Quantum Computing Coursework 2
(solution to problem 2)

2. Suppose the states $|u\rangle$ for $u \in T$ are eigenstates of U with eigenvalue ϕ_u. The phase estimation algorithm (Section 9 in the lecture notes) maps the normalized state

$$|0\rangle(\sum_{u\in T} d_u|u\rangle)$$

to the state

$$\sum_{u\in T} d_u|\hat{\phi}_u\rangle|u\rangle,$$

where the state $|\hat{\phi}_u\rangle$ gives a good estimate of ϕ_u. Show that with t chosen as

$$t = s + \left\lceil \log(2 + \frac{1}{2\epsilon}) \right\rceil,$$

the probability of measuring ϕ_u accurate to s bits in the output of the phase estimation algorithm is at least $|d_u|^2(1-\epsilon)$.

Solution:

The easiest way to do this is to estimate the probability that the outcome of measurement m of the first register is close to ϕ_u for each $u \in T$. We use the above value for t. For each $u \in T$ we have:

$$|\hat{\phi}_u\rangle = \sum_{k=0}^{2^t-1} \beta_{u,k}|k\rangle,$$

for some coefficients $\beta_{u,k}$, $k = 0, \cdots, 2^t - 1$. The outcome of the phase estimation circuit is by linearity:

$$\sum_{u\in T}\sum_{k=0}^{2^t-1} d_u\beta_{u,k}|k\rangle|u\rangle,$$

The main point to notice is that in the linear superposition above all vectors are mutually orthogonal: $\langle u'|u\rangle\langle k'|k\rangle = 1$ if $u = u'$ and $k = k'$ and $\langle u'|u\rangle\langle k'|k\rangle = 0$ otherwise. Therefore, the probability that the measurement of the first register lies in a particular range will be given by the square of the absolute values of all the coefficients of the basis vectors $|k\rangle$ with k lying in that range (see the similar example on page 32 of the notes). Now fix $u_0 \in T$ and let $a_{u_0}/2^t$ be the best t bit approximation less than or equal to ϕ_{u_0}. Let

$$K(u_0, s) = \{k \,|\, 0 \leq k \leq 2^t - 1 \text{ with } k \text{ and } a_{u_0} \text{ equal up to } s \text{ bits}\}.$$

If we measure the first register the probablility that the outcome m is within s bits of of ϕ_{u_0} is given by

$$\sum_{u \in T} \sum_{k \in K(u_0,s)} |d_u|^2 |\beta_{u,k}|^2 \geq \sum_{k \in K(u_0,s)} |d_{u_0}|^2 |\beta_{u_0,k}|^2 \geq |d_{u_0}|^2 \sum_{k \in K(u_0,s)} |\beta_{u_0,k}|^2 \geq |d_{u_0}|^2 (1-\epsilon),$$

since, as computed in the notes, we have:

$$\sum_{k \in K(u_0,s)} |\beta_{u_0,k}|^2 \geq 1 - \epsilon.$$