicity, we only consider Banach spaces over the reals. The origin in a Banach space is denoted by $\mathbf{0}$.

4.1 Balls in Banach spaces

With the definition $d(x, y) = \|x - y\|$, every Banach space is a complete metric space. Hence a natural candidate for endowing Banach spaces with a computability structure is the domain of formal balls which we met in Section 3. For normed vector spaces, in particular for Banach spaces, the theory simplifies significantly: For these spaces the poset of formal balls coincides with the set of closed balls, ordered by reversed set inclusion (Theorem 2.14 of [6]). Recall that we use the notation

$$C(x, r) = \{y \in X \mid \|x - y\| \leq r\}$$

for the closed ball of radius r around the point $x \in X$. So the *domain of balls* is the set

$$B(X) = \{\perp\} \cup \{C(x,r) \mid x \in X,\ r \geq 0\}$$

endowed with the order $\perp \sqsubseteq A$ for all $A \in B(X)$ and

$$\begin{aligned} C(x,r) \sqsubseteq C(y,s) &\iff C(x,r) \supseteq C(y,s) \\ &\iff \|x-y\| + s \leq r. \end{aligned}$$

Recall that the order of approximation $\ll$ is given by $\perp \ll A$ for all $A \in B(X)$ and

$$C(x,r) \ll C(y,s) \iff \|x-y\| + s < r.$$

We are now going to extend the vector space operations and the norm to the balls. Addition $+\colon B(X) \times B(X) \to B(X)$ is defined by

$$C(x,r) + C(y,s) := C(x+y, r+s)$$

and $\perp + B = B + \perp := \perp$ for all $B \in B(X)$. Scalar multiplication in this setting is of type $\mathcal{I} \times B(X) \to B(X)$. What should the product $[\alpha-\varepsilon, \alpha+\varepsilon] \cdot C(x,r)$ be? If $\varepsilon = 0$, we certainly would expect $C(\alpha x, |\alpha| r)$ as the result. The case $r = 0$ is equally simple, we get $[\alpha-\varepsilon, \alpha+\varepsilon] \cdot C(x,0) = C(\alpha x, \varepsilon\|x\|)$. Setting $x = \mathbf{0}$ yields $[\alpha-\varepsilon, \alpha+\varepsilon] \cdot C(\mathbf{0}, r) = C(\mathbf{0}, r(|\alpha| + \varepsilon))$. Combining these yields our general definition

$$[\alpha-\varepsilon, \alpha+\varepsilon] \cdot C(x,r) := C(\alpha x, \varepsilon\|x\| + |\alpha| r + \varepsilon r)$$

and $\perp \cdot B = I \cdot \perp := \perp$ for all $B \in B(X)$ and $I \in \mathcal{I}$. Proposition 9 below will make this definition even more transparent. Finally, for the norm, we set

$$\|C(x,r)\| := [\|x\| - r, \|x\| + r]$$

and $\|\perp\| := \perp$ to get $\|\cdot\|\colon B(X) \to \mathcal{I}$.

Proposition 8 *Addition, scalar multiplication, and the norm as defined above are Scott-continuous. Restricted to maximal elements, they yield the usual operations.*

PROOF. Monotonicity of the operations is easily verified from the definitions. For scalar multiplication, we have to use the fact that $C(x,r) \sqsubseteq C(y,s)$ implies $\|x\| + r \geq \|y\| + s$. In order to verify that $[\alpha-\varepsilon, \alpha+\varepsilon]C(x,r) \sqsubseteq [\alpha-\varepsilon, \alpha+\varepsilon]C(y,s)$ we calculate

$$\begin{aligned}\|\alpha x-\alpha y\|+\varepsilon\|y\|+|\alpha|s+\varepsilon s &= |\alpha|(\|x-y\|+s)+\varepsilon(s+\|y\|)\\ &\le |\alpha|r+\varepsilon(r+\|x\|)\end{aligned}$$

by the above observation. Monotonicity in the first argument is shown in a similar fashion. Preservation of directed suprema for all three operations is immediate by the characterisation in terms of limits: If, for example, $C(x,r) = \bigvee^{\uparrow}_{n\in\mathbb{N}} C(x_n, r_n)$ then $x = \lim_{n\to\infty}$ and $r = \lim_{n\to\infty} r_n$. Thus $x+y = \lim_{n\to\infty}(x_n+y)$ and $r+s = \lim_{n\to\infty}(r_n+s)$, so $C(x,r)+C(y,s) = \bigvee^{\uparrow}_{n\in\mathbb{N}}(C(x_n,r_n)+C(y,s))$. Hence addition is Scott-continuous. (Recall that separate continuity is equivalent to joint continuity for functions $f\colon D\times D'\to E$ between domains.) It is obvious that we get the usual operations in the case that $r = s = \varepsilon = 0$. $\square$

The extension of the norm can also be seen as a special case of the extension of Lipschitz-continuous functions to the domain of balls for metric spaces (cf. the end of Section 3.1, and, for computability, Theorem 18). For every Banach space X, the norm $\|\cdot\|\colon X\to\mathbb{R}$ is Lipschitz-continuous with constant $M = 1$. The extension to the balls is exactly the norm for $B(X)$ as defined above.

We defined the operations on balls by referring to centres and radii. As we are dealing with subsets, another possibility would be to consider the pointwise operations. For addition, the two approaches coincide as the next proposition shows. In the case of scalar multiplication, however, the pointwise operation does not necessarily yield a ball as result. But we have made the best possible choice:

Proposition 9 *Suppose $(X,\|\cdot\|)$ is a Banach space. Then*
1) $C(x,r)+C(y,s) = \{x'+y' \mid x'\in C(x,r)\ \&\ y'\in C(y,s)\}$.
2) $[\alpha-\varepsilon,\alpha+\varepsilon]C(x,r) \supseteq \{\beta y \mid \beta\in[\alpha-\varepsilon,\alpha+\varepsilon]\ \&\ y\in C(x,r)\}$.
3) There is no smaller ball than $[\alpha-\varepsilon,\alpha+\varepsilon]C(x,r)$ which is centred at αx and contains the pointwise scalar product as in (2).
4) $\|C(x,r)\| \supseteq \{\|y\| \mid y\in C(x,r)\}$.
5) There is no smaller interval than $\|C(x,r)\|$ which is centred at $\|x\|$ and contains the pointwise norm as in (4).

PROOF. Note that $y\in C(x,r)$ iff there is x' with $\|x'\|\le r$ and $y = x+x'$. Using this, one readily sees that the right-to-left inclusions hold. As an example, we show (2), the most involved case. Suppose $y\in C(x,r)$ and $\beta\in[\alpha-\varepsilon,\alpha+\varepsilon]$. Write $y = x+x'$ with $\|x'\|\le r$. Then $\beta y = \beta x+\beta x' = \alpha x+(\beta-\alpha)x+\beta x'$ and $\|(\beta-\alpha)x+\beta x'\| \le |\beta-\alpha|\|x\|+|\beta|r \le \varepsilon\|x\|+(|\alpha|+\varepsilon)r$ since $|\alpha-\beta|\le\varepsilon$. For $\subseteq$ in (1) assume that $z\in C(x,r)+C(y,s)$, i.e. that $z = x+y+w$ with $\|w\|\le r+s$. We split w as $w = \frac{r}{r+s}w+\frac{s}{r+s}w$ and observe that $\|\frac{r}{r+s}w\| = \frac{r}{r+s}\|w\|\le r$ which implies $z\in\{x'+y' \mid x'\in C(x,r)\ \&\ y'\in$

$C(y,s)\}$. To see (3) for the case of $\alpha \geq 0$ note that $x + \frac{r}{\|x\|}x \in C(x,r)$ and $(\alpha+\varepsilon)(x+\frac{r}{\|x\|}x) = \alpha x + (\varepsilon\|x\| + \varepsilon r + \alpha r)\frac{x}{\|x\|}$ so the radius could not be smaller. Similarly for $\alpha \leq 0$, with $-\varepsilon$ and $-\frac{r}{\|x\|}x$ replacing ε and $\frac{r}{\|x\|}x$. Finally (5). It suffices to note that $x+\frac{r}{\|x\|}x$ is an element of $C(x,r)$ and has norm $\|x\|+r$. $\square$

For the case of singleton intervals, the scalar multiplication is indeed the pointwise defined scalar product as the latter is a ball. We will write $\alpha C(x,r)$ to abbreviate $\{\alpha\}C(x,r) = C(\alpha x, |\alpha| r)$. Also, as usual, $A - B$ stands for $A + (-B)$, i.e. $A + \{-1\}B$.

4.2 Effectively given Banach spaces

Definition 10 *An* effectively given Banach space *is a Banach space X with an effective structure for the continuous domain $B(X)$ of balls such that addition, scalar multiplication and the norm are computable functions. We denote the effective basis by $\{A_0, A_1, A_2, \ldots\}$ and require further that the set $\{n \in \mathbb{N} \mid A_n = \bot\}$ be recursive.*

For simplicity, we denote the resulting enumeration of computable elements of $B(X)$ with ξ_X rather than $\xi_{B(X)}$. Note that the term "effectively given Banach space" always refers to a given enumeration of a basis for the domain of balls. Hence a Banach space may be effectively given in two distinct ways.

Recall that for an interval $[a,b] \in \mathcal{I}$, we denote with $|[a,b]|$ its length $b-a$. Hence for a ball $A \in B(X)$, the number $|||A|||$ denotes its diameter. In fact, we calculate $|||C(x,r)||| = |[\|x\| - r, \|x\| + r]| = 2r$. Being a composition of computable functions, it is clear that this function is computable for effectively given Banach spaces.

Proposition 11 *Suppose that $(X, \|\cdot\|)$ is an effectively given Banach space. Then $\{\mathbf{0}\}$ and the unit ball $C(\mathbf{0},1)$ are computable elements. Moreover, indices for these elements can be obtained from the effective structure.*

PROOF. If $A = C(x,r)$ then $A - A = C(\mathbf{0}, 2r)$ and $|||A||| = |[\|x\| - r, \|x\| + r]| = 2r$. Thus $0(A - A) = \{\mathbf{0}\}$ and $\frac{1}{|||A|||}(A - A) = C(\mathbf{0},1)$ for $r > 0$. As the operations are computable, this enables us to find indices for the origin and the unit ball by searching for i with $A_i \neq \bot$ and $|||A_i||| > 0$ and then calculating $0(A_i - A_i)$ and $\frac{1}{|||A_i|||}(A_i - A_i)$. $\square$

Let us now turn our interest to the Banach space X, i.e. the set of maximal elements of $B(X)$. By Proposition 11 the origin $\mathbf{0}$ is a computable element of X. Are there any other computable elements in the Banach space? Can we lay our hands on them? The following proposition is the answer to these questions. We are able to find sufficiently many computable elements in X.

Proposition 12 *There is a total recursive function $f: \mathbb{N} \to \mathbb{N}$ such that $\xi_X(f(n))$ is a maximal element above A_n for every $n \in \mathbb{N}$. The resulting sequence $(x_n)_{n\in\mathbb{N}}$ of vectors, where $C(x_n, 0) = \xi_X(f(n))$, is dense in X.*

PROOF. Fix $n \in \mathbb{N}$. We construct a recursive function $g: \mathbb{N} \to \mathbb{N}$ which gives an increasing sequence $A_{g(0)} \sqsubseteq A_{g(1)} \sqsubseteq \cdots$ with lub in the maximal elements above A_n. Then $f(n)$ is defined to be the derived index of the function g and the first part of the proposition is proved. For the construction of g, set $g(0) = n$. To define $g(m+1)$ for $m \in \mathbb{N}$, consider the set $M_m = \{g(m)\} \cup \{i \in \mathbb{N} \mid A_{g(m)} \ll A_i\}$. This certainly is r.e. and hence so is $M'_m = \{i \in M_m \mid |||A_i||| < \frac{1}{2^m}\}$. Note that the set M'_m is not empty: If $g(m) \notin M'_m$ then $A_{g(m)}$ is not maximal and so there is a maximal element of $B(X)$ way-above it. This certainly implies the existence of a basis element A_i with $|||A_i||| < \frac{1}{2^m}$ way-above $A_{g(m)}$, then $i \in M'_m$. So, using the Selection Theorem, we may define $g(m+1)$ to be some element of M'_m. It is clear that the total recursive function g defined by this procedure has the desired property.

To see that the sequence $(x_n)_{n\in\mathbb{N}}$ is dense in X, suppose $x \in X$ and $\varepsilon > 0$. Now $C(x, \varepsilon) \ll \{x\}$, hence there is $n \in \mathbb{N}$ with $C(x, \varepsilon) \ll A_n \ll \{x\}$. This implies $A_n \subseteq C(x, \varepsilon)$ and as $x_n \in A_n$ we certainly have $\|x - x_n\| \leq \varepsilon$. $\square$

Observe that this construction is not possible if the basis elements A_n are replaced by the computable elements $\xi_X(n)$ as the relation $\xi_X(n) \ll A_m$ is *not* r.e. In fact, there is no recursive enumeration of all computable singletons.

The constructed sequence of computable maximal elements allows us to pass to a more convenient basis for $B(X)$. Let $B_0 = \perp$ and

$$B_{\langle n,m\rangle+1} = C(x_n, |q_m|)$$

with the sequence $(x_n)_{n\in\mathbb{N}}$ from Proposition 12.

Theorem 13 *Every effectively given Banach space is an effectively given metric space. In particular, the above constructed set $\{B_0, B_1, B_2, \ldots\}$ is an effective basis for $B(X)$ which makes the operations computable.*

PROOF. The sequence $(x_n)_{n\in\mathbb{N}}$ is dense in X. Furthermore the relation

$$\|x_n - x_m\| < q_k$$

is r.e. in n, m, k as the norm is computable. Thus we are dealing with an effectively given metric space by Theorem 2. It remains to show computability of the operations. This is an easy consequence of computability with respect to the given basis $\{A_0, A_1, \ldots\}$. E.g., for addition we have

$$\begin{aligned} &C(x_i, |q_j|) \ll C(x_n, |q_m|) + C(x_{n'}, |q_{m'}|) \\ &\iff \|x_i - (x_n + x_{n'})\| + |q_m| + |q_{m'}| < |q_j| \end{aligned}$$

which is clearly r.e. in i, j, n, m, n', m'. $\square$

So the set comprising all the B_n does in fact constitute a computability structure for X. How does it relate to the original structure? Let us denote the two resulting effective domains by $(B(X), A)$ and $(B(X), B)$, respectively.

Proposition 14 *The identity* $\mathsf{id}\colon B(X) \to B(X)$ *is* B-A *computable.*

PROOF. Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is the function defined in Proposition 12 and that $n_0 \in \mathbb{N}$ is the code for $C(\mathbf{0}, 1)$ from Proposition 11. Then

$$\begin{aligned} B_{\langle n,m\rangle+1} &= C(x_n, |q_m|) \\ &= C(x_n, 0) + C(\mathbf{0}, |q_m|) \\ &= \xi_X(f(n)) + |q_m| \cdot \xi_X(n_0). \end{aligned}$$

As the operations are computable and f is recursive, this shows that the relation $A_k \ll B_\ell$ is r.e. by the remark at the end of Section 2.3 $\square$

This need not be true for the other direction, in particular there might be basis elements A_n which are not computable w.r.t. to the B-structure. But we are not really interested in the balls, we are interested in the Banach space. The following proposition shows that the identity $\mathsf{id}\colon X \to X$ has an extension $B(X) \to B(X)$ which is A-B computable. Hence the two computability theories for $(X, \|\cdot\|)$ coincide and we may pass to the more convenient basis $\{B_0, B_1, \ldots\}$.

Proposition 15 *There is a total recursive function* $f\colon \mathbb{N} \to \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *the function* $\phi_{f(n)}$ *gives an increasing chain* $B_{\phi_{f(n)}(0)} \sqsubseteq B_{\phi_{f(n)}(1)} \sqsubseteq \cdots$ *with least upper bound* $C_n := \bigvee^{\uparrow}{}_{i\in\mathbb{N}} B_{\phi_{f(n)}(i)}$ *below* $\xi_X(n)$. *If* $\xi_X(n)$ *is maximal then* $C_n = \xi_X(n)$.

PROOF. The idea how to define f is essentially the same as in the proof of Theorem 15 of [7]. Employing Proposition 12, we can avoid partiality of f. In detail we proceed as follows:

(1) Start with a natural number n.
(2) This describes a chain $A_{\phi_{\eta_X(n)}(0)} \sqsubseteq A_{\phi_{\eta_X(n)}(1)} \sqsubseteq \cdots$ with least upper bound $\xi_X(n)$. (The function η_X was introduced in Section 2.3 on Page 5.)
(3) Apply Proposition 12 to get a sequence $(x_{h(i)})_{i\in\mathbb{N}}$ with $x_{h(i)} \in A_{\phi_{\eta_X(n)}(i)}$ for all $i \in \mathbb{N}$.
(4) Define $g\colon \mathbb{N} \to \mathbb{N}$ recursive such that $|||A_{\phi_{\eta_X(n)}(i)}||| < q_{g(i)} < |||A_{\phi_{\eta_X(n)}(i)}|||+ \frac{1}{2^i}$.
(5) Then $B_{\langle h(i),g(i)\rangle+1} = C(x_{h(i)}, q_{g(i)}) \supseteq A_{\phi_{\eta_X(n)}(i)}$.
(6) Now we have to make the sequence $B_0, B_1, \ldots$ increasing. We use the same method as in the algorithm for obtaining the enumeration of computable elements in an effectively given domain described in [7] just after Proposition 3:
   (a) Define $j(0) = \langle h(0), g(0)\rangle + 1$.
   (b) Start with $i = 0$ and $k = 1$.
   (c) The set $P = \{\ell \geq k \mid B_{j(i)} \ll B_{\langle h(\ell),g(\ell)\rangle+1}\} \subseteq \mathbb{N}$ is r.e.
   (d) Write $P = \bigcup_{m\in\mathbb{N}} P_m$ where the test $\ell \in P_m$ is recursive in ℓ, m.
   (e) Now, starting with $m = 1$, each of these sets is tested for the existence of $\ell \leq m$ with $\ell \in P_m$. Whenever no such element is found, we let $j(i + m) = j(i)$ and check for the next value of m.
   (f) If, at some stage, there is $\ell \leq m$ with $\ell \in P_m$ then set $j(i + m) = \langle h(\ell), g(\ell)\rangle + 1$, increment i by m, set $k = \ell + 1$, and go to step (6c).
(7) Now $B_{j(0)} \sqsubseteq B_{j(1)} \sqsubseteq \cdots$ and $\bigvee^{\uparrow}{}_{i\in\mathbb{N}} B_{j(i)} \sqsubseteq \xi_X(n)$.

We define $f(n)$ to be the derived index of the function j. It remains to verify that $C_n = \xi_X(n)$ if the latter is maximal. Maximality of $\xi_X(n)$ means that $\lim_{i\in\mathbb{N}} |||A_{\phi_{\eta_X(n)}(i)}||| = 0$, so we also have

$$\lim_{i\in\mathbb{N}} q_{g(i)} = 0 \tag{3}$$

as $q_{g(i)} < |||A_{\phi_{\eta_X(n)}(i)}||| + \frac{1}{2^i}$ by construction (step(5)). Therefore, it suffices to show that for all $i, k \in \mathbb{N}$ there is $\ell \geq k$ such that $B_{\langle h(i),g(i)\rangle+1} \ll B_{\langle h(\ell),g(\ell)\rangle+1}$. This ensures that the sequence $(B_{j(i)})_{i\in\mathbb{N}}$ is not eventually constant and thus its limit is maximal by (3). As $q_{g(i)} > |||A_{\phi_{\eta_X(n)}(i)}|||$, by (3) there is $\ell \geq i, k$ with $q_{g(i)} - |||A_{\phi_{\eta_X(n)}(i)}||| > q_{g(\ell)}$. As $\ell \geq i$, we have $x_{h(\ell)} \in A_{\phi_{\eta_X(n)}(\ell)} \subseteq A_{\phi_{\eta_X(n)}(i)}$. But also $x_{h(i)} \in A_{\phi_{\eta_X(n)}(i)}$, hence $\|x_{h(i)} - x_{h(\ell)}\| \leq |||A_{\phi_{\eta_X(n)}(i)}||| < q_{g(i)} - q_{g(\ell)}$. This implies $B_{\langle h(i),g(i)\rangle+1} \ll B_{\langle h(\ell),g(\ell)\rangle+1}$. $\square$

Thus the effective structure on X as a Banach space coincides with the constructed effective metric space structure. From now on, we will assume the more convenient latter form of presentation via a dense sequence $(x_n)_{n\in\mathbb{N}}$ in X and the basis consisting of $\perp$ and the $C(x_i,|q_j|)$ as defined above. The results concerning computability of points and sequences in metric spaces specialise as follows.

Corollary 16 *1) An element $x \in X$ is computable iff it is an effective limit of a sequence $(x_{n_i})_{i\in\mathbb{N}}$, i.e. iff there is $f\colon \mathbb{N} \to \mathbb{N}$ recursive such that $\|x - x_{f(n)}\| \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. This equivalence is effective.*
2) A sequence $(y_n)_{n\in\mathbb{N}}$ is computable if and only if there is a recursive function $r\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $\|x_{r(n,k)} - y_n\| \leq 2^{-k}$ for all $n, k \in \mathbb{N}$. This equivalence is effective.

4.3 Computable operators

As expected, we define those operators to be computable which have a computable extension.

Definition 17 *A linear operator $f\colon X \to Y$ between effectively given Banach spaces is* computable *iff there is an extension $g\colon B(X) \to B(Y)$ (i.e. $g(\{x\}) = \{f(x)\}$ for all $x \in X$) which is computable in the sense of Section 2.3.*

Recall that a linear operator $f\colon X \to Y$ between Banach spaces is *bounded* if there is $M \in \mathbb{R}$ such that $\|f(x)\| \leq M\|x\|$ for all $x \in X$. The smallest such bound M is the *norm* $\|f\|$ of f. By linearity, boundedness is equivalent to Lipschitz-continuity. Even more, continuity at the origin implies boundedness.

Theorem 18 *A linear operator is computable if and only if it maps computable sequences of points to computable sequences and is effectively bounded in the sense that we can construct a bound.*

PROOF. As boundedness is equivalent to Lipschitz-continuity the (if) part is a direct consequence of Theorem 7. For the (only if) part, it is clear that computable sequences are mapped to computable sequences. To obtain a bound, suppose $g\colon B(X) \to B(X)$ is the computable function extending f. Then $M := |||g(C(\mathbf{0},1)|||$ is a bound for f. To see this, suppose $g(C(\mathbf{0},1)) = C(y,r)$ and $x \in X$ with $\|x\| = 1$. Then $C(\mathbf{0},1) \sqsubseteq C(x,0)$, hence $C(y,r) = g(C(\mathbf{0},1)) \sqsubseteq g(C(x,0)) = C(f(x),0)$, thus $\|y - f(x)\| + 0 \leq r$. But we also have $C(\mathbf{0},1) \sqsubseteq C(\mathbf{0},0)$ which implies $C(y,r) \sqsubseteq C(\mathbf{0},0)$, thus $\|y\| \leq r$. Therefore $\|f(x)\| \leq r + r = M$. Linearity implies that M is a bound. □

Remark. One might ask the question why computable operators do not necessarily have a computable norm. Apart from the fact there are bounded operators which appear to be computable but have non-computable norm, [2] this would not be in analogy to the classification of computable functions on the reals. There, effective uniform continuity on computable intervals was crucial. Of course, every bounded operator is effectively uniformly continuous, the modulus given by some computable bound. Demanding that the norm be computable would correspond to demanding the *optimal* modulus of continuity to be effective in the case of the real-valued functions. Certainly, this need not be the case.

4.4 Computability structure in the sense of Pour-El & Richards

We want to compare our notion of computability structure for Banach spaces with the one from [17]. The definition of Pour-El and Richards reads as follows:

Definition 19 *Suppose* $(X, \|\cdot\|)$ *is a Banach space. A* PR-computability structure for X *is a nonempty set* $\mathcal{S}(X)$ *of sequences in* X *(referred to as* computable sequences*) such that the following three axioms are satisfied.*

linear forms *If* $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in \mathcal{S}(X)$*, and* $(r_{nk})_{n,k\in\mathbb{N}}$ *and* $(s_{nk})_{n,k\in\mathbb{N}}$ *are computable double sequences of real numbers and if* $d\colon \mathbb{N} \to \mathbb{N}$ *is recursive then* $(w_n)_{n\in\mathbb{N}}$ *is computable, where*

$$w_n = \sum_{k=0}^{d(n)} (r_{nk} y_k + s_{nk} z_k).$$

limits *If* $(y_{nk})_{n,k\in\mathbb{N}}$ *is a computable double sequence which effectively converges to* $(z_n)_{n\in\mathbb{N}}$ *as* $k \to \infty$*, then* $(z_n)_{n\in\mathbb{N}} \in \mathcal{S}(X)$*.*

norm *If* $(y_n)_{n\in\mathbb{N}} \in \mathcal{S}(X)$ *then* $(\|y_n\|)_{n\in\mathbb{N}}$ *is a computable sequence of real numbers.*

(As usual, a double sequence $(y_{kn})_{k,n\in\mathbb{N}}$ *is computable, if the sequence* $(y_{\pi_1(n),\pi_2(n)})_{n\in\mathbb{N}}$ *is computable.) The space is* effectively separable*, if there is a sequence* $(e_n)_{n\in\mathbb{N}} \in \mathcal{S}(X)$ *such that the linear span of* $\{e_0, e_1, e_2, \ldots\}$*, i.e. the set of all finite linear combination of vectors from this set, is dense in* X*.*

Theorem 20 *The computable sequences in an effectively given Banach space* X *constitute a PR-computability structure for* X*. The resulting space is effectively separable.*

PROOF. Let us start with the (linear forms) axiom. Suppose $\oplus\colon \mathbb{N}^2 \to \mathbb{N}$ is the recursive function defining addition on $B(X)$, i.e. $\xi_X(n) + \xi_X(m) =$

[2] Marian Pour-El, personal communication.

$\xi_X(n \oplus m)$. Similarly, let $\odot\colon \mathbb{N}^2 \to \mathbb{N}$ implement scalar multiplication. In the given situation, there are recursive functions $f_y, f_z\colon \mathbb{N} \to \mathbb{N}$ and $f_r, f_s\colon \mathbb{N}^2 \to \mathbb{N}^2$ such that $y_n = \xi_X(f_y(n))$, $z_n = \xi_X(f_z(n))$, $r_{nk} = \xi_{\mathbb{R}}(f_r(n,k))$, and $s_{nk} = \xi_{\mathbb{R}}(f_s(n,k))$ for all $n, k \in \mathbb{N}$. To determine an index $h(n)$ for the element w_n, we use the following algorithm:

(1) Set $a = n_0$, where n_0 is an index for the origin $C(\mathbf{0}, 0)$.
(2) Set $k = 0$.
(3) Set $b = a \oplus ((f_r(n,k) \odot f_y(k) \oplus (f_s(n,k) \odot f_z(k)))$.
(Hence $\xi_X(b) = \xi_X(a) + r_{nk}y_k + s_{nk}z_k$.)
(4) Set $a = b$. Increment k.
(5) If $k > d(n)$ then set $h(n) = a$. Otherwise go to step (3).

It is clear that this gives the desired result $\xi_X(h(n)) = w_n$. Furthermore, the algorithm is effective in n. Thus $h\colon \mathbb{N} \to \mathbb{N}$ is recursive and so the sequence $(w_n)_{n\in\mathbb{N}}$ is computable.

Now the (limits) axiom. If $(y_{nk})_{n,k\in\mathbb{N}}$ is a computable double sequence in X then, by Corollary 16(2), there is $r\colon \mathbb{N}^2 \to \mathbb{N}$ such that

$$\|x_{r(n,k)} - y_{\pi_1(n),\pi_2(n)}\| \leq \frac{1}{2^k}$$

for all $n, k \in \mathbb{N}$. The double sequence $(y_{nk})_{n,k\in\mathbb{N}}$ converges to $(z_n)_{n\in\mathbb{N}}$, hence we also have

$$\|y_{ni} - z_n\| \leq \frac{1}{2^i}$$

for all $n, i \in \mathbb{N}$. Putting these together, we get

$$\|x_{r(\langle n,i\rangle,k)} - z_n\| \leq \|x_{r(\langle n,i\rangle,k)} - y_{ni}\| + \|y_{ni} - z_n\| \leq \frac{1}{2^k} + \frac{1}{2^i}$$

for all $n, k, i \in \mathbb{N}$. Thus setting $r'(n,k) := r(\langle n, k+1\rangle, k+1)$ yields

$$\|x_{r'(n,k)} - z_n\| \leq \frac{1}{2^k}$$

for all $n, k \in \mathbb{N}$, so the limit sequence $(z_n)_{n\in\mathbb{N}}$ is computable by Corollary 16(2).

The (norm) axiom is an immediate consequence of our axioms and the fact that computable functions preserve computability of sequences.

Finally, effective separability follows from Proposition 12. $\square$

So every effectively given Banach space comes with a computability structure in the sense of Pour-El and Richards such that the space is effectively separable. Our aim is to show that the two notions coincide. We start with a

Banach space $(X, \|\cdot\|)$ and a specified set of sequences $\mathcal{S}(X)$ satisfying the axioms to give a PR-computability structure on X. Furthermore, we assume $(e_n)_{n\in\mathbb{N}}$ to be a sequence with dense linear span. In order to be able to define an effective basis for the domain of balls $B(X)$, we need a dense sequence. The natural candidate is the *rational* linear span of $(e_n)_{n\in\mathbb{N}}$. Let us recall how to enumerate the set $\mathbb{N}^*$ of finite sequences over $\mathbb{N}$. (See for example [18, page 71].) For a natural number $n > 0$, let $\psi_n\colon \mathbb{N} \to \mathbb{N}^n$ denote the n-tupling function. (Defined inductively via $\langle\cdot,\cdot\rangle$). Then $\psi\colon \mathbb{N} \to \mathbb{N}^*$ is defined by $\psi(0) = ()$ and $\psi(\langle n, m\rangle + 1) = \psi_m(n)$. Now we define $x_0 = \mathbf{0}$ and

$$x_{n+1} = \sum_{i=0}^{\pi_2(n)-1} q_{a_i} e_i$$

where $\psi(n) = (a_0, a_1, \cdots, a_{\pi_2(n)-1})$.

Lemma 21 *There are total recursive functions $f\colon \mathbb{N} \to \mathbb{N}$ and $\bar{\oplus}, \bar{\odot}\colon \mathbb{N}^2 \to \mathbb{N}$ such that $x_{f(n)} = e_n$, $x_{n\bar{\oplus}m} = x_n + x_m$ and $x_{n\bar{\odot}m} = q_n \cdot x_m$.*

Clearly, $\{x_n \mid n \in \mathbb{N}\}$ is dense in X. We set $B_0 := \perp$ and

$$B_{\langle n,k\rangle+1} := C(x_n, |q_k|),$$

so the set $\{B_0, B_1, \ldots\}$ is a countable basis of $B(X)$. For the proof of the following Theorem, we need the *Effective Density Lemma* from [17]:

Lemma 22 (Theorem 2-1 of [17]) *Suppose that $\mathcal{S}(X)$ is a PR-computability structure for the Banach space $(X, \|\cdot\|)$ and that $(e_n)_{n\in\mathbb{N}} \in \mathcal{S}(X)$ has dense linear span. Then a sequence $(y_n)_{n\in\mathbb{N}}$ is in $\mathcal{S}(X)$ if and only if there is a double sequence $(p_{nk})_{n,k\in\mathbb{N}}$ and recursive functions $d\colon \mathbb{N}^2 \to \mathbb{N}$ and $j\colon \mathbb{N}^3 \to \mathbb{N}$ such that*

$$p_{nk} = \sum_{i=0}^{d(n,k)} q_{j(n,k,i)} e_i$$

and such that $\|p_{nk} - y_n\| \le \frac{1}{2^k}$ for all $n, k \in \mathbb{N}$.

Theorem 23 *The above constructed basis for $B(X)$ makes $(X, \|\cdot\|)$ an effectively given Banach space. The computable sequences on this space are exactly the sequences from $\mathcal{S}(X)$.*

PROOF. We have

$$C(x_n, q_k) \ll C(x_{n'}, q_{k'}) \iff \|x_n - x_{n'}\| + |q_{k'}| < |q_k|.$$

By the (norm) and (linear forms) axiom, this relation is r.e. in n, n', k, k'.

Computability of the operations and the norm is shown in the same manner as in the proof of Theorem 13. Instead of referring to computability w.r.t. the original basis for $B(X)$ we have to refer to the three axioms for sequences, of course. Let us give the details for addition: Using the recursive function $\bar{\oplus}\colon \mathbb{N}^2 \to \mathbb{N}$ from Lemma 21, we see that $C(x_i, |q_{i'}|) \ll C(x_n, |q_{n'}|) + C(x_m, |q_{m'}|)$ iff $C(x_i, |q_{i'}|) \ll C(x_{n\bar{\oplus}m}, |q_{n'}|+|q_{m'}|)$ iff $\|x_i - x_{n\bar{\oplus}m}\| + |q_{n'}| + |q_{m'}| < |q_{i'}|$ which is r.e. in all indices by the (norm) and (linear form) axioms. Hence addition is computable.

So it remains to verify that the resulting set $S'(X)$ of computable sequences coincides with the original $\mathcal{S}(X)$. Assume $(y_n)_{n\in\mathbb{N}} \in \mathcal{S}(X)$. By the Effective Density Lemma 22 it is the limit of a double sequence of recursive rational linear combinations of the e_n. This means that it is the limit of a recursive double sequence of the x_n. By Corollary 16, this implies $(y_n)_{n\in\mathbb{N}} \in \mathcal{S}'(X)$. Conversely, the recursive function f of Lemma 21 shows that $(e_n)_{n\in\mathbb{N}} \in \mathcal{S}'(X)$. We know from Theorem 20 that the set $\mathcal{S}'(X)$ satisfies the (linear forms) and (limits) axioms, so this implies $\mathcal{S}'(X) \subseteq \mathcal{S}(X)$. $\square$

Vice versa we can start with an effective Banach space in the sense of the present paper, take the resulting set of computable sequences, consider this a computability structure in the sense of [17], and finally perform the above construction to get another effective structure for the domain of balls. Propositions 14 and 15 show that this structure is equivalent to the one we started with. This yields

Theorem 24 *The domain-theoretic notion of computability structure for Banach spaces coincides with the notion of Pour-El & Richards from [17].*

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