An Algorithm to Estimate the Hausdorff Dimension of Self-Affine Sets*

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Abstract

We present an algorithm, based on Falconer's results in [4, 6], to effectively estimate the Hausdorff dimension of self-affine sets in $\mathbf{R}^n$: For a given finite set of contracting non-singular linear maps $T_1, \cdots, T_m$, we obtain a decreasing sequence of computable real numbers converging to Falconer's dimension d. For almost all $(a_1, \cdots, a_m) \in \mathbf{R}^{mn}$, the number d is the Hausdorff dimension of the unique non-empty compact subset F satisfying $F = \bigcup_{i=1}^m (T_i(F) + a_i)$. Similarly, we obtain an increasing sequence of computable real numbers converging to Falconer's lower bound d_- which is indeed a lower bound for the Hausdorff dimension of F if the sets $T_i(F)$ are disjoint for $1 \le i \le m$.

1 Introduction

An iterated function system (IFS) is given by a finite collection of contracting maps $S_1, \cdots, S_m$ on $\mathbf{R}^n$, i.e. with $|S_i(x) - S_i(y)| \le c_i|x-y|$ for some $0 < c_i < 1$, all $1 \le i \le m$ and all $x, y \in \mathbf{R}^n$. A theorem of Hutchinson [9] shows that there exists a unique non-empty compact set $F \subset \mathbf{R}^n$, called the invariant set or the attractor of $S_1, \cdots, S_m$, satisfying

$$F = \bigcup_{i=1}^{m} S_i(F).$$

*This work has been supported by EPSRC project "Foundational Structures for Computer Science".
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In fact, F which is usually a fractal can be obtained as follows (see for example [5]). Let B be any large enough closed ball with $S_i(B) \subset B$, for all $1 \leq i \leq m$. Then $F = \cap_{k=1}^{\infty} F_k$ with

$$F_k = \bigcup_{i_1,\cdots,i_k=1}^{m} S_{i_1} S_{i_2} \cdots S_{i_k}(B). \tag{1}$$

Attractors of IFSs present a large and rich class of fractals with various applications in science and technology [5].

The main tool to study fractals is provided by the various notions of dimension. The Hausdorff dimension is mathematically the most satisfactory notion of dimension whereas the box dimension is one which is most easily computed. The problem of estimating the Hausdorff or the box dimension of the attractor of an IFS has been an active area of research [3, 4, 6, 10]. We will recall the definitions of the Hausdorff and box dimensions below.

For any $F \subset \mathbf{R}^n$, $0 \leq s \leq n$ and $\delta > 0$, let

$$\mathcal{H}^s_\delta(F) = \inf\{\sum_{i \in I} |U_i|^s : F \subset \bigcup_{i \in I} U_i, 0 < |U_i| < \delta\}$$

where I is a finite or countable indexing set and $|U|$ denotes the diameter of the set $U \subset \mathbf{R}^n$. The number $\mathcal{H}^s_\delta(F)$ increases as δ decreases and the s-dimensional Hausdorff measure of F is defined as

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

which can be infinite. The Hausdorff dimension of F is

$$\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

The Hausdorff dimension has the desirable properties of dimension, but is in general very difficult to compute. The box dimension, on the other hand, has far less pleasing properties but is easier to calculate. It is defined as follows.

A δ-mesh cube in $\mathbf{R}^n$ is a cube of the form:

$$[m_1\delta, (m_1+1)\delta] \times [m_2\delta, (m_2+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$$

where $m_i \in \mathbf{Z}$ for $i = 1, \cdots, n$. Let $N_\delta(F)$ be the number of δ-mesh cubes which intersect a non-empty bounded set $F \subset \mathbf{R}^n$, then the upper and lower box dimensions are, respectively,

$$\overline{\dim_B}(F) = \limsup_{\delta \to 0} \log N_\delta(F) / -\log \delta$$

$$\underline{\dim}_B(F) = \liminf_{\delta\to 0} \log N_\delta(F)/-\log\delta.$$

In case the upper and lower box dimensions are equal then the common value is the box dimension, $\dim_B(F)$, of F.

In general we have

$$\dim_H(F) \le \underline{\dim}_B(F) \le \overline{\dim}_B(F).$$

The only general algorithmic technique for computing dimension is the box-counting algorithm for estimating the box dimension of a given fractal set F. See for example [1]. It consists of plotting $\log N_\delta(F)$ against $-\log\delta$ for a range of values of δ and then taking the line of best fit. The problem with this method is that we cannot say anything about the error in our estimate of $\dim_B F$. However, for fractals of iterated function systems, there are analytical results for computing the dimension.

2 The dimension of self-affine sets

A map $S:\mathbb{R}^n\to\mathbb{R}^n$ is called a *similarity* if there exists $c>0$ such that $|S(x)-S(y)| = c|x-y|$ for all $x,y\in\mathbb{R}^n$. A similarity transformation is the composition of a rotation, a reflection and a translation.

A *self-similar set* is the invariant set of an IFS consisting of contracting similarity maps, i.e. maps $S_1,\cdots,S_m$ with $|S_i(x)-S_i(y)| = c_i|x-y|$ for all $x,y\in\mathbf{R}^n$, where $1>c_i>0$, for $i=1,\cdots,m$. It is been well-known [11, 9] that the Hausdorff dimension and the box dimension of a self-similar set are equal and given by the unique solution of the equation,

$$c_1^s + c_2^s\cdots + c_m^s = 1,$$

provided that the open set condition holds, i.e. there exists an open set U such that $\bigcup_{i=1}^m S_i(U)\subseteq U$ with the union disjoint.

The problem of calculating the Hausdorff dimension of *self-affine sets* has been more challenging. Let $T_1,\cdots,T_m$ be contracting linear maps on $\mathbf{R}^n$. Then, for $a_i\in\mathbf{R}^n$ ($i=1,\ldots,m$), the invariant set of the IFS consisting of the maps $T_1+a_1, T_2+a_2,\cdots,T_m+a_m$ is called a self-affine set. For these sets, the Hausdorff and Box dimensions can be different [10, 7] and the Hausdorff dimension may not vary continuously even for a smoothly parametrized family of self-affine sets [4, 15, 7]. Several authors [10, 3, 2, 14], have calculated the Hausdorff dimension of particular classes of self-affine sets.

Falconer in [4, 6] developed a new technique to study the dimension of self-affine sets and showed a generic result which we will now explain without giving any proofs. For any non-singular, contracting, linear map T, the *singular values*

$$1 > \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \alpha_n > 0$$

of T are the positive square roots of the eigenvalues of T^tT, where T^t is the transpose of T; equivalently they are the lengths of the principle semi-axes of the ellipsoid $T(B)$ where B is the unit ball centered at the origin. Falconer's *singular value function* is defined by

$$\phi^s(T) = \begin{cases} \alpha_1\alpha_2 \cdots \alpha_{\lceil s\rceil}^{s-\lceil s\rceil+1} & s \leq n \\ (\alpha_1\alpha_2\cdots\alpha_n)^{s/n} & n < s, \end{cases}$$

where $\lceil s\rceil$ is the least integer greater than or equal to s. $\phi^s(T)$ is continuous and strictly decreasing in s and it satisfies the submultiplicative property $\phi^s(TU) \leq \phi^s(T)\phi^s(U)$. For each integer $k \geq 1$, let

$$\Phi_k(s) = \sum_{1\leq i_j\leq m} \phi^s(T_{i_1}T_{i_2}\ldots T_{i_k}). \tag{2}$$

Then $\Phi_k : \mathbf{R}^+ \to \mathbf{R}$ is continuous and strictly decreasing in s. Furthermore, the sequence $\Phi_k(s)$ is submultiplicative: $\Phi_{k+l}(s) \leq \Phi_k(s)\Phi_l(s)$ for all integers $k, l \geq 1$. It follows, from the properties of submultiplicative sequences, that the limit

$$\Phi(s) = \lim_{k\to\infty} (\Phi_k(s))^{1/k}$$

exists and is continuous and strictly decreasing in s. There is therefore a unique number d for which $\Phi(d) = 1$. By constructing a Hausdorff like measure on the product space $\{1, \ldots, m\}^\omega$ and using potential theory, Falconer proved the following theorem.

Theorem 2.1 *[4]*

(i) *For all* $(a_1, \ldots, a_m) \in \mathbf{R}^{mn}$, *we have:* $dim_H(F) \leq d$.

(ii) *For almost all* $(a_1, \ldots, a_m) \in \mathbf{R}^{mn}$ *with respect to the Lebesgue measure on* $\mathbf{R}^{mn}$ *we have:*

$$dim_H(F) = dim_B(F) = min(d, n).$$

In [6], Falconer obtained a lower bound d_- for $\dim_H(F)$ when the sets $T_i(F)$ are disjoint for $i = 1, \ldots, m$. The technique used is similar to the one above for the upper bound d. The singular values of the inverse map T^{-1} are given in terms of those of T, i.e. α_i as above, by:

$$\alpha_n^{-1} \geq \alpha_{n-1}^{-1} \geq \ldots \geq \alpha_1^{-1} > 1.$$

Hence, for $0 \leq s \leq n$,

$$\phi^s(T^{-1}) = \alpha_n^{-1} \ldots \alpha_{n-\lceil s\rceil+2}^{-1} \alpha_{n-\lceil s\rceil+1}^{-(s-\lceil s\rceil+1)} = \phi^{n-s}(T)/(\alpha_1 \ldots \alpha_n).$$

And, for $s > n$, $\phi^s(T^{-1}) = (\det T)^{n/s}$. Then, $(\phi^s(T^{-1}))^{-1}$ is continuous and strictly decreasing in s, and it satisfies the supermultiplicative property

$$(\phi^s((TU)^{-1}))^{-1} \geq (\phi^s(T^{-1}))^{-1}(\phi^s(U^{-1}))^{-1}.$$

For any integer $k \geq 1$, let

$$\Psi_k(s) = \sum_{1 \leq i_j \leq m} (\phi^s(T_{i_1} T_{i_2} \ldots T_{i_k})^{-1})^{-1}.$$

The function $\Psi_k : \mathbf{R}^+ \to \mathbf{R}$ is continuous and strictly decreasing. Furthermore, $\Psi_k(s)$ is a supermultiplicative sequence:

$$\Psi_{k+l}(s) \geq \Psi_k(s)\Psi_l(s),$$

and, therefore, the limit

$$\Psi(s) = \lim_{k\to\infty} (\Psi_k(s))^{1/k}$$

exists and is continuous and strictly decreasing in s. It then follows that there exists a unique number d_- such that $\Psi(d_-) = 1$.

Theorem 2.2 *[6] If the sets $T_i(F)$ are disjoint for $i = 1, \ldots, m$, then $d_- \leq \dim_H(F)$.*

Falconer, in the same paper, provided more results on the upper bound d. He obtained sufficient conditions, involving the projection of F into $n-1$ linear subspaces of $\mathbf{R}^n$, such that $\dim_H(F) = d$. The dimension d is also known to coincide with the box dimension for a large class of connected self-affine sets [3, 6] and also a class of totally disconnected self-affine maps [8].

Despite these analytic results, the problem of actually computing d or d_- is non-trivial. In fact, Falconer states in the concluding section of [4] that this computation is not easy as the rate of convergence of the submultiplicative sequence is not known.

3 Estimating d from above and d_- from below

In this section, we show how one can obtain a computable decreasing sequence which tends to d and a computable increasing sequence which tends to d_-. Although we still cannot say anything about the rate of convergence of these two sequences, they will give successive upper and lower bounds converging to d and d_- respectively.

3.1 The upper bound d

First we consider d. As we have seen the function $\Phi_k : \mathbf{R}^+ \to \mathbf{R}$ is continuous and strictly decreasing in s. Furthermore, $\Phi_k(0) = m^k > 1$ and, for $s \geq n$, $\Phi_k(s) \leq m^k \alpha_1^s$, which implies that for large s, $\Phi_k(s) < 1$. Therefore, there exists a unique real number s_k with $\Phi_k(s_k) = 1$.

We have no analytical formula which directly computes s_k but it is easy to see from Equation 2 that s_k is computable for all $k \geq 0$. In fact, the singular values of any $n \times n$ matrix T can be computed from the eigenvalues of T^tT. This is particularly simple in $\mathbf{R}^2$ where most applications occur. In fact, the singular values of any 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are given by

$$\sqrt{A + B \pm \sqrt{(A-B)^2 + C^2}},$$

where $A = (a^2 + c^2)/2$, $B = (b^2 + d^2)/2$ and $C = ab + cd$. Therefore, the singular values of $T_{i_1}T_{i_2}\ldots T_{i_k}$ for each term in $\Phi_k(s)$ in Equation (2) are computable. Using, for example, the bisection method, one can then compute the unique root s_k of $\Phi_k(s) = 1$. It is easy to see that this computation is exponential in k.

We will now construct a monotonic subsequence of s_k.

Lemma 3.1 *For any sequence of positive integers n_i, we have:*

$$s_{n_1} \geq s_{n_1 n_2} \geq s_{n_1 n_2 n_3} \geq \cdots.$$

Proof From $\Phi_{k+l}(s) \leq \Phi_k(s)\Phi_l(s)$ we get $\Phi_{k+l}(s) \leq 1$ for $s \geq \max(s_k, s_l)$. Since Φ_k is decreasing, it follows that

$$s_{k+l} \leq \max(s_k, s_l),$$

and more generally,

$$s_{k_1+k_2+\dots k_r} \leq \max(s_{k_1}, s_{k_2}, \dots, s_{k_r}).$$

Putting $k_i = q$ for $1 \leq i \leq r$, we get $s_{rq} \leq s_q$ for all integers $r, q \geq 1$. The result now follows immediately. □

In particular, we have $s_r \leq s_1$ for all integers $r \geq 1$.

Proposition 3.2 *For any sequence of positive integers $n_i \geq 2$ we have:* $\lim_{i\to\infty} s_{n_1 n_2 \dots n_i} = d$.

Proof For any integers $q, r \geq 0$, we get by the submultiplicative property:

$$(\Phi_{qr}(s))^{1/qr} \leq ((\Phi_r(s))^q)^{1/qr} = (\Phi_r(s))^{1/r}.$$

Hence, the sequence $f_i(s) = (\Phi_{n_1 n_2 \dots n_i}(s))^{1/n_1 n_2 \dots n_i}$ is decreasing and tends to

$$\Phi(s) = \lim_{k\to\infty} (\Phi_k(s))^{1/k}.$$

Let $t_i = s_{n_1 n_2 \dots n_i}$. Then, $f_i(t_i) = 1 = \Phi(d) \leq f_i(d)$. Thus, $d \leq t_i$ for all $i \geq 0$, i.e. $d \leq \lim_{i\to\infty} t_i$. On the other hand, for any $\epsilon > 0$, $\Phi(d+\epsilon) < 1$ since Φ is continuous and strictly decreasing. Therefore, for large enough i we have: $f_i(d+\epsilon) < 1 = f_i(t_i)$ which implies $t_i < d + \epsilon$ for large i. It follows that $\lim_{i\to\infty} t_i \leq d$. □

Corollary 3.3

$$\inf_{k\geq 1} s_k = \liminf_{k\to\infty} s_k = d.$$

In particular, for $n_i = q \geq 2$, the subsequence s_{q^n} decreases to d. For computation, we can use $q = 2$ to obtain a decreasing sequence of upper bounds for d. However, as the computation of s_k is exponential in k, it is clearly more efficient to use the sequence s'_k defined inductively by $s'_1 = s_1$ and $s'_{k+1} = \min(s'_k, s_{k+1})$. Then, s'_k is decreasing and $\lim_{k\to\infty} s'_k = d$. We do not know if the sequence s_k is in general decreasing and if in fact $\lim_{k\to\infty} s_k = d$.

3.2 The lower bound d_-

There is a similar analysis for d_-. Recall that $\Psi_k : \mathbf{R} \to \mathbf{R}$ is continuous and strictly decreasing. Furthermore, $\Psi_k(0) = m^k > 1$ and for large s, $\Psi_k(s) < 1$. Hence, there is a unique $u_k > 0$ with $\Psi_k(u_k) = 1$. From $\Psi_{k+l}(s) \geq \Psi_k(s)\Psi_l(s)$ we get $\Psi_{k+l}(s) \geq 1$ for $s \leq \max(s_k, s_l)$. It follows that

$$s_{k+l} \geq \max(s_k, s_l),$$

and more generally,

$$s_{k_1+k_2+\ldots k_r} \geq \max(s_{k_1}, s_{k_2}, \ldots, s_{k_r}).$$

Similar to the previous case, we get $u_1 \leq u_r$ for any $r \geq 1$, and for any sequence $n_i \geq 2$ we have:

$$u_{n_1} \leq u_{n_1 n_2} \leq u_{n_1 n_2 n_3} \leq \ldots.$$

and $\lim_{i\to\infty} u_{n_1 n_2 \ldots n_i} = d_-$. We therefore get:

$$\sup_{k\geq 1} u_k = \limsup_{k\to\infty} u_k = d_-.$$

In particular, for $n_i = q \geq 2$, the subsequence u_{q^n} increases to d_-. For computation, we can use $q = 2$ to obtain a decreasing sequence of upper bounds for d or, more efficiently, we can use the sequence u'_k defined inductively by $u'_1 = u_1$ and $u'_{k+1} = \max(u'_k, u_{k+1})$. Clearly, u'_k is increasing and $\lim_{k\to\infty} u'_k = d$.

Here again the elements of the sequence $\langle u_k \rangle_{k \geq o}$ are computable. As before we do not know if $\lim_{k\to\infty} u_k = d_-$. But in the next section we will see an example for which the sequence $\langle u_k \rangle_{k\geq 0}$ is not increasing.

Corollary 3.4 *If the sets $T_i(F)$ $(1 \leq i \leq m)$ are disjoint, then the sequence $\langle [u'_k, s'_k] \rangle_{k\geq 1}$ of intervals is nested and shrinking, and we have $[d_-, d] = \bigcap_{k\geq 1}[u_k, s_k] = \bigcap_{k\geq 1}[u'_k, s'_k]$.*

4 Two examples

A Miranda programme for computing s_k and u_k has been implemented in [12]. In this section we use the above results to estimate d and d_- for a self-similar set where an analytic formula for these quantities exist. Therefore, our estimates can be compared with the actual values.

For a few classes of self-affine sets, there are computable analytic formulas to calculate the Hausdorff dimension [2, 14, 13]. However, in nearly all of them the affine maps induce multiplicative sequences $\langle \Phi_k \rangle_{k\geq 0}$ and $\langle \Psi_k \rangle_{k\geq 0}$, with $\Phi_k(s) = (\Phi_1(s))^k$ and $\Psi_k(s) = (\Psi_1(s))^k$.

This implies that the sequences $\langle s_k \rangle_{k \geq 0}$ and $\langle u_k \rangle_{k \geq 0}$ are constant with values d and d_- respectively.

In order to compare our results with actual values, we need to investigate a non-multiplicative set of maps. Paulson [13] has examined the following self-affine set called the "carpet". It is generated by an IFS with four affine maps S_1, S_2, S_3, S_4 given by

$$S_1(x) = \begin{pmatrix} c & 0 \\ 0 & 1-c \end{pmatrix} (x) + \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S_2(x) = \begin{pmatrix} c & 0 \\ 0 & 1-c \end{pmatrix} (x) + \begin{pmatrix} 1-c \\ c \end{pmatrix},$$

$$S_3(x) = \begin{pmatrix} 1-c & 0 \\ 0 & c \end{pmatrix} (x) + \begin{pmatrix} c \\ 0 \end{pmatrix},$$

$$S_4(x) = \begin{pmatrix} 1-c & 0 \\ 0 & c \end{pmatrix} (x) + \begin{pmatrix} 0 \\ 1-c \end{pmatrix},$$

where $0 < c < 1$ is a constant. He computed the functions $\Psi_k(s)$ analytically for these matrices and obtained a simple formula for the limiting case: the lower bound d_- satisfies the equation

$$4(\sqrt{c - c^2})^{d_-} = 1.$$

In the case where $c = 1/3$ he found the upper bound

$$d = 1.845\cdots$$

and the lower bound

$$d_- = \frac{4 \log 2}{\log(9/2)} = 1.843\cdots$$

Using our programme, run on a PC, we obtain the following values to three decimal places:

k	u_k	s_k
1	1.709	1.892
2	1.798	1.873
3	1.797	1.872
5	1.811	1.866
6	1.818	1.863

As we see, the rate of convergence is not fast. One needs more computer power to reach the actual values to three decimal places. Note that in this example the sequence $\langle u_k \rangle_{k \geq 0}$ is not increasing.

For Barnsley's fern [1], generated by the maps

$$S_1(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix} (x) + \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S_2(x) = \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix} (x) + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix},$$

$$S_3(x) = \begin{pmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} (x) + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix},$$

$$S_4(x) = \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} (x) + \begin{pmatrix} 0 \\ 0.44 \end{pmatrix},$$

we have computed $u_1 = 1.822$, $s_1 = 1.843$, $u_4 = 1.824$ and $s_4 = 1.841$. We do not know if the disjointness condition holds here. Therefore, all that can be said is that the Hausdorff dimension is less that s_k for all $k \geq 1$; in particular it is less than 1.841.

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