Countable and uncountable sets
Structure of the lecture course

- Abbas Edalat will give the first part of the course with 9 lectures and the first assessed course work followed by the second part with another 9 lectures and the second course work by Pete Harisson.
- For the first part of the course, the General Lecture Notes (by Istvan Maros) will be used mostly to review the material you studied in your first year.
- In addition, there will be lecture notes covering new material and providing proofs for some of results in the General Lecture Notes.
Textbooks and Videos

- As well as the three textbooks recommended in the description of the course on the departmental web-page for the course, you can look at:

- You can also watch Gilbert Strang’s lectures at MIT on video online.
Countable sets

We say an infinite (i.e., a non-finite) set $S$ is **countable** if there exists an onto map (i.e., a surjection)

$$f : \mathbb{N} \rightarrow S,$$

where $\mathbb{N} = \{0, 1, 2, \ldots,\}$ is the set of natural numbers.

Such a map $f$ is called an **enumeration** of $S$.

Given such an enumeration $f$ we can construct an enumeration

$$g : \mathbb{N} \rightarrow S,$$

which would be 1-1 as well. Such $g$ will have the same range as $f$ (namely $S$) but it will map distinct elements to distinct elements.

Here is an inductive definition of $g$:

- Let $g(0) := f(0)$.
- For $i > 0$, assume inductively that $g(i - 1)$ has been defined and $g(i - 1) = f(j)$ for some $j \in \mathbb{N}$. Put $g(i) = f(j')$ where $j'$ is the least integer greater than $j$ (i.e., $j' > j$) such that $f(j') \neq f(n)$ for $n < j'$.

It is easy to check that $g$ is onto and 1-1.
Rational numbers are countable

Consider the two dimensional array of fractional numbers below, where every fraction on the nth row has n in the numerator and every fraction in the mth column has m in the denominator.

We count the elements of the array as in the diagram by discarding fractions that are not in reduced form.

This gives a 1-1 correspondence between natural numbers and positive rational numbers.

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Exercises: examples of countable sets

(i) The set of all positive integers is countable.
(ii) The set of all integers is countable.
(iii) We can show by induction on $n$ that the set of ordered lists of natural numbers that have length $n$ is countable.
(iv) We can then use (iii) to show that the set of all finite ordered lists of natural numbers is countable.
(v) Any non-finite subset of a countable set is countable.
(vi) If $S$ is countable then $S^n$, i.e., the collection of all $n$-tuples of elements of $S$, is countable.
(vii) From (vi), we can deduce that the set of integer polynomials (i.e., polynomials with integer co-efficients) is countable.
(viii) From (vii) it follows that the set of roots of integer polynomials, the so-called algebraic numbers, is also countable.
Real numbers are not countable

- The set of real numbers in \([0, 1]\) is not countable.
- Suppose, for the sake of deriving a contradiction, that real numbers in \([0, 1]\) are countable, given by \(a_1, a_2, a_3, \ldots\).
- Write each of these in its decimal expansion:
  \(a_m = 0.a_{m1}a_{m2}a_{m3}\ldots\) where \(a_{mn} \in \{0, 1, 2, \ldots, 9\}\) is the \(n\)th digit in the decimal expansion of \(a_m\).
- We then obtain:
  \[
  a_1 = 0.a_{11}a_{12}a_{13}\ldots a_{1m}\ldots \\
a_2 = 0.a_{21}a_{22}a_{23}\ldots a_{2m}\ldots \\
  \ldots \ldots \ldots \ldots \ldots \\
a_m = 0.a_{m1}a_{m2}a_{m3}\ldots a_{mm}\ldots \\
  \ldots \ldots \ldots \ldots \ldots 
  \]
- Define \(b \in [0, 1]\) with decimal expansion \(b = 0.b_1b_2b_3\ldots\) by putting: \(b_m = 1\) if \(a_{mm} \neq 1\) and \(b_m = 2\) if \(a_{mm} = 1\).
- Then, for each \(m = 1, 2, 3, \ldots\), the \(m\)th digit of \(b\) differs from the \(m\)th digit of \(a_m\) and therefore we have \(b \neq a_m\).
- Thus, \(b \in [0, 1]\) but \(b \neq a_m\) for any \(m\), a contradiction.