Counterexample to Lemma 3.4

Note first that the "zero containment predicate" $\mathsf{Z}(b, \mathbb{F}^{m\times n})$:

$$\mathsf{Z}(b, \mathbb{F}^{m\times n}) \equiv \forall \epsilon > 0. \forall v \in S. \exists A \in b. \|Av\| \leq \epsilon$$

is equivalent to any of the following three statements:

- $\forall u \in \mathbb{F}^m. 0 \in ub = \{uB : B \in b\}$
- $\forall u \in \mathbb{F}^m \forall v \in \mathbb{F}^n. 0 \in ubv = \{uBv : B \in b\}$
- $\forall u \in \mathbb{F}^m \forall v \in \mathbb{F}^n. 0 \in \{\sum_{i=1}^m \sum_{j=1}^n B_{ij} u_i v_j : B \in b\}$

The latter condition is weaker than $\forall w \in \mathbb{F}^{m\times n}. 0 \in Bw = \{\sum_{i=1}^m \sum_{j=1}^n B_{ij} w_{ij} : B \in b\}$. This latter condition of course implies $0 \in b$.

Consider the set of tensor products $T = \{u \otimes v : u \in \mathbb{F}^m, v \in \mathbb{F}^n\}$ where $u \otimes v \in F^{m\times n}$ with $(u \times v)_{ij} = u_i v_j$. Then, T spans $F^{m\times n}$ but it is **not dense** in $\mathbb{F}^{m\times n}$.

Therefore, to construct a counter-example to Lemma 3.4 for the simplest case with $\mathbb{F} = \mathbb{R}$ and $m = n = 2$, say, we construct a compact and convex non-empty set b with $0 \notin b$ such that b is contained in an open subset of $\mathbb{F}^{m\times n}$ which does not contain any tensor product $u \otimes v \in T$.

Here is an example. Let $m = n = 2$, and consider the following matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Observe that I is not in the closure of T and A, B, C span a hyperplane orthogonal to I.

Consider:

$$Q = \{I + aA + bB + cC : a \in [-1, 1], b, c \in [-2, 2]\}$$

Note that Q is convex as well as compact and does not contain 0. Take $s \in Q$:

$$s = \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix}$$

Consider $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We have:

$$sv = \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1+a)x + by \\ cx + (1-a)y \end{pmatrix}$$

If $|x| \geq |y|$, taking $a = -1, b = 0, c = -2y/x$ gives $sv = 0$ and if $|x| \leq |y|$, taking $a = 1, b = -2x/y, c = 0$ gives again $sv = 0$. Thus, $\mathsf{Z}(Q, \mathbb{R}^{2\times 2})$ is satisfied but $0 \notin Q$. This counter-example shows that Lemma 3.4 as it stood is false.