233 Computational Techniques

Problem Sheet for Tutorial 2

Problem 1
Which of the following pairs of vectors are orthogonal:

(a) $[1, 2]$ and $[-1, 1]$,

(b) $[2, 5, 1]$ and $[-3, 1, 1]$,

(c) $[3, 5, 3, -4]$ and $[4, -2, 2, 2]$.

Problem 2
For

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

decide which of the following products are defined, and compute them:
(a) $\boldsymbol{Au}$, (b) $\boldsymbol{Av}$, (c) $\boldsymbol{A}^T\boldsymbol{v}$, ($d$) $\boldsymbol{u}^T\boldsymbol{v}$, ($e$) $\boldsymbol{uv}^T$.

Problem 3
From the pair of vectors in problem 2(b), construct an orthonormal set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ such that two of them are multiples of the given pair.

Problem 4
Matrix representation of linear maps: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map and let $\mathbf{e}_1, \mathbf{e}_2$ be a basis for $\mathbb{R}^2$. Suppose

$$f(\mathbf{e}_1) = 5\mathbf{e}_1 - 6\mathbf{e}_2 \quad f(\mathbf{e}_2) = \mathbf{e}_2 + 3\mathbf{e}_1.$$

- Find the matrix $\boldsymbol{A}$ representing f with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$.
- If $\boldsymbol{v} \in \mathbb{R}^2$ is given by $\boldsymbol{v} = 2\mathbf{e}_1 - \mathbf{e}_2$. Find $f(\boldsymbol{v})$ and check that the matrix $\boldsymbol{A}$ representing f correctly computes the coordinates of $f(\mathbf{v})$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$.

Problem 5
Matrix multiplication is not commutative: that is, $\boldsymbol{AB} \neq \boldsymbol{BA}$ in general. As an illustration, prove that a square 2×2 matrix $\boldsymbol{A}$ satisfying $\boldsymbol{AX} = \boldsymbol{XA}$ for every 2×2 matrix $\boldsymbol{X}$ must be a multiple of the unit matrix $\boldsymbol{I}_2$. In other words, prove the following:

$$\boldsymbol{A} \in \mathbb{R}^{2\times 2} \text{ and } \boldsymbol{AX} = \boldsymbol{XA} \text{ for all } \boldsymbol{X} \in \mathbb{R}^{2\times 2} \iff \exists \lambda \in \mathbb{R} \text{ such that } \boldsymbol{A} = \lambda \boldsymbol{I}_2 \,.$$

(This is true for square matrices of any size!) *Hint*: Compare $\boldsymbol{AX}$ and $\boldsymbol{XA}$ for matrices X which have one entry equal to 1 and all others zero; for instance for

$$\boldsymbol{E}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{E}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note: The formulation was changed slightly in order to clarify the problem.