233 Computational Techniques

Problem Sheet for Tutorial 4

Problem 1
In the standard basis of $\mathbb{R}^2$, let the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ have matrix representation

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of $\boldsymbol{A}$. Hence find the basis with respect to which $\boldsymbol{A}$ is a diagonal matrix and find the matrix for this change of basis.

Problem 2
Find the singular value decomposition of the matrix.

$$\boldsymbol{A} = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}.$$

Problem 3
Show that, for any matrix $\boldsymbol{A} \in \mathbb{R}^{m\times m}$, if $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}^T\boldsymbol{A}$ with eigenvalue $\lambda \neq 0$, then $\boldsymbol{A}\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}\boldsymbol{A}^T$ with the same eigenvalue. (Why do we need $\lambda \neq 0$ here?) Show that if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$, then $\boldsymbol{A}\boldsymbol{v}_1$ and $\boldsymbol{A}\boldsymbol{v}_2$ are orthogonal. State and prove a similar result for eigenvectors of $\boldsymbol{A}\boldsymbol{A}^T$. Deduce that for any matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, the two matrices $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ have the same set of non-zero eigenvalues.

Problem 4

(i) Show that an orthogonal transformation preserves the angle between any two vectors.

(ii) Show that an orthogonal transformation preserves the ℓ_2 norm of a vector. Hence, use the SVD representation of any matrix $\boldsymbol{A}$ to show that the $\|\boldsymbol{A}\|_2 := \sup_{\|x\|_2=1} \|\boldsymbol{A}\boldsymbol{x}\|_2$ is equal to σ_1 the largest singular value of $\boldsymbol{A}$.

Problem 5
The purpose of this exercise is to show you an application of eigenvalues and eigenvectors to a topic which, at first glance, might seem totally unrelated: the *Fibonacci series*.
Recall (from the 1st year PPT classes) that the series is defined by $x_0 := 0$, $x_1 := 1$ and

$$x_{n+1} := x_n + x_{n-1} \qquad (1)$$

for $n \geq 1$. This formula is *recursive*, that is, in order to find x_n for higher values of n, you have to know (or compute) the values for smaller n.
In many situations recursive formulae are not good enough, for instance if one wants to know how x_n grows with n. In this exercise you can find a formula for x_n which is *non-recursive* in the sense that it gives x_n as a *function of the index* n rather than as a function of previously computed values. Eigenvalues and -vectors are a good tool for this.
Here is how to do it:
(a) Express (1) as a vector equation of the form

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \tag{2}$$

for some 2×2 matrix $\boldsymbol{A}$. This transforms the original series into a series of two-dimensional vectors.
(b) By recursive application of (2), express $[x_{n+1}, x_n]^T$ as a power of $\boldsymbol{A}$ times the "initial" vector (which one)?
(c) Now, find eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of $\boldsymbol{A}$. (Here the $\boldsymbol{u}_i$ need not be normalized.)
(d) Express the initial vector as a linear combination of the eigenvectors of $\boldsymbol{A}$.
(e) Use the results of *(b)–(d)* and the relation $\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$ to find the vector $[x_{n+1}, x_n]$—and hence x_n itself—as a function of n alone.
(f) Test your formula for $n = 0, \ldots, 4$.

Problem 6
Using the fact that linear independence of the columns (or rows) of a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is invariant under elementary row or column operations, as proved in the notes, show that the column rank and the row rank of a matrix is invariant under elementary row or column operations.
Hint: Consider the column rank of $\boldsymbol{A}$. (i) Elementary column operations: For the elementary operation of swapping two columns or multiplying one by a non-zero real number the assertion is clear. Consider the elementary operation of subtracting $\lambda\boldsymbol{a}_2$ from $\boldsymbol{a}_1$. Take a set S of maximally independent column vectors of the matrix and consider the four cases where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ belong or do not belong to this set. (ii) Elementary row operations: Consider any elementary row operation on the set S of a maximally independent column vectors of the matrix.