Linear Programming for Piecewise Linear Geometric Objects with Function and Derivative Constraints

MEng Individual Project

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Introduction

- **Optimisation** is a branch of applied mathematics used in a variety of practical contexts to determine the “best” element from a set of possible alternatives
- Aim is to minimise (or maximise) a certain objective function subject to a given set of constraints
- **Linear Programming** can be used to efficiently solve a particular class of optimisation problems in which the objective and the constraints are given in terms of linear functions

Linear Programming

- Typical problem involving minimisation of linear objective function in n variables subject to m linear equality and/or inequality constraints:

$$
\begin{array}{lccccccccc}
\text{Minimise} & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & \leqq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & \leqq & b_2 \\
 & & & & \vdots & & & & & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & \leqq & b_m
\end{array}
$$

- Input parameters a_{ij}, b_i, c_j, $i = \overline{1,m}$, $j = \overline{1,n}$ are fixed real constants. The decision variables x_i, $i = \overline{1,n}$ are yet to be determined

Linear Programming

- Matrix form:

$$\begin{array}{ll} \text{Minimise} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \end{array}$$

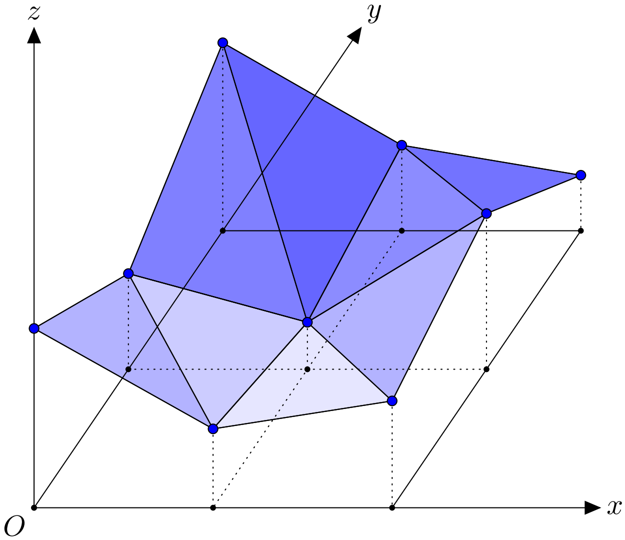
  where $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$

- No analytical formula for solution
- Reliable and efficient algorithms and software
- Computation time proportional to $n^2 m$ if $m \geq n$ (less with structure)
- A mature technology

Application to Piecewise Linear Surfaces

- Piecewise linear surfaces are extensively used in CAD applications to approximate a given map
- A piecewise linear function of 2 variables can be defined by lower and upper bound constraints on its function value and its partial derivatives, respectively
- Constraints are defined for a partitioning of sub-rectangles within the function domain (a closed and compact subset of $\mathbb{R}^2$)
- Linear Programming can be used to determine whether the constraints are *consistent* or not; in addition, the minimal and maximal surfaces can be constructed [A. Edalat, A. Lieutier, D. Pattison, 2014]

Example of Piecewise Linear Surface



Consistency of Function and Derivative Information

- Consider the unit-square $U = [0,1]^2$ as the domain and let:

$$\mathbf{I}\mathbb{R} \;= \Big\{[a,b] \mid a \le b \in \mathbb{R}\Big\} \cup \mathbb{R}$$

$$\mathbf{I}\mathbb{R}^2 = \Big\{[a,b] \times [c,d] \mid a \le b \in \mathbb{R}, c \le d \in \mathbb{R}\Big\} \cup \mathbb{R}^2$$

- Then the lower and upper bound constraints for function and derivative approximations will be given as:

$$f = \left[f^-, f^+\right] \in \mathbf{I}\mathbb{R}$$

$$g = \left(\left[g_1^-, g_1^+\right], \left[g_2^-, g_2^+\right]\right) \in \mathbf{I}\mathbb{R}^2$$

Consistency of Function and Derivative Information

- A pair of functions $(f, g) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{I}\mathbb{R}^2)$ representing function and derivative information respectively, is said to be *consistent* if there exists a third map h such that:

$$f^-(\boldsymbol{x}) \leq h(\boldsymbol{x}) \leq f^+(\boldsymbol{x}) \qquad g_i^-(\boldsymbol{x}) \leq \frac{\partial h(\boldsymbol{x})}{\partial x_i} \leq g_i^+(\boldsymbol{x}),$$

for all $\boldsymbol{x} = [x_1, x_2]^\top \in U$, $i \in \{1, 2\}$

Consistency of Function and Derivative Information

- We impose a grid $(p_0, p_1, \ldots, p_{k-1}) \times (q_0, q_1, \ldots, q_{l-1})$ within the unit square, where all points p_i and q_j lie on the x and y axis, respectively, $i = \overline{0, k-1}$, $j = \overline{0, l-1}$
- Then, for every sub-rectangle $R_{ij} = (p_i, p_{i+1}) \times (q_j, q_{j+1})$ formed by adjacent grid points, the functions $f : U \to \mathbf{I}\mathbb{R}$ and $g : U \to \mathbf{I}\mathbb{R}^2$ are given as follows:

$$f|_{R_{ij}} = \left[c_{ij}^-, c_{ij}^+\right] = c_{ij} \in \mathbf{I}\mathbb{R} \tag{1}$$

$$g|_{R_{ij}} = b_{ij}^1 \times b_{ij}^2 = b_{ij} \in \mathbf{I}\mathbb{R}^2, \tag{2}$$

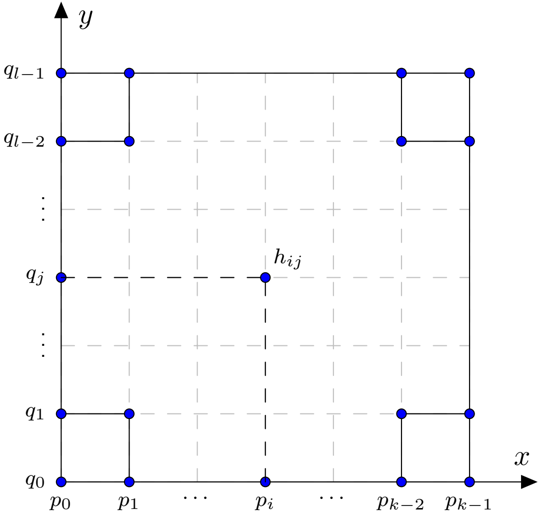
for all $i = \overline{0, k-2}$, $j = \overline{0, l-2}$. Note that each b_{ij}^k, $k \in \{1, 2\}$ is also an interval, as we restrict the value of each partial derivative to lie within a closed and compact interval, so: $b_{ij}^k = \left[b_{ij}^{k-}, b_{ij}^{k+}\right]$ for each k

Consistency of Function and Derivative Information

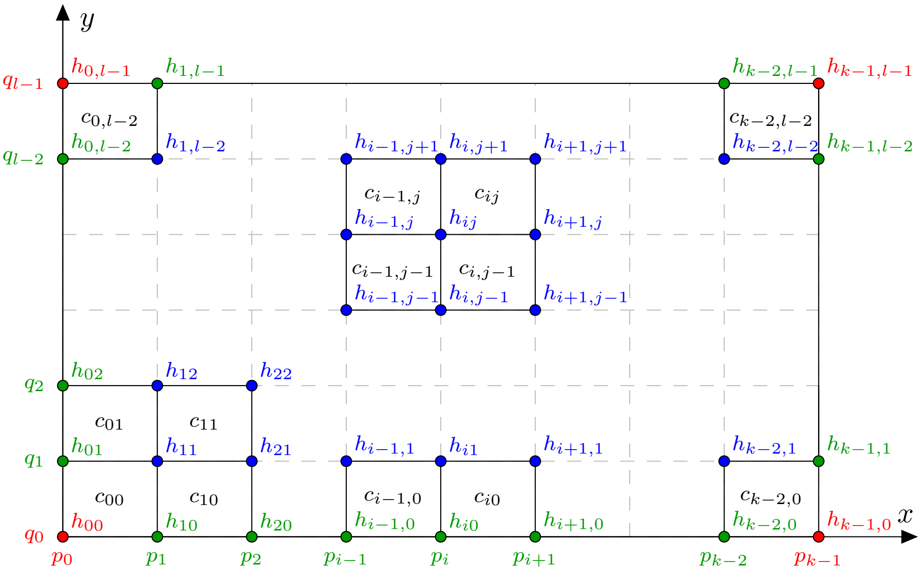
- Checking consistency for a pair (f, g) reduces to determining the existence of heights

$$h_{ij} \equiv h((p_i, q_j))$$

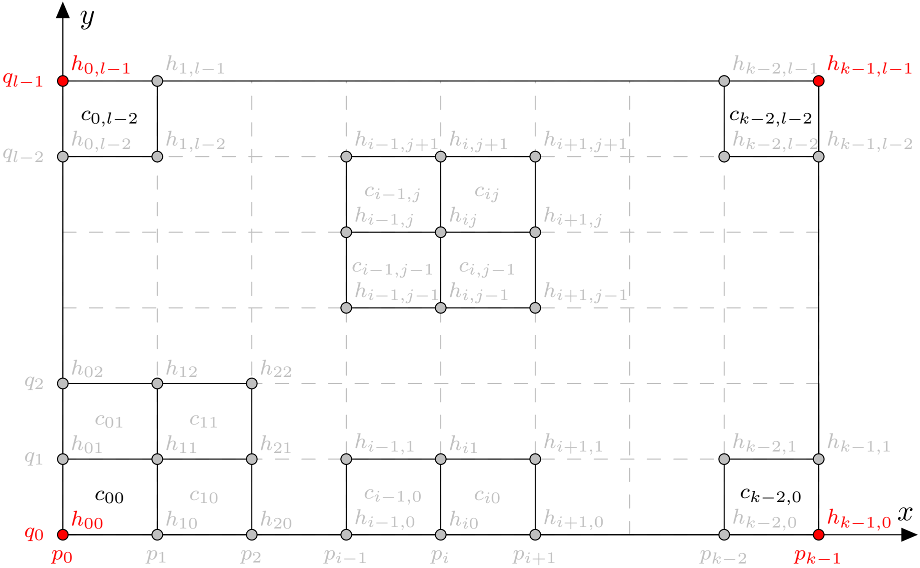
at all the grid points (p_i, q_j) such that the conditions (1) and (2) are satisfied



Function Constraints



Function Constraints (Corner Points)



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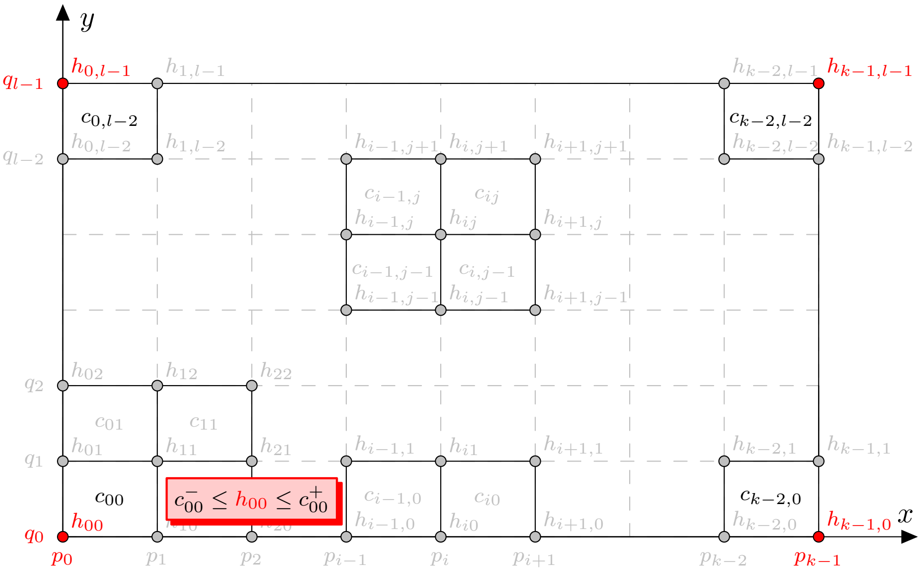
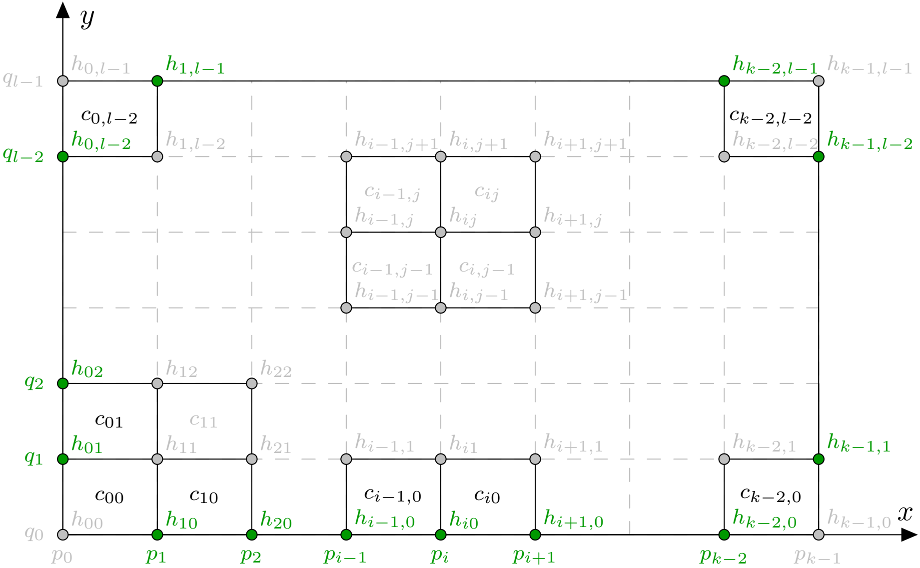
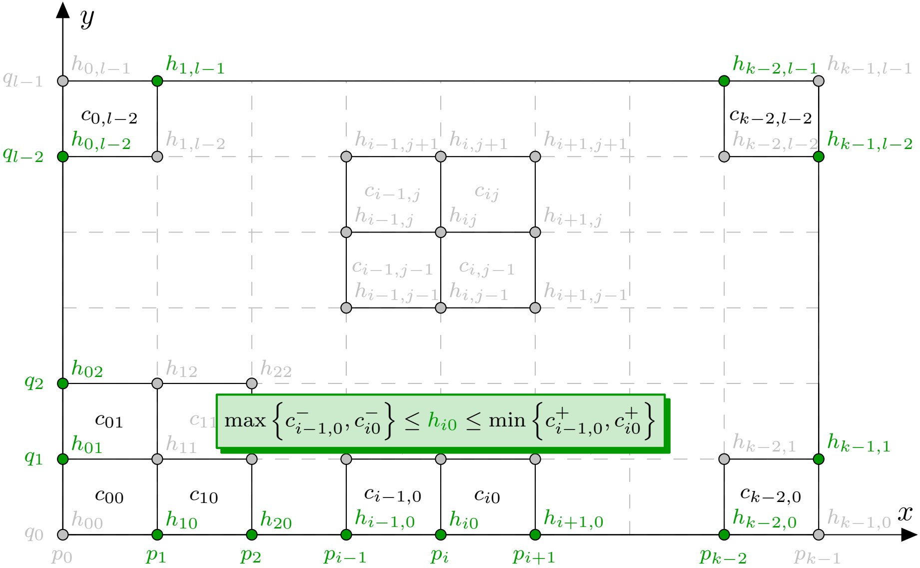


Function Constraints (Border Points)

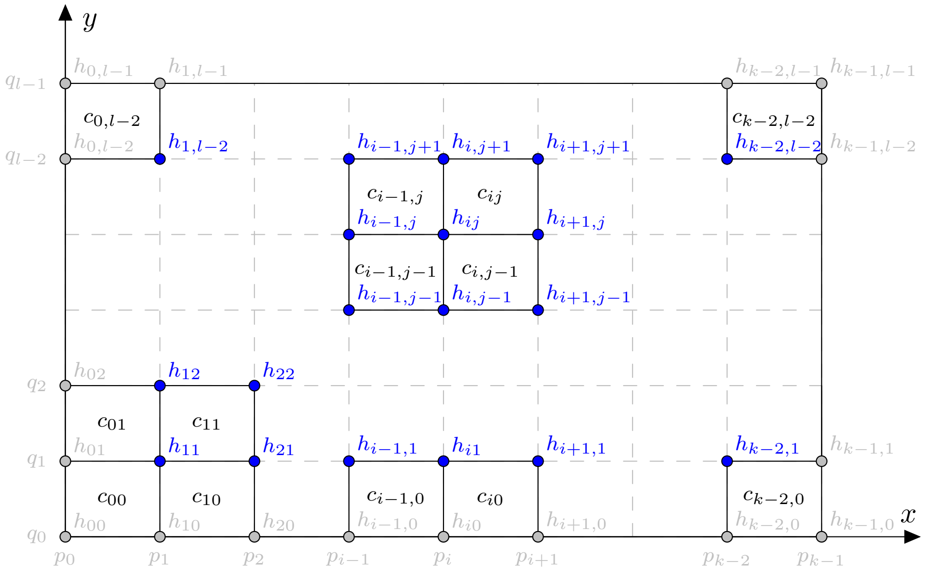


Function Constraints (Border Points)

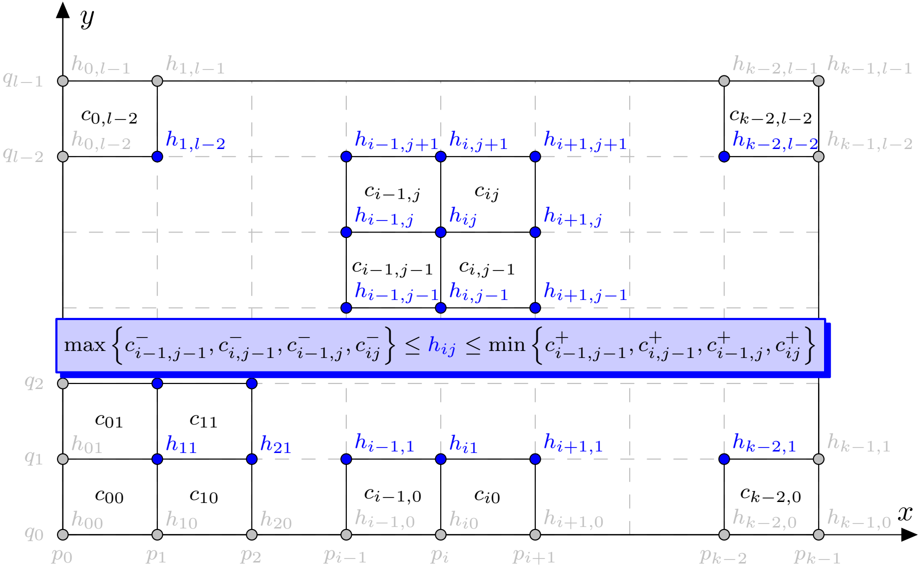


$$\max\left\{c^-_{i-1,0}, c^-_{i0}\right\} \leq h_{i0} \leq \min\left\{c^+_{i-1,0}, c^+_{i0}\right\}$$

Function Constraints (Interior Points)



Function Constraints (Interior Points)



$$\max\left\{c^-_{i-1,j-1}, c^-_{i,j-1}, c^-_{i-1,j}, c^-_{ij}\right\} \leq h_{ij} \leq \min\left\{c^+_{i-1,j-1}, c^+_{i,j-1}, c^+_{i-1,j}, c^+_{ij}\right\}$$

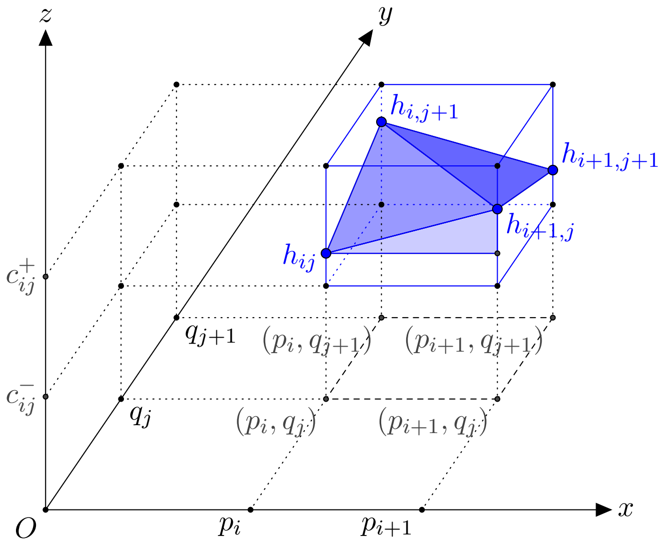
Putting Together All Function Constraints

$$
\begin{array}{ccccc}
c_{00}^{-} & \leq & h_{00} & \leq & c_{00}^{+} \\
c_{k-1,0}^{-} & \leq & h_{k-1,0} & \leq & c_{k-1,0}^{+}
\end{array}
\qquad
\begin{array}{ccccc}
c_{0,l-1}^{-} & \leq & h_{0,l-1} & \leq & c_{0,l-1}^{+} \\
c_{k-1,l-1}^{-} & \leq & h_{k-1,l-1} & \leq & c_{k-1,l-1}^{+}
\end{array}
$$

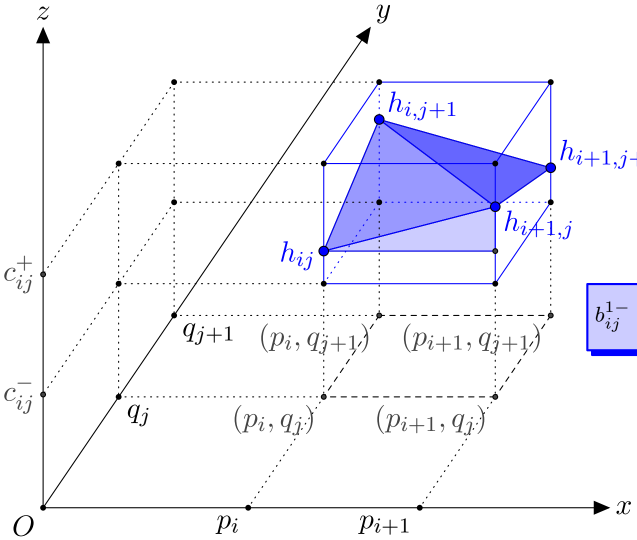
$$
\begin{array}{rcccl}
\max\left\{c_{i-1,0}^{-}, c_{i0}^{-}\right\} & \leq & h_{i0} & \leq & \min\left\{c_{i-1,0}^{+}, c_{i0}^{+}\right\} \\
\max\left\{c_{i-1,l-1}^{-}, c_{i,l-1}^{-}\right\} & \leq & h_{i,l-1} & \leq & \min\left\{c_{i-1,l-1}^{+}, c_{i,l-1}^{+}\right\} \\
\max\left\{c_{0,j-1}^{-}, c_{0j}^{-}\right\} & \leq & h_{0j} & \leq & \min\left\{c_{0,j-1}^{+}, c_{0j}^{+}\right\} \\
\max\left\{c_{k-1,j-1}^{-}, c_{k-1,j}^{-}\right\} & \leq & h_{k-1,j} & \leq & \min\left\{c_{k-1,j-1}^{+}, c_{k-1,j}^{+}\right\}
\end{array}
$$

$$
\max\left\{c_{i-1,j-1}^{-}, c_{i,j-1}^{-}, c_{i-1,j}^{-}, c_{ij}^{-}\right\} \leq h_{ij} \leq \min\left\{c_{i-1,j-1}^{+}, c_{i,j-1}^{+}, c_{i-1,j}^{+}, c_{ij}^{+}\right\}
$$

Derivative Constraints

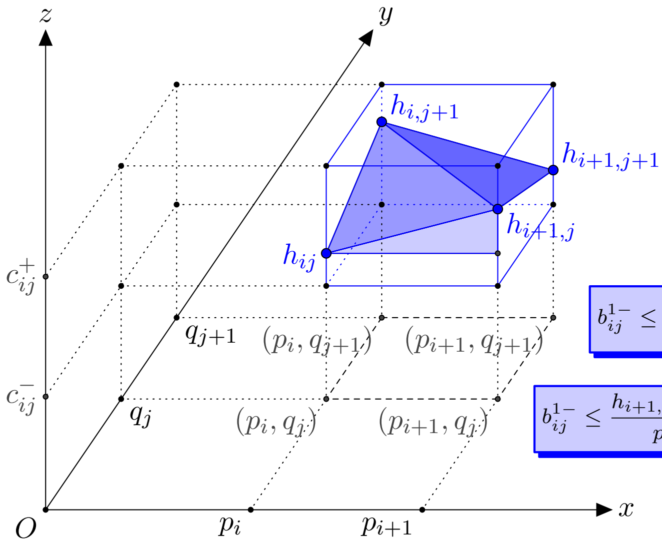


Derivative Constraints



$$b_{ij}^{1-} \leq \frac{h_{i+1,j} - h_{ij}}{p_{i+1} - p_i} \leq b_{ij}^{1+}$$

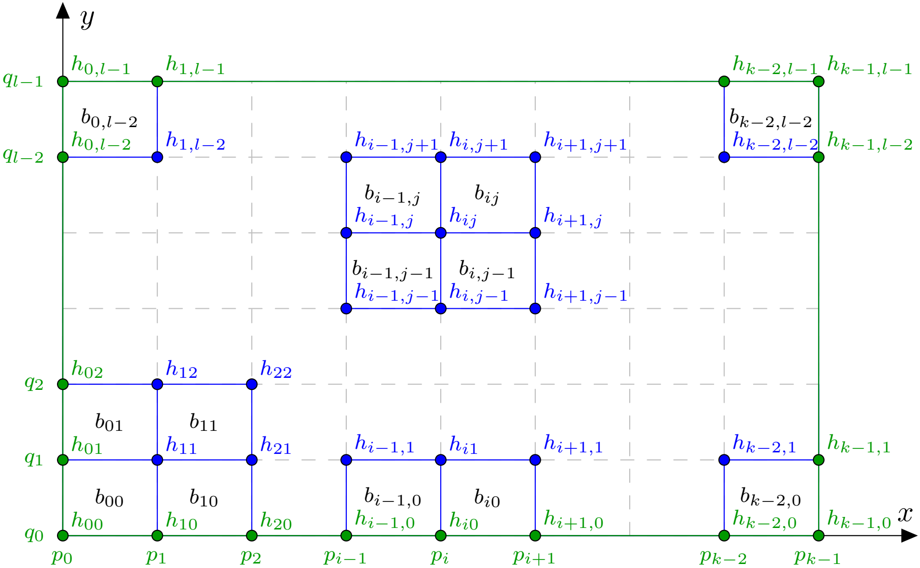
Derivative Constraints



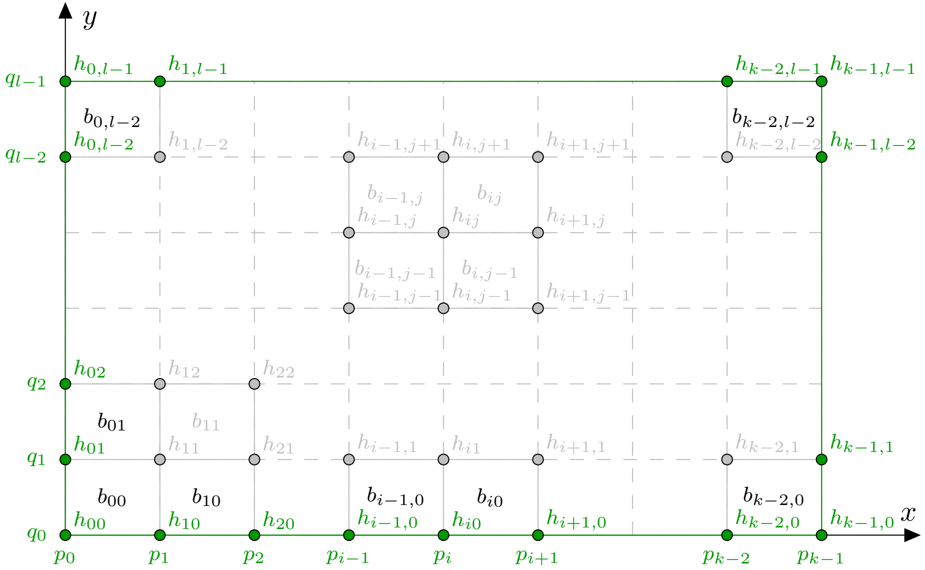
$$b_{ij}^{1-} \leq \frac{h_{i+1,j} - h_{ij}}{p_{i+1} - p_i} \leq b_{ij}^{1+}$$

$$b_{ij}^{1-} \leq \frac{h_{i+1,j+1} - h_{i,j+1}}{p_{i+1} - p_i} \leq b_{ij}^{1+}$$

Derivative Constraints



Derivative Constraints (Border Edges)



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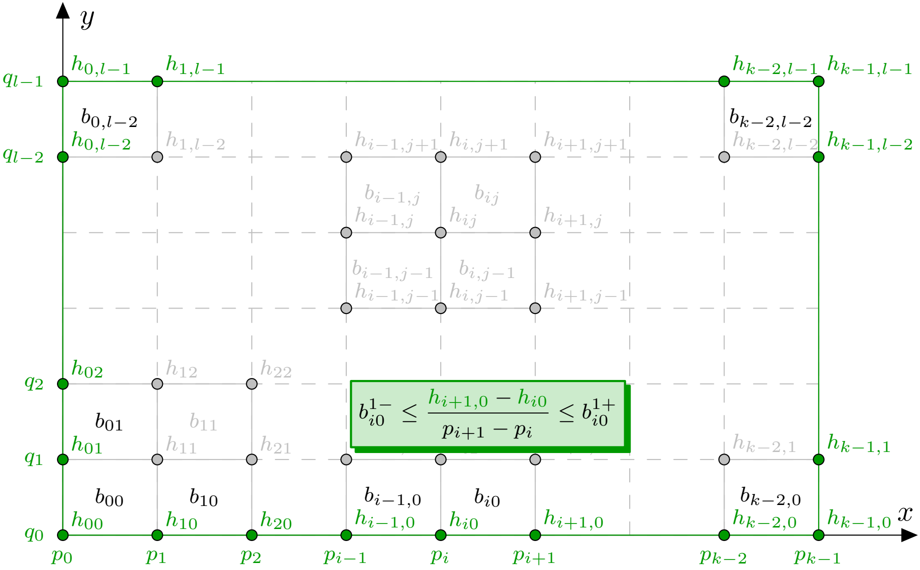
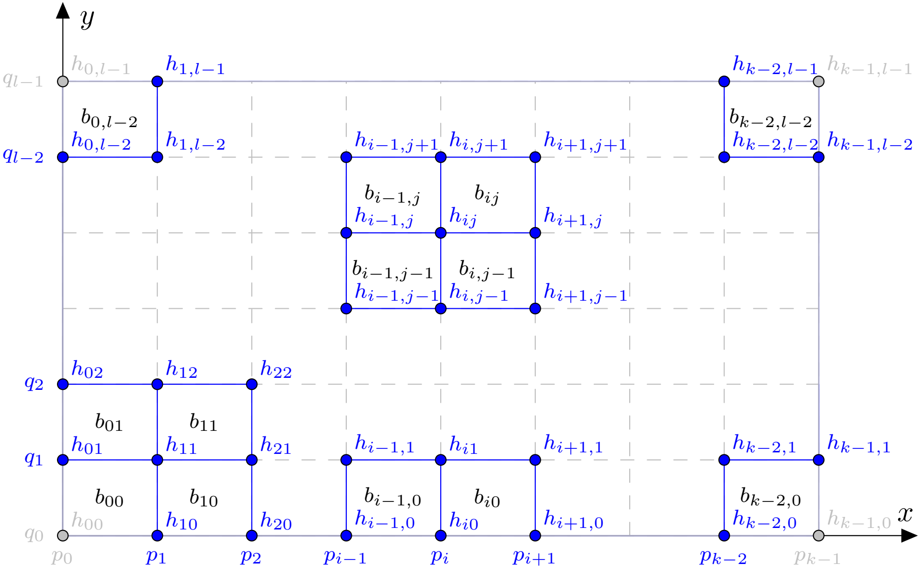
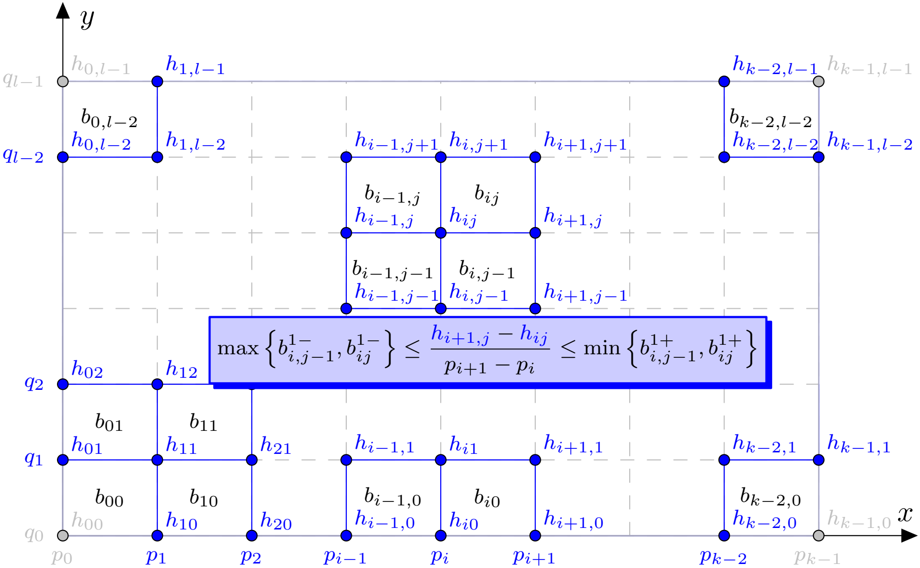


$$b_{i0}^{1-} \leq \frac{h_{i+1,0} - h_{i0}}{p_{i+1} - p_i} \leq b_{i0}^{1+}$$

Derivative Constraints (Interior Edges)



Derivative Constraints (Interior Edges)



$$\max\left\{b^{1-}_{i,j-1}, b^{1-}_{ij}\right\} \leq \frac{h_{i+1,j} - h_{ij}}{p_{i+1} - p_i} \leq \min\left\{b^{1+}_{i,j-1}, b^{1+}_{ij}\right\}$$

Putting Together All Derivative Constraints

$$b^{1-}_{i0} \leq \frac{h_{i+1,0} - h_{i0}}{p_{i+1} - p_i} \leq b^{1+}_{i0}$$

$$b^{1-}_{i,l-2} \leq \frac{h_{i+1,l-1} - h_{i,l-1}}{p_{i+1} - p_i} \leq b^{1+}_{i,l-2}$$

$$b^{2-}_{0j} \leq \frac{h_{0,j+1} - h_{0j}}{q_{j+1} - q_j} \leq b^{2+}_{0j}$$

$$b^{2-}_{k-2,j} \leq \frac{h_{k-1,j+1} - h_{k-1,j}}{q_{j+1} - q_j} \leq b^{2+}_{k-2,j}$$

$$\max\left\{b^{1-}_{i,j-1}, b^{1-}_{ij}\right\} \leq \frac{h_{i+1,j} - h_{ij}}{p_{i+1} - p_i} \leq \min\left\{b^{1+}_{i,j-1}, b^{1+}_{ij}\right\}$$

$$\max\left\{b^{2-}_{i-1,j}, b^{2-}_{ij}\right\} \leq \frac{h_{i,j+1} - h_{ij}}{q_{j+1} - q_j} \leq \min\left\{b^{2+}_{i-1,j}, b^{2+}_{ij}\right\}$$

Summary of Inequality Constraints

- The inequalities developed from the function and derivative constraints can be written as:

$$\begin{array}{ccccc} C_{ij}^{-} & \leq & h_{ij} & \leq & C_{ij}^{+} \\ B_{ij}^{1-} & \leq & h_{i+1,j} - h_{ij} & \leq & B_{ij}^{1+} \\ B_{ij}^{2-} & \leq & h_{i,j+1} - h_{ij} & \leq & B_{ij}^{2+} \end{array}$$

for suitable indices i, j and $C_{ij}^{-}, C_{ij}^{+}, B_{ij}^{i-}, B_{ij}^{i+} \in \mathbb{R}$, where $i \in \{1, 2\}$

- The question of consistency reduces to solving a finite set of inequalities for the $k \times l$ unknowns h_{ij}, which is decidable. In fact, these inequalities represent the constraints of a Linear Programming problem

Minimal and Maximal Surfaces

- The LP problem which gives the least consistent witness can be obtained by minimising the sum of all heights:

$$\text{Minimise} \quad \sum_{\substack{0 \leq i \leq k-1 \\ 0 \leq j \leq l-1}} h_{ij}$$

$$\text{subject to} \quad \begin{array}{ccccc} C_{ij}^{-} & \leq & h_{ij} & \leq & C_{ij}^{+} \\ B_{ij}^{1-} & \leq & h_{i+1,j} - h_{ij} & \leq & B_{ij}^{1+} \\ B_{ij}^{2-} & \leq & h_{i,j+1} - h_{ij} & \leq & B_{ij}^{2+} \end{array}$$

- Similarly, maximising the sum of all heights subject to the same constraints yields the greatest consistent witness

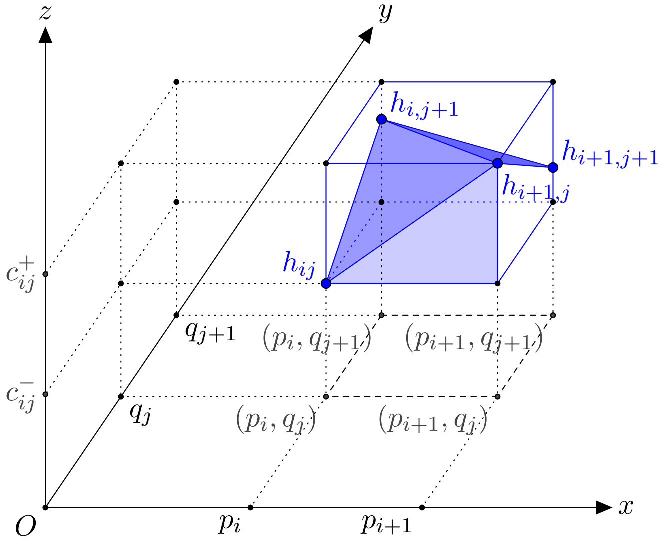
Generalisation to Higher Dimensions

- One can easily extend the above LP problems to determine whether consistency holds for a pair $(f, g) \in (U \to \mathbf{I}\mathbb{R}) \times (U \to \mathbf{I}\mathbb{R}^n)$, $n \geq 2$
- Consider a similar grid within the unit cube $U \subset \mathbb{R}^n$ with sub-hyper-rectangles in which the f and g values are constant
- Derive inequalities from both the function and derivative constraints
- Likewise, the least and greatest consistent witnesses will be obtained by minimising and respectively maximising the sum of heights at all the grid points

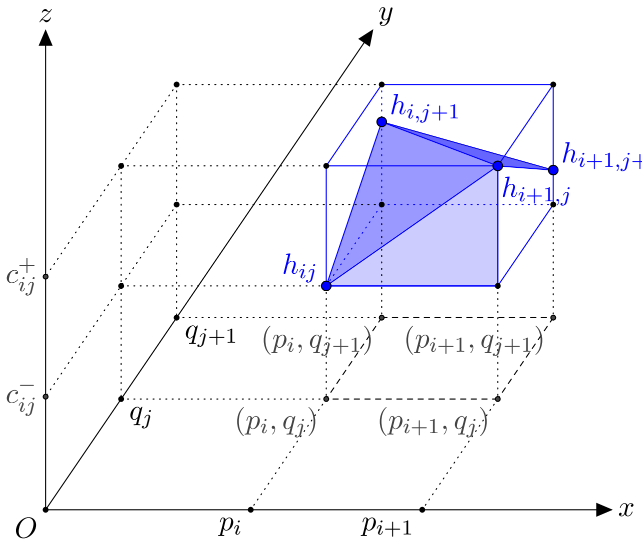
Evaluation

- Need to ensure that the implementation of the LP algorithm is correctly classifying both consistent and inconsistent input
- Reverse engineer the constraints to generate both consistent and inconsistent data
- Start by generating random heights h_{ij} at all the grid points (p_i, q_j) and then determine appropriate bounding boxes from which function and derivative constraints can be constructed
- Make the minimal change to generate inconsistent input from consistent information by breaking a random derivative constraint at a random sub-rectangle in the unit grid

Evaluation (Inconsistent Input)



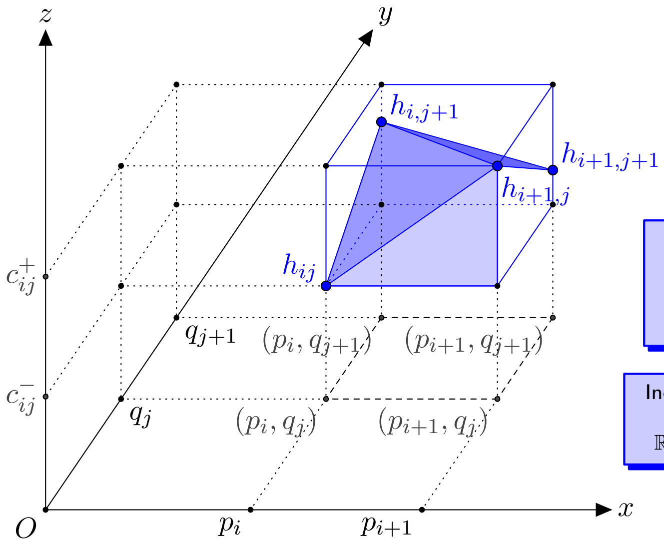
Evaluation (Inconsistent Input)



Maximum Slope:

$$s_{\max} = \frac{c_{ij}^{+} - c_{ij}^{-}}{p_{i+1} - p_i}$$

Evaluation (Inconsistent Input)



Maximum Slope:

$$s_{\max} = \frac{c_{ij}^{+} - c_{ij}^{-}}{p_{i+1} - p_i}$$

Inconsistent Interval:

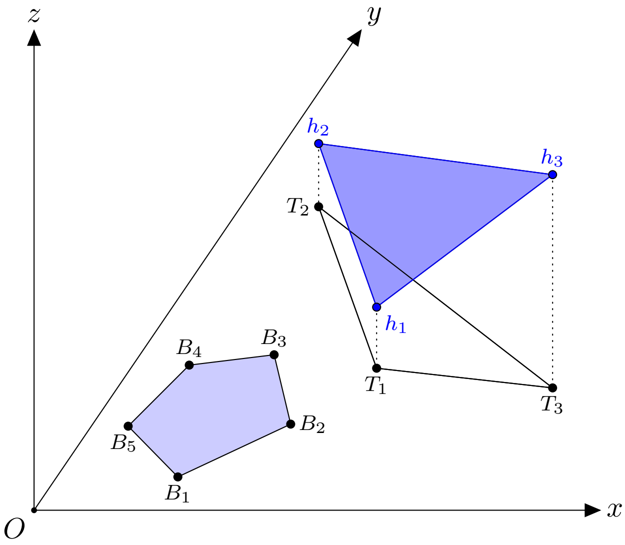
$$\mathbb{R} \setminus [-s_{\max}, s_{\max}]$$

Demo

Extension to Convex Derivative Constraints

- A much more challenging problem is when the derivative constraints lie within a convex polygon B, rather than a simple rectangle
- In the case when the domain is specified as a triangle T_{123}, then a Linear Programming algorithm can be developed to determine whether consistency holds [A. Edalat, 2015]
- A witness can be constructed by considering heights at the three vertices of the triangle and checking if the gradient of the plane is contained within B; if this test fails, then a complex triangulation of T_{123} can be performed to determine a consistent witness

Extension to Convex Derivative Constraints



Extension to Convex Derivative Constraints