Dynamical systems, Measures and Fractals via Domain Theory*

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Abstract

We introduce domain theory in dynamical systems, iterated function systems (fractals) and measure theory. For a discrete dynamical system given by the action of a continuous map $f : X \to X$ on a metric space X, we study the extended dynamical systems (VX, Vf), (UX, Uf) and (LX, Lf) where V, U and L are respectively the Vietoris hyperspace, the upper hyperspace and the lower hyperspace functors. We show that if (X, f) is chaotic, then so is (UX, Uf). When X is locally compact UX, is a continuous bounded complete dcpo. If X is second countable as well, then UX will be ω-continuous and can be given an effective structure. We show how strange attractors, attractors of iterated function systems (fractals) and Julia sets are obtained effectively as fixed points of deterministic functions on UX or fixed points of non-deterministic functions on CUX where C is the convex (Plotkin) power domain. We also show that the set, $M(X)$, of finite Borel measures on X can be embedded in PUX, where P is the probabilistic power domain. This provides an effective framework for measure theory. We then prove that the invariant measure of an hyperbolic iterated function system with probabilities can be obtained as the unique fixed point of an associated continuous function on PUX.

1 Introduction

Domain theory, introduced by Dana Scott [Scott, 1970] as a mathematical theory of semantics of programming languages, has been extensively studied in theoretical computer science in the past two decades. In this paper, we will show that some mainstream branches of mathematics have natural domain-theoretic computational models. We concentrate here on dynamical systems, iterated function systems and measure theory. It will be shown that the domain-theoretic construction of these mathematical disciplines results in the following.

- It provides them with effective structures which can make these subjects constructive.
- It gives new, simple and constructive proofs for the existence and uniqueness of certain fixed points.

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- It gives rise to new algorithms in these fields.
- It brings about new structures for research in these areas.

We will give an overview of our work in the introduction which is followed by our results in the later sections.

A *discrete dynamical system* (X, f) is given by the action of a continuous function $f : X \to X$ on a topological space X, which is usually a metric space (X, d). The (forward) *orbit* of a point $x \in X$ is the sequence $\langle f^n(x)\rangle_{n\geq 0}$. One is interested in the behaviour of orbits of the system. In the interesting cases, the system has one or several indecomposable basic sets in X, on each of which f is invariant and has non-trivial dynamics. An example of such a set is an *attractor*, which attracts all nearby orbits. These can be fixed or periodic points or they can be *strange* or *chaotic* attractors, which have very complex structure. Attractors are observed in physical systems and, therefore, their study is of fundamental importance.

Continuous dynamical systems, e.g. differential equations in $\mathbb{R}^n$, can also be studied via discrete systems. In fact, any *continuous dynamical system*, given by a differentiable vector field on a smooth manifold, gives rise to a continuous flow on the manifold and the *time-one map* of the flow induces a discrete system on the manifold. One can then investigate the behaviour of the continuous system by studying the induced discrete system. Dynamical systems are used to model dynamic behaviour in all branches of science. In fact, they are remarkable for the diversity of their application areas. For an example of this wide ranging application, see [Edalat and Zeeman, 1992]. For an introduction to dynamical systems, see for example [Devaney, 1989; Szlenk, 1984].

In this paper, we consider discrete dynamical systems from a computational point of view. Since the invariant sets, and hence the attractors, of a discrete system (X, f), with X Hausdorff, are closed and, in the interesting cases, compact, it is natural in computing these sets to study the associated dynamical systems (VX, Vf), (UX, Uf) and (LX, Lf) on the three hyperspaces VX, UX and LX which we will now define.

- VX, the *Vietoris space*, is the set of all non-empty compact subsets of X with the *Vietoris (finite) topology* which has a subbase given by the collections of the form $\Box a = \{C \in VX \mid C \subseteq a\}$ and $\Diamond a = \{C \in VX \mid C \cap a \neq \emptyset\}$, where $a \in \Omega X$ is an open set of X. This topology is Hausdorff and, when X is a metric space, it is equivalent to the one induced by the *Hausdorff metric* d_h defined by $d_h(A, B) = \max(d(A, B), d(B, A))$, where for compact sets C and D, $d(C, D)$ is the infimum of positive numbers δ such that C is contained in the δ-*parallel body* of D defined as $D_\delta = \{x \in X \mid \exists y \in D.\, d(x, y) \leq \delta\}$. See [Smyth, 1992, page 737] for the proofs of these statements.
- UX, the *upper space*, is the set of all non-empty compact subsets of X with the base of *upper topology* given by collections $\Box a = \{C \in UX \mid C \subseteq a\}$ $(a \in \Omega X)$. This topology is T_0 and, when X is a metric space, is equivalent to the topology induced by the quasi-metric d_u defined by $d_u(A, B) = d(A, B)$. The specialisation ordering $\sqsubseteq_u$ of UX is reverse inclusion, i.e.

  $$A \sqsubseteq_u B \stackrel{\text{def}}{\iff} \forall a \in \Omega X[A \subseteq a \Rightarrow B \subseteq a] \iff A \supseteq B.$$

  $(UX, \supseteq)$ is a dcpo, in which the least upper bound (lub) of a directed set of compact subsets is their intersection. When X is locally compact, $(UX, \supseteq)$ is a continuous dcpo. See Section 3 for proofs.

- LX, the *lower space*, is the set of all closed subsets of X with the subbase of *lower topology* given by the collections $\Diamond a = \{C \in LX \mid C \cap a \neq \emptyset\}$, $(a \in \Omega X)$. This topology is again T_0 and, when X is a compact metric space, is equivalent to the topology induced by the quasi-metric d_l defined by $d_l(A, B) = d(B, A)$. The specialisation ordering $\sqsubseteq_l$ of LX is subset inclusion, i.e.

$$A \sqsubseteq_l B \stackrel{\text{def}}{\Longleftrightarrow} \forall a \in \Omega X [A \cap a \neq \emptyset \Rightarrow B \cap a \neq \emptyset] \iff A \subseteq B.$$

  $(LX, \sqsubseteq_l)$ is a complete lattice, where the lub of a set of closed sets is the closure of their union. See Section 4 for proofs.

The maps Vf, Uf and Lf are induced by f on the corresponding spaces, i.e. $Vf(A) = f(A)$, $Uf(A) = f(A)$ and $Lf(A) = \overline{f(A)}$, where $f(A)$ is the image of $A \subseteq X$ under $f : X \to X$ and $\overline{B}$ is the closure of B. See for example [Smyth, 1983; Smyth, 1992] for the general definition of these spaces.

The work of Hutchinson, and later that of Barnsley, on *iterated function systems* [Hutchinson, 1981; Barnsley, 1988] has already established one important application of dynamical systems on the Vietoris space VX. A nonstandard treatment can also be found in [Wicks, 1991]. We briefly review Hutchinson's and Barnsley's work here.

Definition 1.1 An *iterated function system (IFS)*, $\{X; f_1, f_2, \ldots, f_N\}$, is given by a finite set of continuous maps $f_i : X \to X$ $(i = 1, \ldots, N)$ on a complete metric space (X, d). The IFS is *hyperbolic* if the maps f_i are all contracting. □

An IFS induces a function $F : VX \to VX$ on the complete space VX, which is defined by $F(A) = f_1(A) \cup f_2(A) \cup \ldots \cup f_N(A)$. If the IFS is hyperbolic, then F will be contracting with contractivity factor $s = \max_i s_i$ where s_i is the contractivity factor of f_i $(1 \leq i \leq N)$. Therefore, by the contracting mapping theorem, F has a unique fixed point in VX, which is called the *attractor* of the IFS. In the interesting cases, there are $N \geq 2$ contracting affine maps on $\mathbb{R}^n$. Then the attractor is usually a *fractal*, i.e. it has fine, complicated and non-smooth local structure, some form of self-similarity and, usually, a non-integral *Hausdorff dimension*.

Conversely, given any image regarded as a compact set in the plane, one can use a self-tiling of the image and Barnsley's collage theorem to find an IFS with contracting affine transformations, whose attractor approximates the image.

There is also a probabilistic version of the theory that produces coloured images.

Definition 1.2 An *IFS with probabilities* $\{X; f_1, \ldots, f_N; p_1, \ldots, p_N\}$ is an IFS $\{X; f_1, f_2, \ldots, f_N\}$ such that each f_i $(1 \leq i \leq N)$ is assigned a weight $0 < p_i < 1$, with

$$\sum_{i=1}^{N} p_i = 1. \qquad \square$$

Given an IFS with probabilities, the *Markov operator* $T : M^1(X) \to M^1(X)$ is defined on the set, $M^1(X)$, of normalised Borel measures of X, by

$$T(\mu)(B) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(B))$$

for any Borel subset $B \subseteq X$. When X is compact, the *Hutchinson metric* d_H can be defined on $M^1(X)$ as follows [Hutchinson, 1981]:

$$d_H(\mu, \nu) = \sup\{\int_X f d\mu - \int_X f d\nu \mid f : X \to \mathbb{R}, \ |f(x) - f(y)| \leq d(x, y), \forall x, y \in X\}$$

Then, using some Banach space theory, including Alaoglu's theorem, it is shown that the weak* topology and the metric topology on $M^1(X)$ coincide, thereby making it a compact metric space. If the IFS is hyperbolic, T will be a contracting map. The unique fixed point of T then defines a probability distribution on X whose support is the attractor of $\{X; f_1, \ldots, f_N\}$. This distribution gives different point densities in different regions of the attractor, and using a colouring scheme, one can colour the attractor accordingly. These results have found important applications in computer graphics and image compression, and also in learning automata [Barnsley and Sloan, 1988; Barnsley *et al.*, 1988; Bressloff and Stark, 1991].

Our thesis is that in studying any dynamical system (X, f), the induced dynamical systems (VX, Vf) (the *Vietoris dynamical system*), (UX, Uf) (*the upper dynamical system*) and (LX, Lf) (*the lower dynamical system*) on the three hyperspaces are of considerable interest. The invariant sets and in particular the attractors of (X, f) are closed and in the interesting cases compact and are, therefore, fixed points of $Vf : VX \to VX$ and $Uf : UX \to UX$. In this paper, we will in fact only study compact attractors.

There are two canonical ways of computing the invariant sets and in particular the attractors of a dynamical system. The first, and in fact the more fundamental one, is to find a decreasing sequence of compact sets which shrink down to the invariant set. The smaller the compact set in the sequence, the better it approximates the invariant set. This ordering corresponds to the specialisation ordering of the upper space UX: For non-empty compact sets A and B we have $A \sqsubseteq_u B$ iff $A \supseteq B$.

The first work in this area was by Hayashi [Hayashi, 1985]. For an IFS $\{f_1, f_2, \ldots, f_N\}$ on a compact Hausdorff space X, Hayashi considered the cpo $(UX, \supseteq)$ and observed that the map $F : UX \to UX$, where $F(A) = f_1(A) \cup \ldots \cup f_N(A)$, is continuous and has,, therefore, a least fixed point. He also showed that if the IFS is hyperbolic, then the least fixed point of F is its unique fixed point, i.e. the attractor of the IFS. Furthermore, he noted that some Julia sets can be obtained in this way.

In this paper, we show that a non-deterministic computation of the attractor of an hyperbolic IFS and of some Julia sets can also be made: The attractor or the Julia set is obtained as the unique fixed point of a continuous map on CUX, where C is the convex (Plotkin) powerdomain functor. Many other invariant sets of dynamical systems can be computed by this method. We also extend the notion of chaos to the case of dynamics on quasi-metric spaces, and show that if (X, f) is chaotic, then (UX, Uf) is also chaotic.

We propose, in particular, that the upper space construction provides a suitable framework for an effective theory of dynamical systems. Points of X are, in effect, maximal elements of UX. When X is locally compact, $(UX, \supseteq)$ is a continuous dcpo, and a point of X, considered as a singleton, can be identified with a shrinking sequence of compact sets. If X is also second countable, i.e. if it has a countable base, then $(UX, \supseteq)$ is ω-continuous and an effective theory of dynamical systems can be constructed.

The other canonical method for computing the invariant set is to obtain an increasing sequence of compact sets whose union has the invariant set as its closure. In this case, the bigger the compact set in the sequence, the better the approximation to the invariant set. This ordering corresponds to the specialisation ordering of the lower space LX, i.e. for non-empty compact sets A and B, we have $A \sqsubseteq_l B$ iff $A \subseteq B$. This method can be used for example to generate the attractor of an IFS $\{X; f_1, \ldots, f_N\}$ starting with, say, the set of fixed points of the maps f_i.

Finally, we develop an exciting link between measure theory and domain theory, which can lead to a constructive measure theory. We will use it in this paper to obtain, constructively, Barnsley's invariant measure for an IFS with probabilities.

The link is established by "well-behaved" measures on "well-behaved" topological spaces.

Definition 1.3 [Rudin, 1966, page 50] A positive Borel measure μ on a locally compact Hausdorff space is *regular* if for all Borel subsets B of X we have:

$$\mu(B) = \inf\{\mu(O) \mid B \subseteq O, O \text{ open}\} = \sup\{\mu(K) \mid B \supseteq K, K \text{ compact}\}. \qquad \square$$

When a measure is regular, the measure of any Borel set is determined by the topologically important open sets or alternatively the compact sets. Any measure which is not regular is considered to be pathological. We want to work with spaces on which Borel measures are regular. If X is locally compact and second countable (i.e. countably based), then every open set will be *σ-compact* [Lang, 1969, page 344], i.e. it is the countable union of compact sets. It then follows from a well-known theorem in measure theory that any measure on X which is finite on compact sets is in fact regular [Rudin, 1966, page 50]. We conclude that locally compact second countable Hausdorff spaces provide a suitable setting for well-behaved measures.

But we have already seen that a locally compact second countable Hausdorff space also has the interesting property that its upper space is an ω-continuous dcpo. This gives us a link with the theory of *valuations*.

Definition 1.4 [Birkhoff, 1967; Saheb-Djahromi, 1980; Lawson, 1982; Jones, 1989]. A *valuation* on a topological space Y is a map $\nu : \Omega Y \to [0, \infty)$ which satisfies:

(i) $\nu(a) + \nu(b) = \nu(a \cup b) + \nu(a \cap b)$,

(ii) $\nu(\emptyset) = 0$, and

(iii) $a \subseteq b \Rightarrow \nu(a) \leq \nu(b)$.

A *continuous* valuation [Lawson, 1982; Jones and Plotkin, 1989; Jones, 1989] is a valuation such that whenever $A \subseteq \Omega(Y)$ is a directed set (wrt $\subseteq$) of open sets of Y, then

$$\nu\Big(\bigcup_{O \in A} O\Big) = \sup\nolimits_{O \in A} \nu(O). \qquad \square$$

For any $b \in Y$, the *point valuation* based at b is the valuation $\eta_b : \Omega(Y) \to [0, \infty)$ defined by

$$\eta_b(O) = \begin{cases} 1 & \text{if } b \in O \\ 0 & \text{otherwise.} \end{cases}$$

Any finite linear combination

$$\sum_{i=1}^{n} r_i \eta_{b_i}$$

of point valuations η_{b_i} with constant coefficients $r_i \in [0, \infty)$, $(1 \leq i \leq n)$ is also a valuation on Y.

Definition 1.5 The *probabilistic power domain*, PY, of a topological space Y consists of the set of continuous valuations ν on Y with $\nu(Y) \leq 1$ and is ordered as follows:

$$\mu \sqsubseteq \nu \text{ iff for all open sets } O \text{ of } Y, \ \mu(O) \leq \nu(O). \qquad \square$$

The partial order $(PY, \sqsubseteq)$ is a dcpo with bottom in which the lub of a directed set $\langle \mu_i \rangle_{i \in I}$ is given by $\bigsqcup_i \mu_i = \nu$ where for $O \in \Omega(Y)$ we have

$$\nu(O) = \sup_{i \in I} \mu_i(O).$$

Moreover, if Y is an (ω-)continuous dcpo, then PY is also an ω-continuous dcpo and has a basis consisting of linear combinations of point valuations [Jones and Plotkin, 1989; Jones, 1989].

Let $M(X)$ be the set of all positive Borel measures μ with $\mu(X) \leq 1$ on the locally compact second countable Hausdorff space X. We will show in this paper that $M(X)$ can be identified with PX and can be embedded in PUX. The image of the embedding consists of all valuations on UX which are supported in the set of its maximal elements, i.e. the singletons of X. Since PUX is an ω-continuous dcpo, we obtain a domain-theoretic computational model for the set of Borel measures on X.

As an application, we consider any hyperbolic IFS with probabilities $\{X; f_1, \ldots, f_N; p_1, \ldots, p_N\}$. This induces a continuous map on the subdcpo of normalised valuations P^1UX. The unique fixed point of this map has support in X and gives us the invariant measure of the IFS.

2 Chaos and the Vietoris space

In this section, let $f : X \to X$ be a continuous map on a metric space (X, d). We recall some definitions, say from [Devaney, 1989]. In this paper, a *neighbourhood* of a point or of a subset of a topological space is any open set containing that point or subset.

Definition 2.1 $f : X \to X$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in X$ and any neighbourhood N of x, there exists $y \in N$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$. $\square$

Therefore, $f : X \to X$ has sensitive dependence on initial conditions if, for some $\delta > 0$, there are points arbitrarily close to any point $x \in X$ which eventually separate from x by at least δ. This condition gives rise to unpredictable dynamics. In fact, small errors either in measuring data or in computation due to round-off may become magnified under iteration, so that the result of a numerical computation of an orbit, no matter how accurate, may in the long term bear no resemblance to the real orbit.

Definition 2.2 $f : X \to X$ is *topologically transitive* or *mixing* if for any pair of non-empty open sets $a, b \subseteq X$ there exists $n > 0$ such that $f^n(a) \cap b \neq \emptyset$. $\square$

That is to say, a topologically transitive map has points which move under iteration from one arbitrarily small neighbourhood to any other. This means that it is not possible to decompose the dynamical system to two disjoint open sets which are invariant under the map. Note that the above definition does not use the metric and in fact one can make the same definition for any topological space X. Recall that $x \in X$ is a *periodic* point of $f : X \to X$ if $f^n(x) = x$ for some $n \geq 1$. There are many possible definitions of chaos; here we adopt Devaney's definition, which is quite widely accepted.

Definition 2.3 $f : X \to X$ is *chaotic* on X if

(i) f has sensitive dependence on initial conditions,

(ii) f is topologically transitive, and

(iii) the periodic points of f are dense in X. $\Box$

Intuitively, a map is chaotic if, in Devaney's words, it combines (i) *unpredictability* and (ii) *indecomposability* with an element of (iii) *regularity*.

It has recently been shown that (i) follows from (ii) and (iii) [Banks *et al.*, 1992]. However, we use the above classical definition, as in the next section we will extend it to the notion of chaotic dynamics on quasi-metric spaces.

Example 2.4 Let Σ be a finite set. Let $\Sigma^{\mathbb{N}}$ be the Cantor space, i.e. the set of infinite sequences $a = a_0 a_1 a_2 \ldots$ with $a_i \in \Sigma$ for all $i \geq 0$, with the product topology. We can define a metric d on Σ by

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ \frac{1}{2^n} & \text{otherwise} \end{cases}$$

where $n \geq 0$ is the least integer such that $a_n \neq b_n$. Consider the *shift* map $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ defined by $\sigma(a_0 a_1 a_2 a_3 \ldots) = a_1 a_2 a_3 \ldots$. Thus, σ removes the first element and shifts all other elements one place to the left. In other words, $\sigma(a)$ is the *tail* of the infinite list a. Then it can be shown that σ is chaotic on $\Sigma^{\mathbb{N}}$ (see [Devaney, 1989]). $\Box$

The shift map has basic importance in dynamical systems as it models many chaotic systems. In computer science, one can view a Turing machine as a dynamical system. In fact, Christopher Moore [Moore, 1991] has shown that Turing machines are equivalent to a class of *generalised shifts*.

Recall that a *relatively compact* subset of a topological space is one whose closure is compact.

Definition 2.5 Given a dynamical system (X, f), a closed region $C \subseteq X$ is said to be a *trapping region* for f if $f(C)$ is contained in the interior of C. A non-empty subset $\Lambda \subseteq X$ is an *attractor* for f if there is a relatively compact neighbourhood N of Λ for which the closure, $\overline{N}$, of N is a trapping region for f and $\Lambda = \bigcap_{i \geq 0} f^i(\overline{N})$. We say Λ is a *strange* attractor if, furthermore, f is chaotic on Λ. $\Box$

Note that with our definition, attractors are always compact; this is in fact the case in all physical systems.

Now consider the dynamical system (VX, Vf) as defined in the introduction. Note that a basic open set of VX is of the form $\Box a \wedge \Diamond a_1 \wedge \ldots \wedge \Diamond a_n$, where $a, a_i \in \Omega X$ $(1 \leq i \leq n)$ and we have

$$(Vf)^{-1}(\Box a \wedge \Diamond a_1 \wedge \ldots \wedge \Diamond a_n) = \Box f^{-1} a \wedge \Diamond f^{-1} a_1 \wedge \ldots \wedge \Diamond f^{-1} a_n$$

and therefore $Vf : VX \to VX$ is in fact a continuous map.

Lemma 2.6 *Let X be a Hausdorff space and $f : X \to X$ a continuous map. If $\langle C_i \rangle_{i \in I}$ is a family of non-empty compact subsets of X which is directed with respect to reverse inclusion, then $\bigcap_{i \in I} C_i$ is non-empty and compact, and we have $f(\bigcap_{i \in I} C_i) = \bigcap_{i \in I} f(C_i)$.*

Proof The set $\bigcap_i C_i$, being the intersection of closed sets, is closed and hence, as a closed subset of any one of the compact sets C_i, is compact. It also follows, by the finite intersection property applied to any one of the compact sets C_i, that $\bigcap_i C_i$ is non-empty. Clearly, $f(\bigcap_i C_i) \subseteq \bigcap_i f(C_i)$. Suppose $y \in \bigcap_i f(C_i)$. For each $i \in I$, there exists $x_i \in C_i$ such that $f(x_i) = y$. Fix $i_0 \in I$. We can assume without loss of generality that $C_{i_0} \supseteq C_i$ for all $i \in I$. Then the filterbase determined by the net $\langle x_i \rangle_{i \in I}$ has an accumulation point x in the compact set C_{i_0}. Therefore, $x \in \bigcap_i C_i$ and, by the continuity of f, we have $f(x) = y$. $\square$

Corollary 2.7 *An attractor of* (X, f) *is a fixed point of* (VX, Vf). $\square$

Example 2.8 Consider the solid two dimensional torus $T = S \times D$, where S is the unit circle and D is the unit disc in the plane. The smooth embedding $f : T \to T$ is defined as follows. Regard T as a rubber ring, and stretch, twist and fold it to fit inside itself as in the Figure 1.

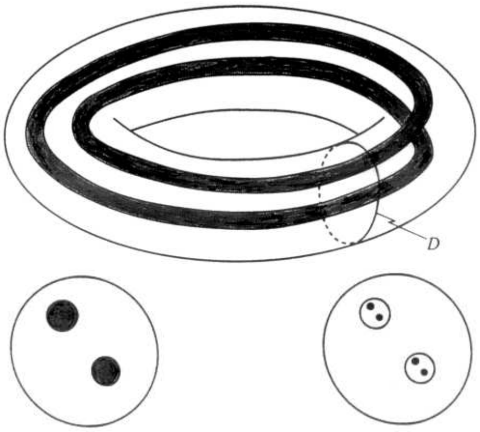

Figure 1. Top: $f(T)$ sitting in T. Bottom left: $f(T) \cap D$, Bottom right: $f^2(T) \cap D$

Repeated application of f produces longer and longer tori, wrapped around T increasingly many times. It is easy to see that

$$D \cap (\bigcap_{i \geq 0} f^i(T)) = D \cap (\bigcap_{i \geq 0} (Vf)^i(T))$$

is a Cantor set. The attractor $\Lambda = \bigcap_{i \geq 0} (Vf)^i(T) = \lim_{i \to \infty} (Vf)^i(T)$ is called the *solenoid* and is a fixed point of $Vf : VT \to VT$. It can be shown that the dynamics of f on Λ is chaotic (see for example [Devaney, 1989]), in particular the periodic points of f are dense in Λ. Note that $(Vf)^i(T)$, $i \geq 0$, is a shrinking sequence of compact neighbourhoods of the attractor Λ, which converges to the latter. $\square$

The strange attractors of the *Hénon* map and the *Plykin* map [Devaney, 1989] are obtained in a similar way.

3 The upper space as a computational model

The upper space, as defined in the introduction, provides a suitable computational model for dynamical systems. The definition and the examples of attractors in the previous section motivate us to study the dynamics on this space.

3.1 Basic Properties of the upper space

We need the results in the following three propositions for which we have not been able to find a reference in the literature. (See however [Smyth, 1992, page 751, exercise 15].) It is therefore justified to give a complete proof of these results for the case of Hausdorff spaces which we need in this paper.

Proposition 3.1 *Let X be a Hausdorff space.*

(i) *The specialisation ordering $\sqsubseteq_u$ of UX is reverse inclusion.*

(ii) *$(UX, \supseteq)$ is a bounded complete dcpo with a Scott continuous meet operation.*

(iii) *The Scott topology of $(UX, \supseteq)$ refines the upper topology.*

Proof (i) Clearly, if $A \supseteq B$ for $A, B \in UX$, then $A \subseteq a$ implies $B \subseteq a$ for all $a \in \Omega X$. On the other hand if $A \not\supseteq B$, then there exists $x \in B - A$. Since A is compact, we can find an open set $a \in \Omega X$ with $A \subseteq a$ and $x \notin a$, i.e. $B \not\subseteq a$. This shows that the specialisation ordering of UX is reverse inclusion.

(ii) If $\langle C_i \rangle_{i \in I}$ is a directed family, wrt reverse inclusion, of non-empty compact sets, then by Lemma 2.6, $\bigcap_i C_i$ is non-empty and compact, and hence clearly the lub of the family. Therefore, $(UX, \supseteq)$ is a dcpo. If we have any set of elements $A_i \in UX$ $(i \in I)$ with $A_i \supseteq C$ for some $C \in UX$, then $\bigcap_{i \in I} A_i$ contains C and is clearly the least upper bound of $\{A_i \mid i \in I\}$, i.e. $(UX, \supseteq)$ is bounded complete. Clearly $A \cup B$ is the meet of $A, B \in UX$. The continuity of the meet operation follows from the equality $A \cup (\bigcap_{i \in I} B_i) = \bigcap_{i \in I} (A \cup B_i)$ which holds for any subset A and any family of subsets $\langle B_i \rangle_{i \in I}$ of X.

(iii) Let $\Box a$ be a basic open set and suppose $\langle C_i \rangle_{i \in I}$ is a directed family of non-empty compact sets with $\bigcap_i C_i \subseteq a$. Then $(\bigcap_i C_i) \cap a' = \emptyset$, where a' is the complement of a. Hence, $\bigcap_i (C_i \cap a') = \emptyset$, where $\langle C_i \cap a' \rangle_{i \in I}$ is a directed family of compact sets. It follows that, for some $i \in I$, we have $C_i \cap a' = \emptyset$ i.e. $C_i \subseteq a$. Therefore, $\Box a$ is Scott open as it is clearly upward closed. $\Box$

Note that if $f : X \to Y$ is a continuous map of Hausdorff spaces X and Y, then by Lemma 2.6 the map $Uf : (UX, \supseteq) \to (UY, \supseteq)$ is Scott continuous.

Recall that a quasi-metric r on a space X is a non-negative real-valued distance function which satisfies:

(i) $r(x, x) = 0$.

(ii) $r(x, z) \leq r(x, y) + r(y, z)$.

(iii) $r(x, y) = 0 \;\&\; r(y, x) = 0 \;\Rightarrow\; x = y$.

That is, a quasi-metric unlike a metric is not necessarily symmetric. We define the open ball of radius ϵ centred at x to be the set $B_\epsilon(x)\{y \mid r(y, x) < \epsilon\}$. The collection of all such open balls defines the base of a topology on X, which we call the *quasi-metric topology* induced by r. Note that in general we obtain a different topology if we use $r(y, x)$ instead of $r(x, y)$ in the definition of an open ball above. However, we fix ideas by adopting the above definition of an open ball. When X is a metric space, the quasi-metric d_u on UX captures the upper topology as follows.

Proposition 3.2 *Let X be a metric space.*

(i) *The upper topology of UX coincides with the quasi-metric topology induced by d_u.*

(ii) *The lub A of an ω-chain $\langle A_i \rangle_{i\geq 0}$ in $(UX, \supseteq)$ satisfies $d_u(A_i, A) \to 0$ as $i \to \infty$.*

Proof (i) Let $A \in \Box a$ for some $a \in \Omega X$. Then $A \subseteq a$ and, by compactness of A, there exists $\epsilon > 0$ such that $A_\epsilon \subseteq a$, where A_ϵ is the ϵ-parallel body of A. Then we have

$$d_u(C, A) < \epsilon \Rightarrow d(C, A) < \epsilon \Rightarrow C \subseteq A_\epsilon \Rightarrow C \subseteq a \Rightarrow C \in \Box a$$

i.e. $\Box a$ contains the quasi-metric ball of radius ϵ centred at A. Conversely, we have

$$d_u(B, A) < \epsilon \iff B \subseteq N_\epsilon(A)$$

where $N_\epsilon(A)$ is the ϵ-open neighbourhood of A in X. Therefore, the open quasi-metric ball in UX of radius ϵ centred at A is in fact the open set $\Box b$ where $b = N_\epsilon(A)$.

(ii) Suppose $\epsilon > 0$ is given. Let $a = N_\epsilon(A)$ be the ϵ-open neighbourhood of A in X. Note that, by Proposition 3.1(iii), $\Box a$ is a Scott open subset in $(UX, \supseteq)$. Since it contains $A = \bigcap_i A_i$, it must contain A_N for some $N \geq 0$. Therefore $d_u(A_i, A) < \epsilon$ for all $i \geq N$. $\Box$

Proposition 3.3 *Let X be a locally compact Hausdorff space. Then the dcpo $(UX, \supseteq)$ is continuous, and the Scott topology coincides with the upper topology. The way-below relation $B \ll C$ holds if and only if B contains a neighbourhood of C.*

Proof For each $C \in UX$, let $W(C) = \{A \mid \exists a \in \Omega X.\ C \subseteq a \subseteq A\}$. Since C is a compact subset of the locally compact space X, it has a relatively compact neighbourhood, and hence $W(C)$ is not empty. Furthermore, if $A_1, A_2 \in W(C)$, then $A_1 \cap A_2 \in W(C)$ and therefore $W(C)$ is directed. We claim that $\bigcap_{A \in W(C)} A = C$. Clearly $\bigcap_{A \in W(C)} A \supseteq C$. If $x \in X - C$, then, since X is Hausdorff and C compact, there exists a relatively compact neighbourhood of C whose closure does not contain x. Therefore, $\bigcap_{A \in W(C)} A \subseteq C$.

We now show that $W(C) = \{B \mid B \ll C\}$. Suppose $B \ll C$. Since $\bigcap_{A \in W(C)} A \subseteq C$, there exists $A \in W(C)$ with $B \supseteq A$. It follows that $B \in W(C)$. On the other hand, if $A \in W(C)$, there exists $a \in \Omega X$ with $C \subseteq a \subseteq A$. Hence, we have $C \in \Box a \subseteq \uparrow A$. By Proposition 3.1(iii), $\Box a$ is Scott open. Therefore, $A \ll C$.

We have shown that $(UX, \supseteq)$ is a continuous dcpo and characterised the way-below relation. It remains to show that any Scott open subset is open in the upper topology. Let O be a Scott open subset of $(UX, \supseteq)$. Assume $C \in O$. Since $\bigcap_{A \in W(C)} A = C$, we have $W(C) \cap O \neq \emptyset$, i.e. there exist $A \in UX$ and $a \in \Omega X$ with $C \subseteq a \subseteq A \in O$. Therefore, $C \in \Box a \subseteq O$, i.e. O is open in the upper topology. $\Box$

Clearly, X can be embedded onto the set of maximal points of UX by the singleton map $s : X \to UX$ with $x \mapsto \{x\}$, and for any collection of open sets a_i $(i \in I)$ of X, we have

$$s(X) \cap (\bigcup_{i \in I} \Box a_i) = s(\bigcup_{i \in I} a_i)$$

i.e. the subspace topology induced by UX on the set of maximal elements $s(X)$ is, in effect, the topology of X. Furthermore, any point of X, regarded as the maximal point $\{x\}$ of UX, can be approximated by the closures of its relatively compact neighbourhoods.

3.2 An Effective Structure

If X is also assumed to be second countable, then we can construct an effective structure for UX.

Proposition 3.4 *Let X be a second countable locally compact Hausdorff space.*

(i) *UX is ω-continuous with an order basis consisting of finite unions of closures of relatively compact open sets of X.*

(ii) *If X is compact, then $(UX, \supseteq)$ is Lawson-compact, and hence a coherent domain.*

(iii) *If X is compact and zero dimensional then UX is ω-algebraic, i.e. it is a Scott domain.*

Proof (i) Since X is a second countable, locally compact Hausdorff space, it has a countable basis consisting of relatively compact open sets [Dugundji, 1966, page 238]. Let S be the set of finite unions of closures of this basis. Then S is a countable subset of UX. For each $C \in UX$, we let $W_1(C) = \{B \in S \mid B \ll C\}$. As in the case of $W(C)$ in the proof of Proposition 3.3, it follows that $W_1(C)$ is directed with lub C. Therefore, S is a countable basis for UX.

(ii) If X is compact, then $(UX, \supseteq)$ is a Scott-continuous retract of a Scott domain. Since Scott domains are Lawson-compact, it follows that $(UX, \supseteq)$ is also Lawson-compact [Abramsky and Jung, 1994, Proposition 4.2.20].

(iii) Since X is zero dimensional, it has a countable basis consisting of closed-open subsets. Since X is also compact, the closed-open subsets are compact-open. Since finite union of compact-open sets is also compact-open, it follows that the countable basis S of UX, as in the proof of (ii) above, consists of compact-open subsets of X. But a compact-open subset of X is easily seen to be a compact element of $(UX, \supseteq)$. Therefore, UX is ω-algebraic. $\square$

Corollary 3.5 *The upper space of the Cantor space $\Sigma^{\mathbb{N}}$ is a Scott domain.* $\square$

An ω-continuous dcpo can be given an *effective structure* wrt an enumeration of a given order basis, in particular by using the effective theory developed in [Edalat, 1991], which generalises the corresponding theory for ω-algebraic dcpo's in [Plotkin, 1981]. This enables us to have the notion of a *computable* element of an effectively given ω-continuous dcpo and that of a *computable* function between two effectively given ω-continuous dcpo's. It then follows that the least fixed point of a computable function on an effectively given dcpo with bottom is computable.

Let X be a second countable locally compact Hausdorff space. We take a countable basis of relatively compact open sets of X, fix an enumeration e of the induced order basis of UX and assume that UX is effectively given wrt e. All notions of computability, defined below, are understood to be with respect to this fixed enumeration e. We say that a compact set $C \subseteq X$ is *computable* if C is a computable element of UX. This provides an effective theory of second countable locally compact Hausdorff spaces. It can also be extended to second countable locally compact metric spaces and, hence, to other branches of analysis. Here, however, we will only note how it can be readily used to give an effective framework for dynamical systems.

We say that a dynamical system (X, f) is *computable* if the continuous function $Uf : UX \to UX$ is computable. Then we can deduce the following.

Proposition 3.6 *An attractor of a computable dynamical system is computable.* $\square$

As noted previously, attractors are the most important objects studied in dynamical systems. Our theory, therefore, provides a constructive framework for their study.

3.3 Chaos and the Upper Space

In order to handle chaotic systems in the framework of UX, we need to define sensitive dependence on initial conditions for dynamical systems on quasi-metric spaces. Given a quasi-metric space (Y, d), we say that a continuous map $g : Y \to Y$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in X$ and any neighbourhood N of x, there exists $y \in N$ and $n \geq 0$ such that $\max(d(g^n(x), g^n(y)), d(g^n(y), g^n(x))) > \delta$. Clearly this definition is consistent with the previous one if d is actually a metric. We say, as before, that $g : Y \to Y$ is *chaotic* if, (i) it has sensitive dependence on initial conditions, (ii) it is topologically transitive, and (iii) its periodic points are dense. We can now show the following.

Theorem 3.7 *Let $f : X \to X$ be a continuous map on the metric space X. Then U preserves the three conditions of chaos separately; in particular, if f is chaotic on X then Uf is chaotic on UX.*

Proof (i) Suppose $f : X \to X$ is sensitive to initial conditions with constant $\delta > 0$. We claim that $Uf : UX \to UX$ is also sensitive to initial conditions with the same constant δ. Let $C \in \Box a \in UX$. Then $C \subseteq a$. Choose $x \in C$. There exist $y \in a$ and $n \geq 0$ such that $d(f^n x, f^n y) > \delta$. Hence,

$$d_u(f^n C, f^n\{y\}) = d(f^n C, f^n\{y\}) \geq d(f^n x, f^n y) > \delta.$$

Therefore, $\max(d_u(f^n C, f^n\{y\}), d_u(f^n\{y\}, f^n C)) > \delta$.

(ii) Suppose $f : X \to X$ is topologically transitive. Let $\Box a$ and $\Box b$ be basic open sets with $a, b \neq \emptyset$. Then there exists $n \geq 0$ such that $a \cap f^n(b) \neq \emptyset$. It follows that $\Box a \cap f^n(\Box b) \neq \emptyset$. Therefore, $Uf : UX \to UX$ is topologically transitive.

(iii) Suppose the periodic orbits of $f : X \to X$ are dense in X. Let $\Box a$ be any basic open set of UX. By our assumption, f has a periodic point $p \in a$. But then $\{p\} \in \Box a$ is a periodic point of $Uf : UX \to UX$. $\Box$

Example 3.8 Consider the solenoid again (Example 2.8), now in the context of the upper space. The torus T is the least element, i.e. the bottom, of UT. Furthermore, Λ is the least fixed point of the continuous map $Uf : UT \to UT$ and is obtained as the lub of the chain

$$T \ll Uf(T) \ll (Uf)^2(T) \ll (Uf)^3(T) \ll \ldots$$

Uf has infinitely many other fixed points as any periodic orbit gives a fixed point of Uf. $\Box$

3.4 The Plotkin Power Domain

In this section we will show that the Plotkin power domain of the upper space of X is the proper framework to study IFSs (Definition 1.1) on X.

Proposition 3.9 *Let $f_n : X \to Y$, $n = 1, 2, \ldots, N$, be a finite set of continuous maps of the Hausdorff spaces X and Y. Then the map*

$$\begin{array}{rccl} F: & UX & \to & UY \\ & C & \mapsto & f_1(C) \cup f_2(C) \cup \ldots \cup f_N(C) \end{array}$$

is Scott continuous.

Proof By Lemma 2.6, each $Uf_i : UX \to UY$ is Scott continuous. By Proposition 3.1(ii), the meet operation $\cup$ in $(UY, \supseteq)$ is continuous, which implies that F is Scott continuous. $\square$

We now need a lemma. Recall that a closed set is *regular* if it is the closure of its interior.

Lemma 3.10 *Let $\{X; f_1, \ldots, f_N\}$ be an hyperbolic IFS on a complete and locally compact metric space X. Then, there exists a regular compact subset $A \subseteq X$ such that $F(A) \subseteq A$, where $F : UX \to UX$ is the induced map on the upper space, i.e. $F(C) = f_1(C) \cup \ldots \cup f_N(C)$ for $C \in UX$.*

Proof Let B be the unique attractor of the IFS. Since B is compact, there exists $r > 0$ such that the r-neighbourhood, $N_r(B) = \{x \in X \mid \exists y \in B.\, d(x, y) < r\}$, of B is relatively compact. Put $A = \overline{N_r(B)}$. Then A is a regular compact set. If $x \in A$, then by compactness of A, there exists $y \in B$ such that $d(x, y) = d(x, B) \leq r$. We have $f_i(B) \subseteq B$ for $1 \leq i \leq N$, since $\bigcup_i f_i(B) = B$. Hence, $f_i(y) \in B$. But for each i, the map f_i is contracting, which implies $d(f_i(x), f_i(y)) \leq d(x, y) \leq r$. It follows that $f_i(x) \in A$, for each $x \in A$, i.e. $F(A) \subseteq A$. $\square$

In particular, if X is $\mathbb{R}^n$ with the Euclidean metric and s_i, $0 \leq s_i < 1$, is the contractivity factor of f_i $(1 \leq i \leq N)$, then it is easy to check that we have $F(A) \subseteq A$, where A is any closed ball of radius R with

$$R > \max_i \frac{|f_i(0)|}{1 - s_i}.$$

As a result of the above lemma, we can assume without loss of generality for an hyperbolic IFS that X is compact.

Let $\{X; f_1, \ldots, f_N\}$ be an IFS, with X compact. Then the induced map $F : UX \to UX$ has a least fixed point, namely $Y(F) = \bigcap_{m \geq 0} F^m(X)$. We will shortly prove that if the IFS is hyperbolic, then this fixed point, i.e. the attractor of the IFS, is unique. Our proof has a domain-theoretic form but it is essentially similar to those in [Hutchinson, 1981; Hayashi, 1985].

We can obtain the attractor of an hyperbolic IFS as a result of a non-deterministic computation, which gives us the "history" of the points of the attractor. For this, we use the Plotkin power domain CUX of the upper space UX.

Suppose X is a second countable locally compact Hausdorff space. Then CUX is an ω-continuous dcpo and has an order basis consisting of equivalence classes of finite sets, $\{A_1, A_2, \ldots, A_K\}$, of basis elements A_i of UX $(1 \leq i \leq K)$ under the Egli-Milner preorder $\sqsubseteq_{\mathrm{EM}}$. We denote this equivalence class by $[\{A_1, A_2, \ldots, A_K\}]$.

Now UX can be embedded in CUX by the singleton map $[.] : UX \to CUX$, defined by $A \mapsto [\{A\}]$. In the other direction, the union map $\bigcup : CUX \to UX$, defined on the above order basis by $[D] \mapsto \bigcup_{A \in D} A$ is the corresponding projection. We note that these maps can also be defined from general categorical considerations.

Given an IFS $\{X; f_1, \ldots, f_N\}$, there is a continuous map $f : CUX \to CUX$ which is defined on the above order basis by:

$$f([\{A_1, A_2, \ldots, A_K\}]) = [\{f_i(A_j) \mid 1 \leq i \leq N, 1 \leq j \leq K\}].$$

Suppose now that X is compact. Then the least fixed point $\bigsqcup_{i \geq 0} f^i([\{X\}])$ of f is (the equivalence class of) the set of lubs of the infinite branches of the finitary branching tree in Figure 2, which we call the *IFS tree*.

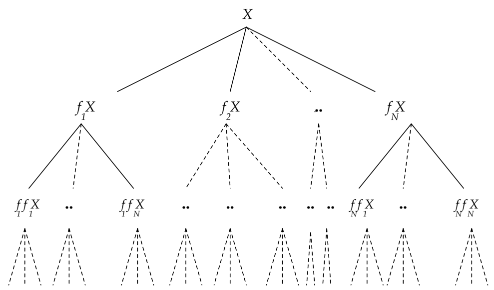

Figure 2. The IFS tree

If the IFS is hyperbolic with contractivity c, then the diameter of a compact set A of depth n in any infinite branch of the tree will satisfy $\mathrm{diam}(A) \leq c^n \mathrm{diam}(X)$. Hence the lub of any infinite branch is a singleton set. It then follows easily that the least fixed point $Y(f)$ of f is a maximal element of CUX, and hence it is the unique fixed point of f.

It is easy to check that the following diagram commutes:

$$\begin{tikzcd} UX \arrow[d, "F"'] & CUX \arrow[l, "\bigcup"'] \arrow[d, "f"] \\ UX & CUX \arrow[l, "\bigcup"'] \end{tikzcd}$$

Since the least fixed point operator is a uniform fixed point operator and $\bigcup : CUX \to UX$ is a strict map, it follows that $\bigcup(Y(f)) = Y(F)$. In the other direction, consider the shattering map

$$\begin{array}{rrcl} \mathsf{S}: & UX & \to & CUX \\ & A & \mapsto & [\{\{x\} \mid x \in A\}] \end{array}$$

Note that for any $A \in UX$, the subset $\{\{x\} \mid x \in A\} \subseteq UX$ is clearly non-empty and convex. It is also Lawson-closed: If $\mathcal{C}_n$ is, for each $n \geq 1$, a finite set of closed balls of radius $1/n$ which intersect and cover the compact set A, then for each $B \in \mathcal{C}_n$ the set $\uparrow B \subseteq UX$ is Lawson-closed and we have

$$\{\{x\} \mid x \in A\} = \bigcap_{n \geq 1} \bigcup_{B \in \mathcal{C}_n} \uparrow B.$$

Since, by Proposition 3.4(ii), $(UX, \supseteq)$ is Lawson-compact, it follows that $\{\{x\} \mid x \in A\}$ is a non-empty convex and Lawson-compact subset of UX and, therefore, represents an element of CUX [Abramsky and Jung, 1994, Theorem 6.2.22]. Hence, S is well-defined. We can immediately see from the IFS tree that $\mathsf{S}(Y(F)) = Y(f)$. We note that the shattering map S is not monotone and in fact $\mathsf{S}(A)$ is always maximal in CUX. However, for any $A \in UX$, the Scott continuity of f implies

$$f(\mathsf{S}(A)) = f([\{\{x\} \mid x \in A\}]) = [\{\{f_i(x)\} \mid x \in X, 1 \leq i \leq N\}] = \mathsf{S}(F(A)).$$

In other words, the following diagram commutes.

$$\begin{tikzcd} UX \arrow[r, "\mathsf{S}"] \arrow[d, "F"'] & CUX \arrow[d, "f"] \\ UX \arrow[r, "\mathsf{S}"] & CUX \end{tikzcd}$$

It follows that if A is a fixed point of F, then $\mathsf{S}(A)$ is a fixed point of f. Since S is easily seen to be one-to-one, we deduce the following.

Theorem 3.11 *Let $\{X; f_1, \ldots, f_N\}$ be an IFS with X compact. Then $F : UX \to UX$ and $f : CUX \to CUX$ each have a least fixed point, and these fixed points are mapped to each other by the shattering and the union maps. If the IFS is hyperbolic, these fixed points are unique.* $\square$

If the IFS is hyperbolic, any point $x \in \bigcap_{m \geq 0} F^m(X)$ of the attractor is the lub of an infinite branch of the corresponding IFS tree as in the following:

$$\begin{array}{c} X \\ \downarrow \\ f_{i_1}(X) \\ \downarrow \\ f_{i_1} f_{i_2}(X) \\ \downarrow \\ f_{i_1} f_{i_2} f_{i_3}(X) \\ \vdots \end{array}$$

The infinite sequence $i_1 i_2 i_3 i_4 \ldots$ therefore gives us a code for the point x of the attractor, whereas, for each $m \geq 1$, the finite sequence $i_1 i_2 \ldots i_m$ codes the compact neighbourhood $f_{i_1} f_{i_2} \ldots f_{i_m}(X)$ of x. If $f_i(X) \cap f_j(X) = \emptyset$ for $i \neq j$, then it is easy to see that the code of any point x of the attractor is unique. Otherwise, a point can have more than one code.

The above theorem is the basis of a new algorithm [Edalat, 1993c], implemented in [Nikolaou, 1993; Kakos, 1993], for generating the digitised image of the attractor of an hyperbolic IFS on the plane.

3.5 Julia sets

Julia sets, together with the Mandelbrot set, have been the object of intensive research in the last decade. See [Blanchard, 1984] for an overview and an extensive bibliography. Many researchers have investigated the question of computability of these sets. In particular, a team of dynamical system theorists, including Steven Smale, whose work in the 60's generated the ongoing surge of interest in dynamical systems, have developed a theory of computability over real numbers with which they have studied the Julia sets [Blum *et al.*, 1989]. In their framework, all real numbers are taken to be fundamental elements and, hence, computable. They show that most Julia sets are not R.E. over real numbers.

The upper space provides a framework for investigating the computability of these sets using the usual theory of computability over natural numbers. In this paper, we confine ourselves to the Julia sets of the family of quadratic complex maps of the form $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = z^2 + c$, where $c \in \mathbb{C}$ is a fixed complex number. This family is of fundamental importance as it gives a model for the emergence of chaos in physical systems [Feigenbaum, 1979]. The *filled Julia set* is defined to be the set of all points which do not escape to infinity under iteration by f. The *Julia set* itself is the boundary of the filled

Julia set. We need the following lemma, whose analogue for the Vietoris space can be found in [Barnsley, 1988, page 291]:

Proposition 3.12 *Let $f : Y \to Y$ be a continuous map on a Hausdorff space Y. Suppose the compact set $X \subseteq Y$ satisfies $f^{-1}(X) \subseteq X \subseteq fX$. Then the map $f^{-1} : UX \to UX$, which takes any compact set to its pre-image, is well-defined and continuous.*

Proof Let $A \in UX$. Then $A \subseteq X$ and hence $f^{-1}A \subseteq f^{-1}X \subseteq X$. Since A is closed and f is continuous, $f^{-1}A$ is a closed subset of the compact set X, and is therefore compact. Since A is non-empty and $A \subseteq X \subseteq fX$, it follows that $f^{-1}A$ is also non-empty. Hence, $f^{-1} : UX \to UX$ is well-defined. It remains to show that f^{-1} is continuous. Since X is compact, the upper topology and the Scott topology on UX coincide by Proposition 3.3. Therefore, it is sufficient to show that the monotone function $f^{-1} : (UX, \supseteq) \to (UX, \supseteq)$ preserves lubs of directed sets. But the equality

$$f^{-1} \bigcap_{i \in I} A_i = \bigcap_{i \in I} f^{-1} A_i$$

holds in fact for any mapping $f : X \to X$ and any family of sets $\langle A_i \rangle_{i \in I}$. □

Suppose now $f : \mathbb{C} \to \mathbb{C}$ is a quadratic complex map with $f(z) = z^2 + c$. We can choose, say, a large solid square X around the origin, with sides parallel to the axes and with vertices at points with rational coordinates, such that $f^{-1}(X) \subseteq X \subseteq fX$.

Proposition 3.13 *The filled Julia set of the quadratic map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 + c$, is the least fixed point of $f^{-1} : UX \to UX$.*

Proof A sketch of the proof can be constructed from [Barnsley, 1988, page 272]. For a direct proof see [Edalat, 1993b]. □

By introducing a cut in the complex plane, we can make the two maps

$$\begin{array}{rccc} g^+ : & X & \to & X \\ & w & \mapsto & \sqrt{w-c} \end{array} \qquad \begin{array}{rccc} g^- : & X & \to & X \\ & w & \mapsto & -\sqrt{w-c} \end{array}$$

continuous. Then, we have a non-hyperbolic IFS

$$\{X; g^+, g^-\}$$

and we can make a non-deterministic computation of Julia sets using the Plotkin power domain. This has been implemented in [Jones, 1993]. This method works for all cases in which an explicit formula for finding the roots of the polynomial exists, i.e. it works for quadratic, cubic and quartic polynomials and also for special polynomials, for example of the form $z \mapsto z^n + c$ where n is any positive integer.

Since UX is ω-continuous, it can be given an effective structure as explained earlier in this section. A convenient countable order basis for $U\mathbb{C}$ is given by the collection of finite unions of rectangles with sides parallel to the axes and vertices at points with rational coordinates. Since the least fixed point of a computable function is computable, it follows that the filled Julia set is computable if the map $f^{-1} : UX \to UX$ is computable with respect to an enumeration of the above basis. This provides a framework for studying the computability of Julia sets.

In Figure 3, you can see the first few iterates of f^{-1} for three different quadratic maps, as c decreases from -1 to -3, taken with permission from [Barnsley, 1988].

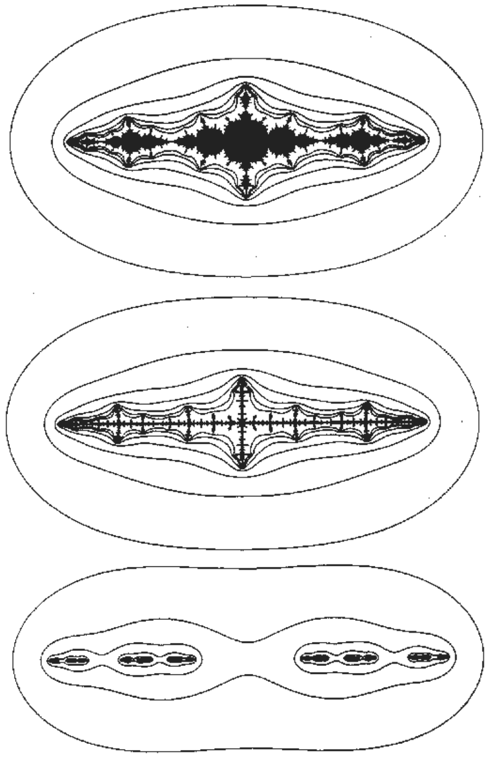

Figure 3. Generating three Julia sets in the upper space

4 The lower space

The lower space can be used in computing the invariant sets, and in particular the attractors of a dynamical system, when it is possible to generate a dense orbit of the invariant set. Hyperbolic IFSs provide a good example. First, we need some basic properties of the lower space. It is easy to see that the specialisation ordering of the lower space is subset inclusion. If $A, B \in LX$, where X is a Hausdorff space, then clearly $A \subseteq B$ implies $A \cap a \neq \emptyset \Rightarrow B \cap a \neq \emptyset$, for any open set $a \in \Omega X$. On the other hand, if $x \in A - B$ then the complement a of B is an open neighbourhood of x with $a \cap B = \emptyset$. This shows that the specialisation ordering is indeed subset inclusion. Clearly then $(LX, \subseteq)$ is a complete lattice such that the lub of a set of closed sets is the closure of their union.

Proposition 4.1 *Let X be a compact metric space.*

(i) *The lower topology of LX coincides with the quasi-metric topology induced by d_l.*

(ii) *If $\langle A_n \rangle_{n \geq 0}$ is an increasing chain in $(LX, \subseteq)$ with lub A, then $d_l(A_n, A) \to 0$ as $n \to \infty$.*

Proof (i) Suppose $A \in \Diamond a$, where $a \in \Omega X$. Then $a \cap A \neq \emptyset$ and therefore a contains an open ball $B_\epsilon(x)$ with $x \in A$ and $\epsilon > 0$. It follows that

$$d_l(C, A) < \epsilon \;\Rightarrow\; d(A, C) < \epsilon \;\Rightarrow d(\{x\}, C) < \epsilon \;\Rightarrow\; a \cap C \neq \emptyset \;\Rightarrow\; C \in \Diamond a.$$

Therefore a basic open set of the lower topology is open in the quasi-metric topology. Conversely, consider the quasi-metric ball of radius $\epsilon > 0$ centred at $A \in LX$. Since A is compact, there exist open balls b_i $(1 \leq i \leq k)$ of radius $\frac{\epsilon}{2}$ in X which cover A. Let $O = \bigcap_i \Diamond b_i$. Then it is easy to see that $C \in O$ implies $d_l(C, A) < \epsilon$. Therefore, any quasi-metric open ball is open in the upper topology.

(ii) Suppose $\epsilon > 0$ is given. Put $O = \bigcap_i \Diamond b_i$, where the open balls b_i $(1 \leq i \leq k)$ of radius $\frac{\epsilon}{2}$ in X cover A. Since $A = \overline{\bigcup_{i \geq 0} A_i}$, there exists $N \geq 0$ such that $A_N \cap b_n \neq \emptyset$ for $1 \leq n \leq k$. It follows that $A_i \in O$ for $i \geq N$, which implies $d_l(A_i, A) < \epsilon$ for $i \geq N$. □

Proposition 4.2 *Let $f_i : X \to Y$, $i = 1, 2, \ldots, N$, be a finite set of continuous maps of the Hausdorff spaces X and Y. Then the map*

$$\begin{array}{rccl} F: & LX & \to & LY \\ & C & \mapsto & \overline{f_1(C)} \cup \overline{f_2(C)} \cup \ldots \cup \overline{f_N(C)} \end{array}$$

is Scott continuous.

Proof For every continuous function $f : X \to Y$, the function $Lf : LX \to LY$ with $Lf(C) = \overline{f(C)}$ is continuous. In fact, if $\langle C_i \rangle_{i \in I}$ is a directed set in LX, we have

$$\overline{f(\overline{\bigcup_i C_i})} = \overline{\overline{f(\bigcup_i C_i)}} = \overline{f(\bigcup_i C_i)} = \overline{\bigcup_i f(C_i)} = \overline{\bigcup_i \overline{f(C_i)}}$$

using $f(\overline{A}) \subseteq \overline{f(A)}$. Since union is join in LX, and the join operation is Scott continuous, it follows that F is Scott continuous. □

Let $\{X; f_1, \ldots, f_N\}$ be an hyperbolic IFS with X compact and let $x_i \in X$ be the unique fixed points of the contracting maps f_i, $1 \leq i \leq N$. The map $F : LX \to LX$ defined above is now given by $F(A) = f_1(A) \cup \ldots \cup f_N(A)$, since A and hence $f(A)$ is compact. Suppose $B \subseteq \{x_i \mid 1 \leq i \leq N\}$ is any non-empty subset of the fixed points. Then, we have:

Proposition 4.3 *The attractor of the IFS is the unique fixed point of $F : LX \to LX$ and is the lub of the chain*

$$B \subseteq F(B) \subseteq F^2(B) \subseteq \ldots$$

Proof Since B consists of fixed points of the maps f_i, we have $B \subseteq F(B)$, and therefore, by monotonicity of F, we indeed have a chain as above. The continuity of F implies that the lub of this chain is a fixed point of F and is therefore the attractor of the IFS. □

However, the lower space is not in general continuous and, therefore, cannot be given a simple effective structure.

5 Measure theory via domain theory

We denote the set of all Borel sets of a topological space Y by $\mathcal{B}(Y)$. Recall that a *positive* Borel measure on a topological space Y is a Borel measure μ such that $\mu(B) \geq 0$ for all Borel sets $B \in \mathcal{B}(Y)$. A positive Borel measure μ is *finite* if $\mu(B) < \infty$ for all $B \in \mathcal{B}(Y)$. Recall also Definition 1.4 of a continuous valuation.

In this section, we will show that the set of finite measures on a locally compact second countable Hausdorff space X can be embedded into the set of continuous valuations on UX, and consequently any finite positive Borel measure on X can be identified as the lub of a chain of linear combinations of pointwise valuations on UX. This will establish a fundamental link between measure theory and domain theory.

Our results are based on an important property of locally compact Hausdorff spaces, which we will establish now. We need the following result of Pettis which holds under more general conditions than stated here.

Theorem 5.1 *[Pettis, 1951, page 193]. Let X be a Hausdorff space and $\mu : \Omega X \to [0,\infty)$ be a valuation. Suppose that given $O \in \Omega X$ and $\epsilon > 0$, there exists $O' \in \Omega X$ and a compact set C such that $O' \subseteq C \subseteq O$ and $\mu(O) \leq \mu(O') + \epsilon$. Then μ has a unique extension to a measure on X.* □

We can now easily show the following.

Proposition 5.2 *Any continuous valuation on a locally compact Hausdorff space has a unique extension to a measure.*

Proof It is well known that the lattice ΩX of open sets of a locally compact space X is continuous with $O' \ll O$ iff there exists a compact set C with $O' \subseteq C \subseteq O$ [Gierz *et al.*, 1980, page 42]. Suppose $\mu : \Omega X \to [0,\infty)$ is a continuous valuation. Let $O \in \Omega X$ and $\epsilon > 0$ be given. Then the collection of open sets $\{O' \mid O' \ll O\}$ is directed and $O = \bigcup\{O' \mid O' \ll O\}$. Since μ is continuous, we have $\mu(O) = \sup\{\mu(O') \mid O' \ll O\}$. It follows that there exists $O' \ll O$ with $\mu(O) \leq \mu(O') + \epsilon$. Hence the conditions of Theorem 5.1 are satisfied and μ has a unique extension to a measure. □

Corollary 5.3 *On a second countable locally compact Hausdorff space, finite measures and continuous valuations are in one to one correspondence.*

Proof All we need to show is that a finite measure defines a continuous valuation. Clearly a finite measure, being σ-additive, defines an ω-continuous valuation. Since ΩX is now ω-continuous, any ω-continuous valuation is in fact continuous. □

Assume now that X is a second countable locally compact Hausdorff space. Denote by $M(X)$ the set of all positive Borel measures μ on X with $\mu(X) \leq 1$. For $\mu, \nu \in M(X)$, define $\mu \sqsubseteq \nu$ iff for all $O \in \Omega X$ we have $\mu(O) \leq \nu(O)$. Recall Definition 1.5 of the probabilistic power domain PY of a topological space Y.

Corollary 5.4 *$(M(X), \sqsubseteq)$ is a dcpo and is isomorphic to $(PX, \sqsubseteq)$.*

Proposition 5.5 (i) *Let $\mu, \nu \in M(X)$ with $\mu \sqsubseteq \nu$. Then $\mu(B) \leq \nu(B)$ for all Borel sets $B \in \mathcal{B}(X)$.*

(ii) *Let $\langle \mu_i \rangle_{i\in I}$ be a directed set in $M(X)$ with $\mu = \bigsqcup_{i\in I} \mu_i$. Then $\mu(B) = sup_{i\in I}\mu_i(B)$ for all Borel subsets $B \in \mathcal{B}(X)$.*

Proof (i) By the definition of a regular measure (Definition 1.3), we have

$$\mu(B) = \inf\{\mu(O) \mid B \subseteq O, O \text{ open}\} \leq \inf\{\nu(O) \mid B \subseteq O, O \text{ open}\} = \nu(B).$$

(ii) Define $\nu : \mathcal{B}(X) \to [0,1]$ by $\nu(B) = \sup_{i\in I}\mu_i(B)$. Clearly $\mu(O) = \nu(O)$ for all $O \in \Omega X$. We will show that ν is a Borel measure which, by the uniqueness of the extension of a valuation to a measure on X (Proposition 5.2), proves that $\mu = \nu$. To show that ν is a measure we must check that it is σ-additive. Let $\langle B_j \rangle_{j\geq 0}$ be a countable set of disjoint Borel sets of X. For $i \in I$ and

$n \geq 0$, put

$$t_{in} = \sum_{j=0}^{n} \mu_i(B_j) \qquad \text{and} \qquad t_n = \sum_{j=0}^{n} \nu(B_j).$$

Then

$$\nu(\bigcup_{j\geq 0} B_j) = \mathrm{sup}_i \mu_i(\bigcup_{j\geq 0} B_j) = \mathrm{sup}_i \mathrm{sup}_n t_{in} = \mathrm{sup}_n \mathrm{sup}_i t_{in} = \mathrm{sup}_n t_n = \sum_{j=0}^{\infty} \nu(B_j)$$

since, for bounded subsets of real numbers, suprema commute with each other and taking a supremum commutes with finite addition. □

We now need the following result of J. Lawson. Recall that the lattice ΩY of open sets of a locally quasi-compact sober space Y is a continuous distributive lattice and Y is in fact isomorphic with the spectrum $\mathrm{Spec}(\Omega Y)$, consisting of non-unit prime elements of ΩY with the hull-kernel topology. The Lawson topology on ΩY induces a topology on $\mathrm{Spec}(\Omega Y)$ and hence on Y which is finer than the original topology of Y. (See [Gierz *et al.*, 1980, page 252].)

Theorem 5.6 *([Lawson, 1982, page 221]). Let Y be a second countable locally quasi-compact sober space and suppose μ is a continuous valuation on Y. Then μ extends uniquely to a regular measure on Y equipped with the relative Lawson topology induced from ΩY.* □

If Y is a continuous dcpo, then the relative Lawson topology induced on Y from ΩY coincides with the Lawson topology on Y [Abramsky and Jung, 1994, Exercise 7.3.19.8]. We therefore obtain the following.

Corollary 5.7 *Any continuous valuation on an ω-continuous bounded complete dcpo Y extends uniquely to a finite regular measure on Y equipped with its Lawson topology.*

We note here that in [Jones, 1989, page 110], it is claimed that any continuous valuation on a continuous dcpo Y extends uniquely to a Borel measure on Y. But there is a gap in the proof and we do not know if the stated result is correct.

Corollary 5.8 *For a second countable locally compact Hausdorff space X, any continuous valuation μ on UX extends uniquely to a finite regular measure μ^* on UX equipped with its Lawson topology.* □

Assume for the rest of this section that X is a second countable locally compact Hausdorff space. Since every locally compact Hausdorff space is regular, and since by Urysohn's Theorem [Dugundji, 1966, page 195, Corollary] every second countable regular Hausdorff space is metrizable, we can assume that X is in fact a metric space. We will now examine the relationship between the Borel subsets of X and those of UX.

Proposition 5.9 *Let $a \in \Omega X$ be an open subset of X. There exists a countable decreasing sequence, $\langle O_n \rangle_{n\geq 1}$, of open sets of UX such that $O_n \cap s(X) = s(a)$ for each $n \geq 1$ and $\bigcap_{n\geq 1} O_n = s(a)$.*

Proof For each integer $n \geq 1$, let $\langle b_i \rangle_{i\in I_n}$ be the collection of open balls of radius less than $\frac{1}{n}$ which are contained in a. Let $O_n = \bigcup_{i\in I_n} \Box b_i$. Clearly $O_n \cap s(X) = s(a)$ and $O_n \supseteq O_{n+1}$ for all $n \geq 1$. Now put $A = \bigcap_{n\geq 1} O_n$. We claim that $s(a) = A$. If $x \in a$, then, for each $n \geq 1$, there is some $i \in I_n$ with $x \in b_i$, i.e. $\{x\} \in A$. On the other hand A consists of singletons only. For if $x, y \in C \in UX$ with $d(x, y) > 0$, then $C \notin \bigcup_{i\in I_n} \Box b_i$ if $n > \frac{2}{d(x,y)}$. But clearly $\{x\} \in A$ implies $x \in a$. Therefore $s(a) = A$. □

Corollary 5.10 *For each open set $a \in \Omega X$, $s(a)$ is a G_δ set, and therefore a Borel subset of UX. In particular $s(X)$ is a Borel subset of UX.* $\square$

Corollary 5.11 *If B is a Borel subset of X, then $s(B)$ is a Borel subset of UX.*

Proof Let $\mathcal{D} \subseteq \mathcal{B}(X)$ be the set of all Borel subsets B of X such that $s(B)$ is a Borel subset of UX. We want to show that $\mathcal{D} = \mathcal{B}(X)$. By the "Lemma on Monotone Classes" [Berberian, 1967, page 7], it is sufficient to show that $\mathcal{D}$ contains the field generated by the open subsets of X and it is a monotone class, i.e. it is closed under the union of increasing and the intersection of decreasing sequences. Any element of the field generated by the open subsets of X is the disjoint and finite union of sets of the form $a - b$ where $a, b \in \Omega X$ [Jones, 1989, page 36]. Since $s(a - b) = s(a) - s(b)$, where $s(a)$ and $s(b)$ are both Borel subsets of UX, it follows that $\mathcal{D}$ contains the field generated by the open subsets. On the other hand, it is easy to see that $\mathcal{D}$ is a monotone class: For example, if $\langle B_i \rangle_{i \geq 0}$ is an increasing sequence in $\mathcal{D}$, then $s(\bigcup_i B_i) = \bigcup_i s(B_i)$ is a Borel subset of UX since each $s(B_i)$ is a Borel set. $\square$

We also need the following result.

Proposition 5.12 *Let $C \subseteq X$ be a closed subset. Then $s(C)$ is a G_δ subset of UX.*

Proof For each $n \geq 1$, let $\langle b_i \rangle_{i \in I_n}$ be the collection of all open balls of radius less than $\frac{1}{n}$ in X which intersect C. Put $O_n = \bigcup_{i \in I_n} \Box b_i$. Then $O_n \supseteq O_{n+1}$ for all $n \geq 1$. Put $A = \bigcap_{n \geq 1} O_n$. We claim that $s(C) = A$. As in the proof of Proposition 5.9, A consists of singletons only. Let $x \in C$. Then for each $n \geq 1$ there is some b_i with $i \in I_n$ such that $x \in b_i$. Hence, we have $\{x\} \in A$. If $x \in X - C$, then the distance d of x from the closed set C is positive. It follows easily that $\{x\} \notin O_n$ for $n > \frac{2}{d}$, which proves the claim and completes the proof. $\square$

Conversely, for any Borel subset B of UX, the continuity of s implies that

$$s^{-1}(B) = \bigcup (s(X) \cap B)$$

is a Borel subset of X.

Lemma 5.13 *Let μ and ν be finite positive Borel measures on a topological space Y such that $\mu(O) \leq \nu(O)$ for all open sets $O \in \Omega Y$. Then $\mu(B) \leq \nu(B)$ for all G_δ sets $B \in \mathcal{B}(Y)$.*

Proof Let $B \in \mathcal{B}(Y)$ be a G_δ set. Then $B = \bigcap_{n \geq 1} O_n$ with $O_n \in \Omega Y$. By replacing O_n with $O_1 \cap O_2 \cap \ldots \cap O_n$ we can assume that $O_n \supseteq O_{n+1}$ for all $n \geq 1$. Therefore

$$\mu(B) = \lim_{n \to \infty} \mu(O_n) \leq \lim_{n \to \infty} \nu(O_n) = \nu(B). \quad \square$$

Corollary 5.14 *If μ and ν are valuations on a continuous dcpo Y with $\mu \sqsubseteq \nu$ then their unique extensions μ^* and ν^* to regular Borel measures given by Corollary 5.8 satisfy $\mu^*(B) \leq \nu^*(B)$ for all G_δ sets $B \in \mathcal{B}(Y)$.* $\square$

Definition 5.15 A valuation $\mu \in PUX$ is said to be *supported* in $s(X)$ if $\mu^*(UX - s(X)) = 0$. If μ is supported in $s(X)$, then the *support* of μ is the set of points $y \in s(X)$ such that $\mu(N) > 0$ for any neighbourhood $N \subseteq UX$ of y. The subset of PUX consisting of all valuations which are supported in $s(X)$ is denoted by $S(X)$. $\square$

Proposition 5.16

(i) *A valuation* $\mu \in PUX$ *is supported in* $s(X)$ *iff* $\mu^*(\bigcup_i \Box a_i) = \mu^*(\bigcup_i s(a_i))$ *for all collections of open sets* $a_i \in \Omega(X)$, *with* $i \in I$.

(ii) $S(X)$ *is a subdcpo of* PUX. $\Box$

Proof (i) Suppose μ is supported in $s(X)$. Then we have

$$\mu^*(\bigcup_i \Box a_i) = \mu^*(\bigcup_i \Box a_i \cap s(X)) + \mu^*(\bigcup_i \Box a_i - s(X)) = \mu^*(s(\bigcup_i a_i))$$

since $\mu^*(\bigcup_i \Box a_i - s(X)) \leq \mu^*(UX - s(X)) = 0$. If on the other hand $\mu^*(\Box_X) = \mu^*(s(X))$, it follows immediately that μ is supported in $s(X)$.

(ii) Let $\langle \mu_i \rangle_{i \in I}$ be a directed set of elements in $S(X)$, with $\mu = \bigsqcup_i \mu_i$. Then $\mu_i^*(UX) = \mu_i^*(s(X))$ for all $i \in I$. Hence,

$$\mu^*(UX) = \sup_i \mu_i^*(UX) = \sup_i \mu_i^*(s(X)) \leq \mu^*(s(X)) \leq \mu^*(UX)$$

by Corollaries 5.10 and 5.14. Therefore $\mu^*(UX) = \mu^*(s(X))$, i.e. $\mu \in S(X)$. $\Box$

Definition 5.17 A *normalised* Borel measure on a topological space Y is a Borel measure μ such that $\mu(Y) = 1$. $\Box$

We denote by $M^1(X)$, respectively P^1UX and $S^1(X)$, the subset of $M(X)$, respectively PUX and $S(X)$, consisting of normalised measures (valuations). Given any $\mu \in PUX$ with $\mu(UX) = c \neq 0$, we can define $\mu' \in P^1UX$ by $\mu'(O) = \frac{1}{c}\mu(O)$ for any $O \in \Omega(UX)$. Then $\mu \sqsubseteq \mu'$. Therefore, the maximal elements of PUX are precisely the maximal elements of P^1UX. We also have the following:

Proposition 5.18 *If* $\mu \in S^1(X)$ *then* μ *is a maximal element of* PUX.

Proof Let $\mu \in S^1(X)$ and $\mu \sqsubseteq \nu$. Then $\mu^*(s(X)) = 1$ and by Corollaries 5.10 and 5.14, $\mu^*(s(X)) \leq \nu^*(s(X))$, which implies that $\nu^*(s(X)) = 1$. Therefore, $\nu \in S^1(X)$. Suppose $\mu \neq \nu$. Then there exists $O \in \Omega UX$ such that $\mu(O) < \nu(O)$. By Corollary 5.14, we have $\mu^*(B) \leq \nu^*(B)$ for all G_δ subsets $B \in \mathcal{B}(UX)$. We also know from Proposition 5.9 and Proposition 5.12 respectively that $O \cap s(X)$ and $s(X) - O \cap s(X)$ i.e. $s(X - s^{-1}(O))$ are G_δ subsets of UX. If $\mu^*(O \cap s(X)) < \nu^*(O \cap s(X))$, then

$$\begin{aligned}
\nu^*(s(X)) &= \nu^*(s(X) - O \cap s(X)) + \nu^*(O \cap s(X)) \\
&> \mu^*(s(X) - O \cap s(X)) + \mu^*(O \cap s(X)) \\
&= 1
\end{aligned}$$

which contradicts $\nu^*(s(X)) = 1$. If on the other hand, we have $\mu^*(O \cap s(X)) = \nu^*(O \cap s(X))$, it follows that $\mu^*(O - O \cap s(X)) < \nu^*(O - O \cap s(X))$ and therefore

$$\nu^*(UX - s(X)) \geq \nu^*(O - O \cap s(X)) > 0$$

contradicting $\nu \in S(X)$. We conclude that $\mu = \nu$. $\Box$

We conjecture that the converse of the above result holds as well, i.e. any maximal element of PUX is supported in $s(X)$.

We will now show that the singleton map $s : X \to UX$ induces an isomorphism between $M(X)$ and $S(X)$. Define the map $e : M(X) \to S(X)$ by $e(\mu) = \mu \circ s^{-1}$.

Proposition 5.19 *The map $e : M(X) \to S(X)$ is well-defined and Scott continuous.*

Proof Since $s^{-1} : \Omega UX \to \Omega X$ preserves union and finite intersection, it is easy to see that $e(\mu) = \mu \circ s^{-1}$ is a continuous valuation on UX and therefore belongs to PUX. We need to check that $\nu = e(\mu)$ is supported in $s(X)$. By Proposition 5.9, there exists a countable decreasing sequence $\langle O_i \rangle_{i \geq 0}$ of open sets $O_i \in \Omega UX$, with $s(X) = \bigcap_i O_i$ and $s^{-1}(O_i) = X$ for all $i \in I$. Hence

$$\nu^*(s(X)) = \lim_{i\to\infty} \nu^*(O_i) = \lim_{i\to\infty} \nu(O_i) = \lim_{i\to\infty} \mu(s^{-1}(O_i)) = \mu(X) = \mu(s^{-1}(UX)) = \nu^*(UX).$$

It remains to show that e is continuous. This easily follows from the continuity of composition. $\square$

In the other direction, define $j : S(X) \to M(X)$ by $j(\nu) = \nu^* \circ s$. Note that $\nu^* \circ s$ is a Borel measure on X: In fact ν^* is a Borel measure on UX which is supported in $s(X) \subseteq UX$ and $s : X \to s(X)$ is an isomorphism.

Proposition 5.20 *The map $j : S(X) \to M(X)$ is continuous.*

Proof First note that j is monotone. For, if $\nu_1 \sqsubseteq \nu_2$, then by Corollaries 5.10 and 5.14 we have $\nu_1^*(s(O)) \leq \nu_2^*(s(O))$ for all $O \in \Omega X$. Let $\langle \nu_i \rangle_{i \in I}$ be a directed set in $S(X)$. We want to prove that $j(\bigsqcup_i \nu_i) = \bigsqcup_i j(\nu_i)$. We have to show that they have the same value on open sets of X. Let $a \in \Omega X$. Since $\nu_i, \bigsqcup_i \nu_i \in S(X)$ for all $i \in I$, we have $\nu_i^*(s(a)) = \nu_i(\Box a)$ and $(\bigsqcup_i \nu_i)^*(s(a)) = (\bigsqcup_i \nu_i)(\Box a)$, for all $i \in I$, by Proposition 5.16(i). Therefore,

$$\begin{array}{rcl} (j(\bigsqcup_i \nu_i))(a) & = & (\bigsqcup_i \nu_i)^*(s(a)) = (\bigsqcup_i \nu_i)(\Box a) = \sup_i \nu_i(\Box a) \\ & = & \sup_i \nu_i^*(s(a)) = \sup_i j(\nu_i)(a) = (\bigsqcup_i j(\nu_i))(a) \end{array}$$

as required. $\square$

Theorem 5.21 *The dcpo's $M(X)$ and $S(X)$ are isomorphic via the maps $e : M(X) \to S(X)$ and $j : S(X) \to M(X)$.*

Proof We prove that j is the inverse of e. Let $\mu \in M(X)$. We show that μ and $j \circ e(\mu) = (\mu \circ s^{-1})^* \circ s$ have the same values on open sets of X. For $a \in \Omega X$, we have

$$j \circ e(\mu)(a) = (\mu \circ s^{-1})^* \circ s(a) = (\mu \circ s^{-1})(\Box a) = \mu(a)$$

since $\mu \circ s^{-1} \in S(X)$. Next we show that $e \circ j(\nu) = \nu$. If $\nu \in S(X)$ and $O \in \Omega UX$, we have

$$e \circ j(\nu)(O) = (\nu^* \circ s) \circ s^{-1}(O) = \nu^*(O \cap s(X)) = \nu(O)$$

since $\nu \in S(X)$. $\square$

Corollary 5.22 *The maps e and j restrict to give an isomorphism between $M^1(X)$ and $S^1(X)$.* $\square$

Since UX is ω-continuous, it follows from [Jones and Plotkin, 1989; Jones, 1989] that PUX is also ω-continuous. Furthermore, any valuation $\mu \in PUX$ is the directed lub of linear combinations of point valuations. In fact, if $B \subseteq UX$ is a countable order basis of UX, then PUX has the following countable order basis:

$$\{\sum_{i=1}^{n} r_i \eta_{b_i} \mid b_i \in B,\ r_i \text{ rational and } \sum r_i \leq 1\}.$$

Any finite measure on X is the directed lub of linear combinations of point valuations on UX. We therefore have an effective framework for measures on X. This framework can provide a constructive measure theory for locally compact second countable Hausdorff spaces.

We note that the map $j : S(X) \to M(X)$ can in fact be extended to a monotone map $\bar{\jmath} : PUX \to M(X)$ defined by $\bar{\jmath}(\nu) = \nu^* \circ s$. We then have $\bar{\jmath} \circ e = \mathrm{Id}_{M(X)}$ and $e \circ \bar{\jmath} \sqsubseteq \mathrm{Id}_{PUX}$. However, $\bar{\jmath}$ is not in general continuous. For example, if $X = [0,1]$ is the unit interval with the Euclidean metric and if $b_i = [0, \frac{1}{i}] \in U[0,1]$ $(1 \leq i)$, then the ω-chain $\langle b_i \rangle_{i \geq 0}$ in UX has lub $\{0\} \in UX$, and the ω-chain $\langle \eta_{b_i} \rangle_{i \geq 1}$ in PUX has lub $\eta_{\{0\}} \in PUX$. However, the measure $\bar{\jmath}(\eta_{b_i}) \in M([0,1])$ is zero everywhere, i.e. it is the bottom of $M([0,1])$ for all $i \geq 1$, whereas, $\bar{\jmath}(\eta_{\{0\}}) \in M([0,1])$ is the atomic measure at the point $0 \in [0,1]$. This means that we cannot dispense with UX to construct the effective theory. In fact, had $\bar{\jmath}$ been continuous, $M(X)$ would have been a continuous retract of the ω-continuous dcpo PUX and would, therefore, have been ω-continuous itself, i.e. $M(X)$ would have been sufficient to construct an effective theory.

6 IFSs with probabilities

We now apply the results of the previous section to IFSs with probabilities (Definition 1.2). Let X be a compact metric space. Then X is second countable and the results of the previous section hold for X. The upper space UX has now a bottom element, namely X. Furthermore, P^1UX has also a bottom element, namely, the unit point valuation η_X based at $X \in UX$ with

$$\eta_X(O) = \begin{cases} 1 & \text{if } O = UX \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{X; f_1, \ldots, f_N; p_1, \ldots, p_N\}$ be an iterated function system with probabilities. Define

$$\begin{array}{rccc} T: & P^1UX & \to & P^1UX \\ & \mu & \mapsto & T(\mu) \end{array}$$

by $T(\mu)(O) = \sum_{i=1}^N p_i \mu(f_i^{-1}(O))$. Note that

$$T(\mu)(UX) = \sum_{i=1}^N p_i \mu(f_i^{-1}(UX)) = \sum_{i=1}^N p_i \mu(UX) = \sum_{i=1}^N p_i = 1$$

and hence T is well-defined.

Proposition 6.1 *The map $T : P^1UX \to P^1UX$ is continuous.*

Proof First note that T is monotone. In fact, if $\mu \sqsubseteq \nu$, then $\mu(f_i^{-1}(O)) \leq \nu(f_i^{-1}(O))$ for $O \in \Omega UX$ and $1 \leq i \leq N$. Hence, $T(\mu)(O) \leq T(\nu)(O)$, and therefore $T(\mu) \sqsubseteq T(\nu)$. Suppose $\langle \mu_j \rangle_{j \in J}$ is a directed set of valuations on UX. Then, for any $O \in \Omega UX$, we have

$$\begin{array}{rcl} T(\bigsqcup_j \mu_j)(O) & = & \sum_{i=1}^N p_i (\bigsqcup_j \mu_j)(f_i^{-1}(O)) \\ & = & \sum_{i=1}^N p_i \sup_j \mu_j(f_i^{-1}(O)) \\ & = & \sup_j (\sum_{i=1}^N p_i \mu_j(f_i^{-1}(O))) \\ & = & \bigsqcup_j (T(\mu_j))(O). \end{array}$$

Therefore $T(\bigsqcup_j \mu_j) = \bigsqcup_j T(\mu_j)$, and T is continuous. □

Theorem 6.2 *Let $\{X; f_1, \ldots, f_N; p_1, \ldots, p_N\}$ be an hyperbolic IFS with probabilities. Then the least fixed point of T is supported in $s(X)$ and its support is $s(A)$, where A is the unique attractor of the IFS $\{X; f_1, \ldots, f_N\}$.*

Proof It is easy to check that for any $C \in UX$, we have $T(\eta_C) = \sum_{i=1}^{N} p_i \eta_{f_i(C)}$. It follows that

$$T^m(\eta_X) = \sum_{i_1,i_2,\ldots,i_m=1}^{N} p_{i_1} p_{i_2} \ldots p_{i_m} \eta_{f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_m}(X)}.$$

Let $\mu = \bigsqcup_m T^m(\eta_X)$ be the least fixed point of T. We will show that $\mu(s(X)) = 1$, which proves that μ is supported in $s(X)$. By the proof of Proposition 5.9 we have $s(X) = \bigcap_{n \geq 1} O_n$, where $O_n = \bigcup_{i \in I_n} \Box b_i$ with $\langle b_i \rangle_{i \in I_n}$ being the collection of all open balls of X with radius less than $\frac{1}{n}$. Since $\langle O_n \rangle_{n \geq 1}$ is a decreasing sequence of open sets of UX, we have $\mu(s(X)) = \inf_n \mu(O_n)$. We now calculate $\mu(O_n)$ for each $n \geq 1$. We have $\mu(O_n) = \sup_m T^m(\eta_X)(O_n)$, and

$$T^m(\eta_X)(O_n) = \sum_{i_1,i_2,\ldots,i_m=1}^{N} p_{i_1} p_{i_2} \ldots p_{i_m} \eta_{f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_m}(X)}(O_n).$$

Let d be the diameter of X and c the contractivity factor of the hyperbolic IFS. By definition $0 \leq c < 1$ and each of the maps f_i $(1 \leq i \leq N)$ contracts by at least a factor c. If $m \geq 1$ is large enough so that $c^m d < \frac{1}{n}$, then the diameter of the compact set $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_m}(X)$ (for $1 \leq i_1, i_2, \ldots, i_m \leq N$) in the above equation will be less than $\frac{1}{n}$ and hence is contained in an open ball b_i for some $i \in I_n$. It follows that in each term we have $\eta_{f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_m}(X)}(O_n) = 1$. Since

$$\sum_{i_1,i_2,\ldots,i_m=1}^{N} p_{i_1} p_{i_2} \ldots p_{i_m} = 1$$

we conclude that for each $n \geq 1$ we have $T^m(\eta_X)(O_n) = 1$ for large enough m. Therefore, $\mu(O_n) = 1$ for all $n \geq 1$, which implies $\mu(s(X)) = 1$. This shows that μ is supported in $s(X)$.

It remains to show that the support of μ is $s(A)$ where $A = \bigcap_{n \geq 0} F^n(X)$ is the unique attractor of the IFS. Let $x \in \bigcap_{n \geq 0} F^n X$ and suppose O is any open neighbourhood of $s(x)$ in UX. Then, there exists $\delta > 0$ such that the open ball $B_\delta(x)$ of radius δ centred at x satisfies $\Box B_\delta(x) \subseteq O$. Let $i_1 i_2 i_3 \ldots$ be a code for x as in Section 3. Choose $m \geq 1$ large enough so that $c^m d < \delta$. Then $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_m}(X) \in \Box B_\delta(x)$ (for $1 \leq i_1, i_2, \ldots, i_m \leq N$), and hence

$$\mu(O) \geq \mu(\Box B_\delta(x)) \geq T^m(\eta_X)(\Box B_\delta(x)) \geq p_{i_1} p_{i_2} \ldots p_{i_m} > 0$$

On the other hand, let $x \notin A$ and suppose $d(\{x\}, A) = \delta$. Let $m_0 \geq 1$ be such that $c^{m_0} d < \delta/2$, then for $m \geq m_0$ the diameter of the compact set $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_m}(X)$ (for $1 \leq i_1, i_2, \ldots, i_m \leq N$) is less than $\delta/2$. Since this set contains points of A, it must therefore be disjoint from $B_{\delta/2}(x)$, the ball of radius $\delta/2$ centred at x. Hence, $T^m(\eta_X)(\Box B_{\delta/2}(x)) = 0$ for all $m \geq m_0$. It follows that $\mu(\Box B_{\delta/2}(x)) = 0$. We conclude that $s(A)$ is the support of μ. $\Box$

Since by Proposition 5.18, a valuation $\mu \in P^1 UX$ is a maximal element if it is supported in $s(X)$, we immediately deduce:

Corollary 6.3 *An hyperbolic IFS with probabilities has a unique fixed point.* $\Box$

If μ is this unique fixed point, then $j(\mu) \in M^1(X)$, where j is the map defined in the previous section, is the unique invariant measure of the IFS with probabilities as in Barnsley's work. We have therefore obtained a simple and constructive proof of Hutchinson's result, which does not use the Hutchinson metric. Our proof is the basis of a new algorithm [Edalat, 1993c], implemented in [Nikolaou, 1993; Kakos, 1993], for generating the digitised image of the attractor of an hyperbolic IFS with probabilities on the plane.

7 Conclusion and further work

We have seen that domain theory can be successfully applied in studying dynamical systems, measure theory, and fractals of IFSs. We have obtained computational models, using the upper space and the probabilistic power domains, which lead to constructive theories in these mainstream branches of mathematics. Furthermore, some of the classic results in these fields, such as the existence and uniqueness of the invariant measure of an hyperbolic IFS with probabilities, can be obtained constructively using these models, giving rise to new algorithms.

The prospects for development of the subject look promising. In fact, there are a number of areas for immediate further work; we will briefly mention a few. Based on the effective theory of second countable locally compact Hausdorff spaces presented here, one can obtain a similar theory for second countable locally compact metric spaces. Furthermore, using the results on Borel measures in this paper, we are led to develop a constructive theory of integration of real-valued functions with respect to bounded Borel measures on compact metric spaces [Edalat, 1993a]. This should be compared with other attempts in constructive analysis including the work of Bishop and Bridges [Bishop and Bridges, 1985]. In another direction, we need to further investigate, for any dynamical system, the dynamical and ergodic properties of the associated dynamical systems on the Vietoris and the upper spaces. We also have to establish which Julia sets in our theory are computable and how the result compares with that in [Blum *et al.*, 1989].

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