Countable and uncountable sets

Structure of the lecture course

- Abbas Edalat will give the first part of the course with 9 lectures and the first assessed course work followed by the second part with another 9 lectures and the second course work by Pete Harisson.
- For the first part of the course, the General Lecture Notes (by Istvan Maros) will be used mostly to review the material you studied in your first year.
- In addition, there will be lecture notes covering new material and providing proofs for some of results in the General Lecture Notes.

Textbooks and Videos

- As well as the three textbooks recommended in the description of the course on the departmental web-page for the course, you can look at:
  (i) Strang, Gilbert. Introduction to Linear Algebra. 4th ed. Wellesley, MA: Wellesley-Cambridge Press, February 2009.[an introductory textbook]
  (ii) Strang, Gilbert. Linear Algebra and its Applications. 3rd ed. Harcourth Brace Jovanovich, February 1988. [a more advanced textbook]
- You can also watch Gilbert Strang's lectures at MIT on video online.

Countable sets

- We say an infinite (i.e., a non-finite) set S is **countable** if there exists an onto map (i.e., a surjection)

$$f : \mathbb{N} \to S,$$

  where $\mathbb{N} = \{0, 1, 2, \ldots, \}$ is the set of natural numbers.
- Such a map f is called an **enumeration** of S.
- Given such an enumeration f we can construct an enumeration

$$g : \mathbb{N} \to S,$$

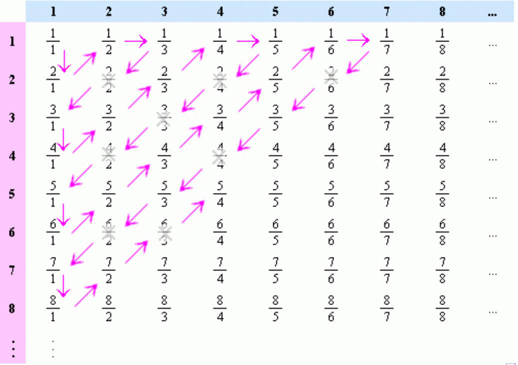
  which would be 1-1 as well. Such g will have the same range as f (namely S) but it will map distinct elements to distinct elements.
- Here is an inductive definition of g:
  - Let $g(0) := f(0)$.
  - For $i > 0$, assume inductively that $g(i-1)$ has been defined and $g(i-1) = f(j)$ for some $j \in \mathbb{N}$. Put $g(i) = f(j')$ where j' is the least integer greater than j (i.e., $j' > j$) such that $f(j') \neq f(n)$ for $n < j'$.
- It is easy to check that g is onto and 1-1.

Rational numbers are countable

Consider the two dimensional array of fractional numbers below, where every fraction on the nth row has n in the numerator and every fraction in the mth column has m in the denominator.

We count the elements of the array as in the diagram by discarding fractions that are not in reduced form.

This gives a 1-1 correspondence between natural numbers and positive rational numbers.



Exercises: examples of countable sets

(i) The set of all positive integers is countable.

(ii) The set of all integers is countable.

(iii) We can show by induction on n that the set of ordered lists of natural numbers that have length n is countable.

(iv) We can then use (iii) to show that the set of all finite ordered lists of natural numbers is countable.

(v) Any non-finite subset of a countable set is countable.

(vi) If S is countable then S^n, i.e., the collection of all n-tuples of elements of S, is countable.

(vii) From (vi), we can deduce that the set of integer polynomials (i.e., polynomials with integer co-efficients) is countable.

(viii) From (vii) it follows that the set of roots of integer polynomials, the so-called algebraic numbers, is also countable.

Real numbers are not countable

- The set of real numbers in $[0, 1]$ is not countable.
- Suppose, for the sake of deriving a contradiction, that real numbers in $[0, 1]$ are countable, given by $a_1, a_2, a_3, \ldots$.
- Write each of these in its decimal expansion: $a_m = 0.a_{m1}a_{m2}a_{m3}\ldots$ where $a_{mn} \in \{0, 1, 2, \cdots, 9\}$ is the nth digit in the decimal expansion of a_m.
- We then obtain:

$$
\begin{aligned}
a_1 &= 0.a_{11}a_{12}a_{13}\ldots a_{1m}\cdots \\
a_2 &= 0.a_{21}a_{22}a_{23}\ldots a_{2m}\cdots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_m &= 0.a_{m1}a_{m2}a_{m3}\ldots a_{mm}\cdots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

- Define $b \in [0, 1]$ with decimal expansion $b = 0.b_1b_2b_3\ldots$ by putting: $b_m = 1$ if $a_{mm} \neq 1$ and $b_m = 2$ if $a_{mm} = 1$.
- Then, for each $m = 1, 2, 3, \ldots$, the mth digit of b differs from the mth digit of a_m and therefore we have $b \neq a_m$.
- Thus, $b \in [0, 1]$ but $b \neq a_m$ for any m, a contradiction.