Set-valued Lipschitz constant for vector-valued functions: Failure of the "natural" extension of the definition for scalar functions

For a scalar function $f : U \subset \mathbb{R}^n \to \mathbb{R}$, the definition of a set-valued Lipschitz constant is the following [1]:

Definition 0.1. The continuous function $f : U \to \mathbb{R}$ has *a non-empty, convex and compact set-valued Lipschitz constant* $b \in \mathbf{C}\mathbb{R}^n$ in an open subset $a \subset U$ if for all $x, y \in a$ we have: $b(x-y) \sqsubseteq f(x) - f(y)$, equivalently $f(x) - f(y) \in b(x-y)$.

Note that in the above definition, we use a compact convex set namely $b(x-y)$ that contains the difference $f(x)-f(y)$ of the values of the function at two points $x, y \in a$, i.e., we require: $b(x-y) \sqsubseteq f(x)-f(y)$, equivalently $f(x)-f(y) \in b(x-y)$. This therefore gives a set-theoretic bound for $f(x) - f(y)$ using b and $x - y$. We now see the similarity with the definition of the **classical** Lipschitz constant k for a map $f : X \to Y$ between normed vector spaces, i.e., $\|f(x) - f(y)\| \leq k\|x - y\|$, which gives an upper bound for the norm of the difference $f(x) - f(y)$ in terms of k and $\|x - y\|$. Thus, it seems natural that for extension to vector functions $f : \mathbb{F}^n \to \mathbb{F}^m$ we follow the same rule and require that the difference $f(x) - f(y)$ be contained in $b(x - y)$.

So, suppose we replace Definition 3.2 in the current paper with the following definition which is a "natural" extension of the above definition of a set-valued Lipschitz constant for scalar functions.

Definition 0.2. The continuous function $f : U \subset \mathbb{F}^n \to \mathbb{F}^m$ has *a non-empty, convex and compact set-valued Lipschitz constant* $b \in \mathbf{C}(\mathbb{F}^{m\times n})$ in an open subset $a \subseteq U$ if for all $x, y \in a$ we have: $b(x - y) \sqsubseteq f(x) - f(y)$, equivalently $f(x) - f(y) \in b(x - y)$.

It is now natural to define the zero-containment predicate $\mathsf{Z}(b, \mathbb{F}^{m\times n})$ in the "natural" setting as:

$$\mathsf{Z}(b, \mathbb{F}^{m\times n}) \equiv \forall \epsilon > 0. \forall v \in S. \exists A \in b. \|Av\| \leq \epsilon$$

We will show here that with this "natural" extension of the notion of set-valued Lipschitz constant from scalar functions to vector functions, Lemma 3.5 (which plays a crucial role in the results of the paper) fails.

It can be shown that the following four conditions are equivalent:

- $\mathsf{Z}(b, \mathbb{F}^{m\times n})$
- $\forall u \in \mathbb{F}^m. 0 \in ub = \{uB : B \in b\}$
- $\forall u \in \mathbb{F}^m \forall v \in \mathbb{F}^n. 0 \in ubv = \{uBv : B \in b\}$
- $\forall u \in \mathbb{F}^m \forall v \in \mathbb{F}^n. 0 \in \{\sum_{i=1}^m \sum_{j=1}^n B_{ij} u_i v_j : B \in b\}$

The latter condition is weaker than $\forall w \in \mathbb{F}^{m\times n}. 0 \in Bw = \{\sum_{i=1}^m \sum_{j=1}^n B_{ij} w_{ij} : B \in b\}$. This latter condition of course implies $0 \in b$.

Consider the set of tensor products $T = \{u \otimes v : u \in \mathbb{F}^m, v \in \mathbb{F}^n\}$ where $u \otimes v \in F^{m\times n}$ with $(u \times v)_{ij} = u_i v_j$. Then, T spans $F^{m\times n}$ but it is **not dense** in $\mathbb{F}^{m\times n}$.

Therefore, to construct a counter-example in the new setting to Lemma 3.5 for the simplest case with $\mathbb{F} = \mathbb{R}$ and $m = n = 2$, say, we construct a compact and convex non-empty set b with $0 \notin b$ such that b is contained in an open subset of $\mathbb{F}^{m\times n}$ which does not contain any tensor product $u \otimes v \in T$.

Here is an example. Let $m = n = 2$, and consider the following matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Observe that I is not in the closure of T and A, B, C span a hyperplane orthogonal to I.

Consider:

$$b = \{I + \alpha A + \beta B + \gamma C : \alpha \in [-1, 1], \beta, \gamma \in [-2, 2]\}$$

Note that b is convex as well as compact and does not contain 0. Take $s \in b$:

$$s = \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1-\alpha \end{pmatrix}$$

Consider $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We have:

$$sv = \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1-\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1+\alpha)x + \beta y \\ \gamma x + (1-\alpha) y \end{pmatrix}$$

If $|x| \geq |y|$, taking $\alpha = -1, \beta = 0, \gamma = -2y/x$ gives $sv = 0$ and if $|x| \leq |y|$, taking $\alpha = 1, \beta = -2x/y, \gamma = 0$ gives again $sv = 0$. Thus, $\mathsf{Z}(b, \mathbb{R}^{2\times 2})$ is satisfied but $0 \notin b$. This counter-example shows that Lemma 3.5 would become false if we chose the above “natural” extension of the notion of set-valued Lipschitz constant.

(I spent many months trying to show that the “natural extension” works as I was convinced it was correct but (in retrospect) obviously could not prove Lemma 3.5 in that setting. Then I found it non-trivial to construct a counter example as above to Lemma 3.5 in that setting.)

References

A. Edalat. A continuous derivative for real-valued functions. In S. B. Cooper, B. Löwe, and A. Sorbi, editors, *New Computational Paradigms, Changing Conceptions of What is Computable*, pages 493–519. Springer, 2008.