Problem 1
Which of the following pairs of vectors are orthogonal:

(a) \([1, 2]\) and \([-1, 1]\),
(b) \([2, 5, 1]\) and \([-3, 1, 1]\),
(c) \([3, 5, 3, -4]\) and \([4, -2, 2, 2]\).

Problem 2
For

\[
A = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 3 \end{bmatrix},
\]

decide which of the following products are defined, and compute them:
(a) \(Au\), (b) \(Av\), (c) \(A^Tv\), (d) \(u^Tv\), (e) \(uv^T\).

Problem 3
From the pair of vectors in problem 1(b), construct an orthonormal set \(\{v_1, v_2, v_3\}\) such that two of them are multiples of the given pair.

Problem 4
Matrix representation of linear maps: Let \(f : \mathbb{R}^2 \to \mathbb{R}^2\) be a linear map and let \(e_1, e_2\) be a basis for \(\mathbb{R}^2\). Suppose

\[
f(e_1) = 5e_1 - 6e_2 \quad f(e_2) = e_2 + 3e_1.
\]

- Find the matrix \(A\) representing \(f\) with respect to the basis \(e_1, e_2\).
- If \(v \in \mathbb{R}^2\) is given by \(v = 2e_1 - e_2\). Find \(f(v)\) and check that the matrix \(A\) representing \(f\) correctly computes the coordinates of \(f(v)\) with respect to the basis \(e_1, e_2\).

Problem 5
Matrix multiplication is not commutative: that is, \(AB \neq BA\) in general. As an illustration, prove that a square \(2 \times 2\) matrix \(A\) satisfying \(AX =XA\) for every \(2 \times 2\) matrix \(X\) must be a multiple of the unit matrix \(I_2\). In other words, prove the following:

\[
A \in \mathbb{R}^{2 \times 2} \text{ and } AX =XA \text{ for all } X \in \mathbb{R}^{2 \times 2} \iff \exists \lambda \in \mathbb{R} \text{ such that } A = \lambda I_2.
\]
(This is true for square matrices of any size!) **Hint:** Compare $AX$ and $XA$ for matrices $X$ which have one entry equal to 1 and all others zero; for instance for

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$ 

**Note:** The formulation was changed slightly in order to clarify the problem.
Solution

Problem 2
Two vectors are orthogonal if their dot product is zero. The dot products are $1 \times (-1) + 2 \times 1 = 1$ for \((a)\), $2 \times (-3) + 5 \times 1 + 1 \times 1 = 0$ for \((b)\) and $3 \times 4 + 5 \times (-2) + 3 \times 2 + (-4) \times 2 = 0$ for \((c)\); so the pairs \((b)\) and \((c)\) are orthogonal, the pair \((a)\) is not.

Problem 3
(a) 
\[ Au = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}. \]

(b) \( Av \) is not defined: the column dimension of \( A \) is 3, while the dimension of \( v \) is only 2.

(c) 
\[ A^T v = \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ 23 \end{bmatrix}. \]

(d) \( u^T v \) is not defined: \( u \) and \( v \) do not have the same dimension.

(e) 
\[ uv^T = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} [2, 3] = \begin{bmatrix} 2 & 3 \\ 4 & 6 \\ -2 & -3 \end{bmatrix} \]
is the outer product of \( u \) and \( v \). It can also be understood as a product of two matrices with dimensions \( 3 \times 1 \) and \( 1 \times 2 \) respectively.

Problem 4
\([2, 5, 1] =: u_1 \) and \([-3, 1, 1] =: u_2 \) are already orthogonal. So the easiest thing to do is to find a third vector \( u_3 \) which is orthogonal to both of them, and then to normalize each of the three vectors, i.e. to divide each of them by its Euclidean norm, resulting in a vector of norm 1. (If \( u \neq 0 \), then its norm is nonzero, and \( v := u/\|u\| \) has Euclidean norm \( \|v\|_2 = 1 \).)

In three dimensions, the first step can be done by taking the vector product\(^{1}\) of \( u_1 \) and \( u_2 \), since the vector product is always orthogonal to both vectors from which it is formed. So
\[ u_3 = u_1 \times u_2 = \begin{bmatrix} 5 \times 1 - 1 \times 1 \\ 1 \times (-3) - 2 \times 1 \\ 2 \times 1 - 5 \times (-3) \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 17 \end{bmatrix} \]

\(^{1}\)The vector product of two vectors \( a = [a_1, a_2, a_3] \) and \( b = [b_1, b_2, b_3] \) is defined as the vector \( [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1] \).
Alternatively, let $u_3^T = [a, b, c]$. Then the orthogonality conditions for the set $\{u_1, u_2, u_3\}$ are

$$0 = u_3^T u_1 = 2a + 5b + c \quad \text{and} \quad 0 = u_3^T u_2 = -3a + b + c.$$ 

We can rearrange the second equation as $c = 3a - b$ and use this to eliminate $c$ from the first equation: $0 = 2a + 5b + 3a - b = 5a - 4b$, or $b = -(5/4)a$. We can now express $c$ in terms of $a$ alone as $c = 3a + (5/4)a = (17/4)a$. So we get $u_3^T = [a, -(5/4)a, (17/4)a] = a[1, -5/4, 17/4]$ and we can check that this vector is really orthogonal to both $u_1$ and $u_2$ for any choice of $a$.

For instance, for $a = 4$, we obtain $u_3^T = [4, -5, 17]$ as before.

The norms of the three vectors are

$$\|u_1\| = \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}, \quad \|u_2\| = \sqrt{(-3)^2 + 1^2} = \sqrt{11},$$

$$\|u_3\| = \sqrt{4^2 + (-5)^2 + 17^2} = \sqrt{330},$$

and so the resulting orthonormal set is

$$v_1^T = [2, 5, 1]/\sqrt{30}, \quad v_2^T = [-3, 1, 1]/\sqrt{11}, \quad v_3^T = [4, -5, 17]/\sqrt{330}.$$

By the way, the $v_i$ are only determined up to sign – orthogonality is a bilinear relation, and the negative of a vector has the same norm as the original vector. (So, for instance, if your third vector is $[-4, 5, -17]/\sqrt{330}$, that’s also correct.)

**Problem 5**

(i)

$$A = \begin{bmatrix} 5 & 3 \\ -6 & 1 \end{bmatrix}$$

(ii)

$$f(v) = 7e_1 - 13e_2 = (7, -13)^T.$$ 

$$Av = \begin{bmatrix} 5 & 3 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -13 \end{bmatrix}$$

**Problem 6**

*Part "⇒":* Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

Then for $X = E_{12}$,

$$AE_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}, \quad E_{12}A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix},$$

and so $AE_{12} = E_{12}A$ if and only if $a = d$ and $c = 0$. Similarly for $X = E_{21}$:

$$AE_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}, \quad E_{21}A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix},$$
and so $AE_{21} = E_{21}A$ if and only if $a = d$ and $b = 0$. So from the hypothesis that $AX =XA$ for all $X$, it follows that $a = d$ and $b = c = 0$, that is, $A$ must be of the form

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \lambda I_2 \text{ for } \lambda = a.$$ 

Part $\Longleftarrow$: The unit matrix satisfies $X I_m = I_m X = X$ for every matrix $X$ and in every dimension $m$; so if $A = \lambda I_2$ for $\lambda \in \mathbb{R}$, then $AX = (\lambda I_2)X = \lambda(I_2X) = \lambda X$ and $XA = X(\lambda I_2) = (X\lambda)I_2 = X\lambda = \lambda X$. 
